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LIMITING BEHAVIOR OF BOUNDARY CROSSING PROBABILITIES FOR DEPENDENT SAMPLE SUMS

By

KEN-ICHI YOSHIHARA

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1. Introduction. Let $\{X_j\}$ be a sequence of random variables defined on a probability space (Ω, A, P). Let $S_n=X_1+\cdots+X_n$ for $n\geqq 1$ with $S_0=0$. Let $\{W(t), t\geqq 0\}$ be a standard Wiener process defined on that probability space. In [6] Robbins and Siegmund proved an invariance theorem which states that if $\{X_i\}$ is i.i.d.r.v's the limit of the probability that $S_n\geqq m^{1/2}g\left(\frac{n}{m}\right)$ for some $n\geqq \tau m$ (or some $n\geqq 1$) is the probability that $W(t)\geqq g(t)$ for some $t\geqq\tau>0$ (or for some $t>0$) where $g(t)$ is a function of a certain class of functions including functions which are $\sim(2t\log\log t)^{1/2}$ as $t\to\infty$.

On the other hand, recently, many approximation theorems for partial sums of independent and weakly dependent random vectors are proved. (See, for examples, Major [4], Philipp and Stout [5] and Berkes and Philipp [1]). Using the theorems, Shorack [8] proved the Darling and Erdös theorem on the maximum of normalized sums of certain dependent rv's (excepting the strong mixing case).

In this paper, modifying Shorack's technique, we prove the above mentioned Robbins and Siegmund theorems hold for some weakly dependent sample sums (including the strong mixing case) (Theorems 1–3). The results contain an extension of Shorack's theorem (Corollary to Theorem 3). Examples are also shown.

2. Results. Define the random function on $[0, \infty)$ by letting $S(t)=S_{\mathrm{int}(t)}$ for $t\geqq 0$, where int$(\cdot)$ denotes the greatest integer function. We always assume that the following conditions hold:

(2.1) $$EX_i=0 \quad \text{and} \quad E|X_i|^2\leqq M<\infty \quad \text{for} \quad i\geqq 1, \quad \text{and}$$

(2.2) $$D(t)=t^{-1/2}|S(t)-W(t)|=O((\log\log t)^{-\alpha}) \quad \text{a.s.} \quad \text{as } t\to\infty \quad \text{for some} \quad \alpha>\frac{1}{2}.$$

Theorem 1. *Suppose that $g(t)$ is continuous for $t\geqq\tau>0$, that $t^{-1/2}g(t)$ is ultimately nondecreasing as $t\to\infty$ and that*

(2.3)
$$\int_{\tau}^{\infty} t^{-3/2}g(t)\exp\{-g^2(t)/2t\}dt<\infty .$$

Then

(2.4)
$$\lim_{m\to\infty} P(S_n \geqq m^{1/2}g(n/m) \quad \textit{for some} \quad n\geqq \tau m)$$
$$=P(W(t)\geqq g(t) \quad \textit{for some} \quad t\geqq \tau).$$

Theorem 2. *Suppose that g is continuous for $t>0$, that $t^{-1/2}g(t)$ is non-increasing for t sufficiently small, and that*

(2.5)
$$\int_{0+}^{1} t^{-3/2}g(t)\exp\{-g^2(t)/2t\}dt<\infty .$$

Then, for all τ (>0) sufficiently small

(2.6)
$$\lim_{m\to\infty} P(S_n \geqq m^{1/2}g(n/m) \quad \textit{for some} \quad 1\leqq n\leqq \tau m)$$
$$=P(W(t)\geqq g(t) \quad \textit{for some} \quad 0<t\leqq \tau).$$

Theorem 3. *Suppose that $g(t)$ is continuous for $0<t\leqq\tau<\infty$ and that the both growth conditions of Theorems* 2 *and* 3 *hold for t sufficiently small. Then,* (2.4) *continues to hold with $n\geqq\tau m$ replaced by $n\geqq 1$ and $t\geqq\tau$ by $t>0$.*

Remarks. (a) The same relations as in Theorems 1–3 are valid if S_n, $W(t)$ are replaced by $|S_n|$, $|W(t)|$.
(b) Instead of assuming that the continuous function g satisfies the indicated growth conditions, it is sufficient to assume that it majorizes some function which does.

To extend Theorem 1 in Shorack [8], we define normalizing functions b and c by

(2.7)
$$b(t)=(2\log\log t)^{1/2}$$

and

(2.8)
$$c(t)=2\log\log t+2^{-1}\log\log\log t-2^{-1}\log(4\pi) \qquad \text{for} \quad t>e^e .$$

Let E_v denote the extreme value distribution function defined by

(2.9)
$$E_v(t)=\exp(-\exp(-t)) \qquad \text{for} \quad -\infty<t<\infty .$$

The following corollary is an extension of Shorack's theorem.

