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ON EXPLOSION AND GROWTH ORDER OF INHOMOGENEOUS DIFFUSION PROCESSES

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1. Introduction

In this paper, we shall discuss the explosion and the growth order of solutions of stochastic differential equations;

$$(1.1) \qquad dX(t)=b(t, X(t))dt+\sigma(t, X(t))dw(t),$$

where $b(t, x)=(b_i(t, x))$, $i=1,\dots, d$, is a d-vector function and $\sigma(t, x)=(\sigma_{ij}(t, x))$, $i, j=1,\dots, d$, is a $d\times d$-matrix, which is defined on $[0, \infty)\times R^d$, is Borel measurable with respect to the complete set of variables, and $w(t)=(w_i(t))$, $i=1,\dots, d$, is a d-dimensional Brownian motion process. Equation (1.1) is equivalent to the system of d equations;

$$(1.1)' \qquad dX_i(t)=b_i(t, X(t))dt+\sum_{j=1}^{d}\sigma_{ij}(t, X(t))dw_j(t),$$

$i=1,\dots, d$. In the following, let $|x|$ be the Euclidean norm of $x\in R^d$, and for a $d\times d$-matrix $M=(m_{ij})$ define

$$|M|=(\sum_{i,j=1}^{d} m_{ij}^2)^{1/2}.$$

When $b(t, x)\equiv b(x)$ and $\sigma(t, x)\equiv\sigma(x)$, that is, the coefficients do not depend on time variable, (1.1) has the pathwise unique global solution on the entire interval $[0, \infty)$, provided only the global Lipschitz condition is satisfied: there exists a positive constant L such that, for all $x, y\in R^d$,

$$|b(x)-b(y)|+|\sigma(x)-\sigma(y)|\leqq L|x-y|.$$

From this global Lipschitz condition *the restriction on the growth* of b and σ follows: there exists a positive constant K such that, for all $x\in R^d$,

$$|b(x)|^2+|\sigma(x)|^2\leqq K(1+|x|^2).$$

So, the satisfaction of a global Lipschitz condition (in fact, the growth restriction) on $b(x)$ and $\sigma(x)$ implies that $X(t)$ does not explode on $[0, \infty)$ almost surely. A second sufficient codition for non-occurrence of an explosion in the case of continuously differentiable σ is that $b\equiv 0$, that is, the systematic part be absent. This follows from application of a sensitive test for explosion discovered by W. Feller. (This test, as well as d-dimensional analogue discovered by Hasminskii, can be found in McKean [4, p. 65]). It should be emphasized that, for $d\geqq 2$, the condition $b\equiv 0$ is no longer in general sufficient to preclude an explosion if the growth of σ is not restricted (see McKean [4, p. 106 (problem 3)]). Hasminskii studied the problem also in the case where the coefficients depended on both time and space in [3, pp. 112–119], but he restricted his attention in situation when the time dependence was not so strong.

On this subject, when the coefficients depend on time, the author ([5], [6]) tried to give sufficient conditions for non-occurrence of explosion, in terms of integral condition and extreme condition of functions which appear in the restriction on the growth of $b(t, x)$ and $\sigma(t, x)$. In section 2, a sufficient condition for infinite explosion time shall be obtained in simpler and best possible way by using a concave function.

In section 3, we shall prove a very simple theorem on the existence of the order of growth of $X(t)$. This theorem is a generalization of the one dimensional result of Gihman and Skorohod [2, p. 130 (lemma 1)] who state in the following way: if $X(t)$ satisfies $dX(t)=dt+\sigma(X(t))dw(t)$, where $w(t)$ is an one dimensional Brownian motion process and $\sigma(x)$ is defined on R^1 such that $\sigma^2(x)\leqq$Const. $(1+|x|^\gamma)$ for some $\gamma<1$, then $X(t)/t\to 1$ as $t\to\infty$ with probability one.

A precise formulation is in the following.