Corollary. *Define*

(2.10)
$$u_m=\sup_{1\leqq s\leqq m}\frac{S(s)}{s^{1/2}} \textit{ and } U_m=\sup_{1\leqq s\leqq m}\frac{|S(s)|}{s^{1/2}} .$$

Then

$$b(m)u_m - c(m) \xrightarrow{D} E_v \quad as \quad m \to \infty \tag{2.11}$$

and

$$b(m)U_m - c(m) \xrightarrow{D} E_v^2 \quad as \quad m \to \infty\,. \tag{2.12}$$

3. Proofs.

We need a lemma which corresponds to Lemma 4 of Robbins and Siegmund [7].

Lemma. *For any* $0<\tau<c<\infty$

$$\lim_{m\to\infty} P\left(S_n \geqq m^{1/2} g\left(\frac{n}{m}\right) \quad for\ some \quad \tau m \leqq n \leqq cm\right)$$
$$= P(W(t) \geqq g(t) \quad for\ some \quad \tau \leqq t \leqq c)\,.$$

Proof. The proof is easily deduced from Theorem 4.1 of Billingsley [1], (2.2) and the fact

$$P(\max_{\tau\leqq t\leqq c}(W(t)-g(t))=0)=0$$

which has been obtained by Ylvisaker [9].

Proof of Theorem 1. By Lemma to prove Theorem 1 it is enough to show that (2.4) holds for τ sufficiently large. Let

$$h(t) = t^{1/2}(\log|\log t|)^{-\alpha}\,.$$

Then there exists a constant M (>1) such that

$$\frac{1}{m^{1/2}} h(n) \leqq Mh\left(\frac{n}{m}\right) \text{ for all } n \geqq \tau m\,.$$

Hence for all m ($\geqq 1$)

$$P\left(W(n) \geqq m^{1/2}\left(g\left(\frac{n}{m}\right) + \varepsilon Mh\left(\frac{n}{m}\right)\right) \text{ for sor some } n \geqq \tau m\right) - p_m$$
$$\leqq P\left(W(n) \geqq m^{1/2} g\left(\frac{n}{m}\right) + \varepsilon h(n) \text{ for some } n \geqq \tau m\right) - p_m$$
$$\leqq P\left(S_n \geqq m^{1/2} g\left(\frac{n}{m}\right) \text{ for some } n \geqq \tau m\right) \tag{3.1}$$
$$\leqq P\left(W(n) \geqq m^{1/2} g\left(\frac{n}{m}\right) - \varepsilon h(n) \text{ for some } n \geqq \tau m\right) + p_m$$
$$\leqq P\left(W(n) \geqq m^{1/2}\left(g\left(\frac{n}{m}\right) - \varepsilon Mh\left(\frac{n}{m}\right)\right) \text{ for some } n \geqq \tau m\right) + p_m$$

where

$$p_m = P(h(n) \geqq \varepsilon \quad \text{for some} \quad n \geqq \tau m). \tag{3.2}$$

It is obvious that by (2.2)

$$\lim_{m\to\infty} p_m = 0. \tag{3.3}$$

Next, since the distribution of $W(n)$ is the same as that of $\sum_{i=1}^{n} Z_i$ where $\{Z_i\}$ is a sequence of i.i.d. $N(0, 1)$-random variables and $g(t) \pm \varepsilon M h(t)$ satisfies conditions of Theorem 1, so applying Theorem 2 of Robbins and Siegmund [7] we have

$$\begin{aligned}\lim_{m\to\infty} P\Big(W(n) \geqq m^{1/2}\Big(g\Big(\frac{n}{m}\Big) \pm \varepsilon Mh\Big(\frac{n}{m}\Big)\Big) \text{ for some } n \geqq \tau m\Big) \\ = P(W(t) \geqq g(t) \pm \varepsilon Mh(t) \quad \text{for some} \quad t \geqq \tau).\end{aligned} \tag{3.4}$$

We note that for any $T\,(>\tau)$ and for any $\varepsilon > 0$

$$\begin{aligned} &P(W(t) \geqq g(t) + \varepsilon Mh(t) \quad \text{for some} \quad t \in [\tau, T]) \\ &\quad \leqq P(W(t) \geqq g(t) \pm \varepsilon Mh(t) \quad \text{for some} \quad t \geqq \tau) \\ &\quad \leqq P(W(t) \geqq g(t) - \varepsilon Mh(t) \quad \text{for some} \quad t \in [\tau, T]) \\ &\qquad + P(W(t) \geqq g(t) - Mh(t) \quad \text{for some} \quad t \geqq T). \end{aligned} \tag{3.5}$$

As $g(t)$ and $h(t)$ are continuous on $[0, T]$, so

$$\begin{aligned} &\lim_{\varepsilon \downarrow 0} P(W(t) \geqq g(t) \pm \varepsilon Mh(t) \quad \text{for some} \quad t \in [\tau, T]) \\ &\quad = P(W(t) \geqq g(t) \quad \text{for some} \quad t \in [\tau, T]). \end{aligned} \tag{3.6}$$

Thus, by (3.5) and (3.6)

$$\begin{aligned} &P(W(t) \geqq g(t) \quad \text{for some} \quad t \in [\tau, T]) \\ &\quad \leqq \liminf_{\varepsilon \downarrow 0} P(W(t) \geqq g(t) \pm \varepsilon Mh(t) \quad \text{for some} \quad t \geqq \tau) \\ &\quad \leqq \limsup_{\varepsilon \downarrow 0} P(W(t) \geqq g(t) \pm \varepsilon Mh(t) \quad \text{for some} \quad t \geqq \tau) \\ &\quad \leqq P(W(t) \geqq g(t) \quad \text{for some} \quad t \in [\tau, T]) \\ &\qquad + P(W(t) \geqq g(t) - Mh(t) \quad \text{for some} \quad t \geqq T). \end{aligned} \tag{3.7}$$

As (2.3) holds for the function $g(t) - Mh(t)$, that is, $g(t) - Mh(t)$ belongs to the upper class for the law of the iterated logarithm of $W(t)$, so

$$\lim_{T\to\infty} P(W(t) \geqq g(t) - Mh(t) \quad \text{for some} \quad t \geqq T) = 0. \tag{3.8}$$

Hence, (2.4) follows from (3.1), (3.3), (3.4), (3.7) and (3.8).

Proof of Theorem 2. It follows from the conditions on g that there is an integer valued function $q(m)$ such that $q(m)\uparrow\infty$, $q(m)=o(m)$ and

$$\sum_{j=1}^{q}\frac{j^2}{mg^2\left(\frac{j}{m}\right)}\longrightarrow 0 \tag{3.9}$$

as $m\to\infty$. We note that for any τ $(0<\tau<\infty)$

$$p_m^{(2)}\leqq P\left(S_n\geqq m^{1/2}g\left(\frac{n}{m}\right)\quad\text{for some}\quad 1\leqq n\leqq\tau m\right)\leqq p_m^{(1)}+p_m^{(2)} \tag{3.10}$$

where

$$p_m^{(1)}=P\left(S_n\geqq m^{1/2}g\left(\frac{n}{m}\right)\quad\text{for some}\quad 1\leqq n\leqq q\right) \tag{3.11}$$

and

$$p_m^{(2)}=P\left(S_n\geqq m^{1/2}g\left(\frac{n}{m}\right)\quad\text{for some}\quad q\leqq n\leqq\tau m\right). \tag{3.12}$$

Now, by (3.9) and Chebyshev's inequality

$$\begin{aligned}p_m^{(1)}&\leqq\sum_{n=1}^{q}P\left(S_n\geqq m^{1/2}g\left(\frac{n}{m}\right)\right)\\&\leqq\sum_{n=1}^{q}m^{-1}\left\{g\left(\frac{n}{m}\right)\right\}^{-2}ES_n^2\\&\leqq Km^{-1}\sum_{n=1}^{q}n^2\left\{g\left(\frac{n}{m}\right)\right\}^{-2}\longrightarrow 0\end{aligned} \tag{3.13}$$

as $m\to\infty$, where K is an absolute constant.

On the other hand, by the method of the proof of Theorem 1

$$\lim_{m\to\infty}p_m^{(2)}=P(W(t)\geqq g(t)\quad\text{for some}\quad 0<t\leqq\tau). \tag{3.14}$$

Hence, (2.6) follows from (3.10), (3.13) and (3.14) and the proof is completed.

For proofs of Theorem 3 and Corollary are on the same lines of those of Theorems 1 and 2 and so are omitted (cf. Shorack [8]).

4. Examples. For a class of functions $\{g\}$ considered in Section 2 many examples were shown in Robbins and Siegmund [7].