Let $(\Omega, \boldsymbol{F}, P)$ be a probability space with an increasing family $\{\boldsymbol{F}_t;\ t\geqq 0\}$ of sub-σ-algebras of $\boldsymbol{F}$ and let $w(t)=(w_i(t))$ be a d- dimensional Brownian motion process adapted to $\boldsymbol{F}_t$. Throughout this paper, we assume the following:

(1.2) $b(t, x)$ and $\sigma(t, x)$ are continuous in (t, x), and for any $T>0$, $R>0$, there exists a constant $C_{TR}>0$ depending only on T and R such that

$$|b(t, x)-b(t, y)|+|\sigma(t, x)-\sigma(t, y)|\leqq C_{TR}|x-y|$$

if $t\leqq T$, $|x|\leqq R$ and $|y|\leqq R$.

For any natural number n, let $g_n(x)$ be the function on R^d such that $g_n(x)=1$ for $|x|\leqq n$; $g_n(x)=2-|x|/n$ for $n<|x|\leqq 2n$; $g_n(x)=0$ for $2n<|x|$. Set

$$b^{(n)}(t, x)=g_n(x)b(t, x) \quad \text{and} \quad \sigma^{(n)}(t, x)=g_n(x)\sigma(t, x).$$

Then there is a constant $K_n>0$ depending only on n such that

$$(1.3)' \qquad |b^{(n)}(t, x)-b^{(n)}(t, y)|+|\sigma^{(n)}(t, x)-\sigma^{(n)}(t, y)| \leqq K_n|x-y|, \quad t\leqq n,\ x\in R^d,\ y\in R^d,$$

$$(1.3)'' \qquad |b^{(n)}(t, x)|^2+|\sigma^{(n)}(t, x)|^2\leqq K_n(1+|x|^2), \quad t\leqq n,\ x\in R^d.$$

As is well known, by (1.3)' and (1.3)'', there exists a pathwise unique solution $X^{(n)}(t)=(X_i^{(n)}(t))$, $i=1,\dots, d$, which is defined up to $t\leqq n$ of the stochastic differential equation

$$(1.4) \qquad dX^{(n)}(t)=b^{(n)}(t, X^{(n)}(t))dt+\sigma^{(n)}(t, X^{(n)}(t))dw(t).$$

By $X^{(n)}(t;\ t_0, x_0)$ we mean the solution of (1.4) with the initial condition $X^{(n)}(t_0)=x_0\in R^d$ $(t_0\geqq 0)$. Define $\tau_n(t_0, x_0)$ by $\tau_n(t_0, x_0)=\inf\{t;\ |X^{(n)}(t;\ t_0, x_0)|\geqq n\}$ (define $\tau_n(t_0, x_0)$ by $\tau_n(t_0, x_0)=\infty$ if there is no such time) and set $e_n(t_0, x_0)=\min\{n, \tau_n(t_0, x_0)\}$. Then, $\{e_n(t_0, x_0);\ n\geqq 1\}$ is monotone increasing family of stopping times, for which

$$\sup_{t_0\leqq t\leqq e_n(t_0,x_0)} |X^{(n)}(t;\ t_0, x_0)-X^{(m)}(t;\ t_0, x_0)|=0$$

with probability one, if $m>n$. Define a random process $X(t;\ t_0, x_0)$ by $X(t;\ t_0, x_0)=X^{(n)}(t;\ t_0, x_0)$ for $t<e_n(t_0, x_0)$ $(n\geqq 1)$. The process $X(t;\ t_0, x_0)$ is called the solution of (1.1) with the initial condition $X(t_0)=x_0$. A random time $e(t_0, x_0)=\lim_{n\to\infty} e_n(t_0, x_0)$ is called *the explosion time* of $X(t;\ t_0, x_0)$.

We are interested in whether $e(t_0, x_0)=\infty$ or not. Also we shall investigate the asymptotic behavior of $X(t;\ t_0, x_0)$ when $e(t_0, x_0)=\infty$.

2. Growth restriction on b and σ

Here we give a sufficient condition of infinite explosion time by using a concave function.

Theorem 1.1. *Let $b(t, x)$ and $\sigma(t, x)$ satisfy* (1.2) *and let*

$$(2.1) \qquad |b(t, x)|^2+|\sigma(t, x)|^2\leqq \alpha(t)\beta(|x|^2)$$

for all $t\in[0, \infty)$ and $x\in R^d$, where α: $[0, \infty)\to[0, \infty)$ is continuous and β: $[0, \infty)\to[0, \infty)$ is monotone increasing, concave such that

$$(2.2) \qquad \int_0^\infty \frac{du}{1+\beta(u)}=\infty .$$

Then, $P(e(t_0, x_0)=\infty)=1$ for all $t_0\in[0, \infty)$ and $x_0\in R^d$.