Philipp and Stout [6] showed that (2.2) holds for certain Gaussian, lacunary trigonometric and ϕ-mixing sequences, and Naresh et al. [5] showed that (2.2) holds for some martingale differences sharping Strassen's result. Further, in

Yoshihara [10], it is proved that (2.2) also holds for sums of some functions of strong mixing sequences. Thus, in these cases all results in Section 2 hold.

We state the hypotheses for these cases precisely.

Example 1. Let $\{X_n, n \geqq 1\}$ be a stationary Gaussian sequence with 0 means and $EX_1X_n = O(n^{-2})$. Then $D(t) = O(t^{-\lambda})$ a.s. as $t \to \infty$ for some $\lambda > 0$.

Example 2. Let $\{n_k\}$ be a lacunary sequence of positive real numbers (not necessarily integers), that is a sequence satisfying

$$n_{k+1}/n_k \geqq q, \quad k = 1, 2, \ldots$$

for some $q > 1$. Let $\{\cos 2\pi n_k \omega\}$ be a sequence of random variables defined on the probability space $([0, 1), B, P)$ where P is Lebesgue-measure and B consists of the Lebesgue-measurable sets of $[0, 1)$. Then

$$S(t) = \sum_{k=1}^{[t]} \sqrt{2} \cos 2\pi n_k \omega \tag{4.1}$$

satisfies (2.2) with $D(t) = O(t^{-\lambda})$ for some $\lambda > 0$.

Example 3. Let $\{X_n\}$ be a stationary ergodic sequence of martingale differences. It will be assumed that $EX_1^2 = 1$. Let $V_n = \sum_{i=1}^{n} E\{X_i^2 \mid M_0^{i-1}\}$, where M_0^j is the σ-algebra generated by $X_1, \ldots, X_j$. For a fixed $\alpha > \frac{1}{2}$, let $f_\alpha(t) = t(\log|\log t|)^{-2\alpha}$ for $t > 0$.

Assume that

$$|V_n - n| = o(f_\alpha(n)) \quad \text{a.s.} \tag{4.2}$$

$$\lim_{n\to\infty} (f_\alpha(n))^{-1} \sum_{k=1}^{n} E\{X_n^2 I[X_k^2 \geqq \delta f_\alpha(k)] \mid M_0^{k-1}\} = 0 \quad \text{a.s.} \tag{4.3}$$

for all $\delta > 0$, and

$$E\{X_1^2(\log|\log X_1^2|)^{2\alpha}\} < \infty\,. \tag{4.4}$$

Then, by Theorem 4.1 in Naresh et al. [5]

$$D(t) = o((\log|\log t|)^{-(2\alpha-1)/2}) \quad \text{a.s.} \tag{4.5}$$

Example 4. Let $\{\xi_n\}$ be a nonstationary strong mixing sequences, i.e., $\{\xi_n\}$ satisfies the strong mixing condition

$$\alpha(n) = \sup_k \sup_{A \in M_{-\infty}^k, B \in M_{k+n}^\infty} |P(A \cap B) - P(A)P(B)| \downarrow 0 \quad (n \to \infty). \tag{4.6}$$

Here, M_a^b denotes the σ-algebra generated by $\xi_a,\ldots,\xi_b$. Let $\{f_n\}$ be a family of measurable mappings from the space of infinite sequences $(a_1, a_2,\ldots)$ of real numbers into the real line. Let $\{X_n\}$ be a sequence of random variables defined by

$$X_n=f_n(\xi_n, \xi_{n+1},\ldots) \quad (n\geqq 1). \tag{4.7}$$

Then, (2.2) holds for the sequence $\{X_n\}$ if the following conditions are satisfied:

$$EX_n=0 \quad \text{and} \quad E|X_n|^{2+c+\delta}\leqq M_1<\infty \quad (n\geqq 1), \tag{4.8}$$

$$\inf_{n\geqq 1} n^{-1}E|\sum_{j=1}^{n} X_j|^2\geqq M_2>0, \tag{4.9}$$

$$\sum_{j=1}^{\infty} j^{c/2}\{\alpha(j)\}^{\delta/(2+c+\delta)}<\infty, \quad \text{and} \tag{4.10}$$

$$\sum_{j=1}^{\infty} j^{c/2}\sup_{n}\{E\,|\,X_n-E\{X_n\,|\,M_n^{n+j}\}|^{2+c}\}^{1/(2+c+\delta)}<\infty. \tag{4.11}$$

Here, c, δ, M_1 and M_2 are some positive constants. (cf. Philipp and Stout [6] and Yoshihara [10]).

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Department of Mathematics,
Faculty of Engineering,
Yokohama National University,
156, Tokiwadai, Hodogaya,
Yokohama, Japan