Proof. We consider the solution $X^{(n)}(t;\ t_0, x_0)$ of (1.4) with the initial condition $X^{(n)}(t_0)=x_0\in R^d$ for $n>\max\{|x_0|, t_0\}$. Let $\tau_n(t_0, x_0)$ be the first exit time from the shell $\{x; |x|<n\}$ for $X^{(n)}(t; t_0, x_0)$ and set $e_n(t_0, x_0)=\min\{n, \tau_n(t_0, x_0)\}$. For notational simplicity we write as $X^{(n)}(t)=X^{(n)}(t; t_0, x_0)$, $\tau_n=\tau_n(t_0, x_0)$ and $e_n=e_n(t_0, x_0)$, omitting t_0 and x_0. Put

$$Q^{(n)}(t)=E(\sup_{t_0\leqq u\leqq t}|X^{(n)}(u)|^2)$$

for $t\in[t_0, n]$. Then, by (1.3)$'$ and (1.3)$''$, $Q^{(n)}(t)$ is bounded (see [1, p. 102]). Observe that $X^{(n)}(t)$ satisfies (1.4). Then, by Schwartz's inequlity, we have,

$$|X^{(n)}(u)|^2\leqq 3\Big[|x_0|^2+u\int_{t_0}^{u}|b^{(n)}(v, X^{(n)}(v))|^2dv + \sum_{i=1}^{d}\Big\{\sum_{j=1}^{d}\int_{t_0}^{u}\sigma_{ij}^{(n)}(v, X^{(n)}(v))dw_j(v)\Big\}^2\Big]$$

for all $u\in[t_0, n]$. Take the supremum and the mathematical expectation in the above. Then, by martingale inequality, we get,

$$E(\sup_{t_0\leqq u\leqq t}|X^{(n)}(u)|^2)\leqq 3\Big[|x_0|^2+t\int_{t_0}^{t}E|b^{(n)}(u, X^{(n)}(u))|^2du + 4\int_{t_0}^{t}E|\sigma^{(n)}(u, X^{(n)}(u))|^2du\Big]$$

for all $t\in[t_0, n]$. By the definition of $b^{(n)}$ and $\sigma^{(n)}$ we note that

$$|b^{(n)}(t, x)|^2+|\sigma^{(n)}(t, x)|^2\leqq|b(t, x)|^2+|\sigma(t, x)|^2\leqq\alpha(t)\beta(|x|^2)$$

for all $t\in[0, \infty)$ and $x\in R^d$, since (2.1) holds. By the assumption, since β is monotone increasing, we also note that

$$\beta(|X^{(n)}(u)|^2)\leqq\beta(\sup_{t_0\leqq v\leqq u}|X^{(n)}(v)|^2), \quad u\leqq n.$$

Hence, we have,

$$Q^{(n)}(t)\leqq 3\Big[|x_0|^2+(t+4)A(t)\int_{t_0}^{t}E\beta(\sup_{t_0\leqq v\leqq u}|X^{(n)}(v)|^2)du\Big]$$

for all $t\in[t_0, n]$, where $A(t)=\max_{0\leqq u\leqq t}\alpha(u)$. By the assumption, since β is concave, by Jensen's inequality, we obtain,

$$Q^{(n)}(t)\leqq 3\Big[|x_0|^2+(t'+4)A(t')\int_{t_0}^{t}\beta(Q^{(n)}(s))\,ds\Big]$$

for all $t\in[t_0, t'](t'<n)$. Note that $1+\beta(Q^{(n)}(t))\geqq\beta(Q^{(n)}(t))$. Then, the above inequality, as is well known, implies

$$\int_{3|x_0|^2}^{Q^{(n)}(t)} \frac{du}{1+\beta(u)} \leqq H(t'; t_0) \tag{2.3}$$

for all $t \in [t_0, t'](t' < n)$, where

$$H(t; t_0) = 3(t+4)A(t)(t-t_0).$$

Now, let's assume that there exist some t_0 and x_0 such that $P(e(t_0, x_0) < T) \equiv \delta > 0$ for some $T < \infty$. Let T' be arbitrary such that $T' > T$ and be fixed. We choose n so large that $n > \max\{|x_0|, T'\}$ previously and consider $X^{(n)}(t; t_0, x_0)$ for such n in the following. Let t be any such that $T < t \leqq T'$. Then, since

$$\{\tau_n < t\} = \{e_n < t\} \supseteqq \{e_n < T\} \supseteqq \{e(t_0, x_0) < T\},$$

we see that

$$\begin{aligned} Q^{(n)}(t) &\geqq E[\sup_{t_0 \leqq u \leqq t} |X^{(n)}(u)|^2; \sup_{t_0 \leqq u \leqq t} |X^{(n)}(u)|^2 > n-1] \\ &\geqq (n-1)P(\sup_{t_0 \leqq u \leqq t} |X^{(n)}(u)| > n) \\ &= (n-1)P(\tau_n < t) \\ &\geqq (n-1)\delta \end{aligned}$$

for all $t \in (T, T']$. Accordingly, by (2.3), we obtain,

$$\int_{3|x_0|^2}^{(n-1)\delta} \frac{du}{1+\beta(u)} \leqq H(T'; t_0). \tag{2.4}$$

Letting n tend to infinity in the above, we see that the right-hand side of (2.4) is finite, while the left-hand side becomes infinity since (2.2) holds. So, we are led to contradiction. Therefore, for any $t_0 \geqq 0$, $x_0 \in R^d$ and T, $P(e(t_0, x_0) \geqq T) = 1$, and the proof is complete.

Example 2.2. Let $b(t, x)$ and $\sigma(t, x)$ satisfy the conditions;

$$\sup_{|x| \leqq 1} \{|b(t, x)|^2 + |\sigma(t, x)|^2\} \leqq k_1(t) \qquad \text{for} \quad t \geqq 0,$$

$$|b(t, x)|^2 + |\sigma(t, x)|^2 \leqq k_2(t)(1+|x|^{2\delta}) \quad (\delta \leqq 1)$$

for $t \geqq 0$ and $|x| \geqq 1$, where k_1 and k_2 are nonnegative and continuous. Then, Theorem 2.1 will apply if we take $\alpha(t) = k_1(t) + k_2(t)$ and $\beta(u) = 1 + u^\delta$ for $0 \leqq \delta \leqq 1$; $\beta(u) = 2$ for $\delta < 0$.

Theorem 2.1 remains true in one dimension and the conditions (2.1) and (2.2) are best possible in a sense, as we can imagine from the following due to McKean

[4, p. 66]: consider the one dimensional stochastic differential equation

$$dX(t)=b(X(t))dt+dw(t),$$

where $b(x)=|x|^\gamma$ near large $|x|$, then the explosion time is almost surely infinite or finite according as $\gamma\leqq 1$ or not.

3. Asymptotic behavior

In this section, let $b(t, x)=(b_i(t, x))$ be such that

$$b_i(t, x)=c_i(t)a_i(x), \quad i=1,\dots, d.$$

For such drift coefficient we treat with the solution $X(t; t_0, x_0)$ of (1.1) with the initial condition $X(t_0)=x_0\in R^d$, whose explosion time is denoted by $e(t_0, x_0)$. In the following we set $a(x)=(a_i(x))$, $i=1,\dots, d$, and for continuous $c_i(t)$, $i=1,\dots, d$, we set

$$B(t)=\left[\sum_{i=1}^{d}\left(\int_{t_0}^{t}|c_i(s)|ds\right)^2\right]^{1/2}.$$

To begin with, we state on a moment estimate for $X(t; t_0, x_0)$.

Lemma 3.1. *Let $b(t, x)=(c_i(t)a_i(x))$ and $\sigma(t, x)$ satisfy* (1.2) *and let the following conditions hold*;

$$|a(x)|\leqq D \quad \text{with a constant} \quad D>0 \tag{3.1}$$

for all $x\in R^d$,

$$|\sigma(t, x)|^2\leqq\alpha(t)\beta(|x|^2) \tag{3.2}$$

for all $t\in[0, \infty)$ and $x\in R^d$, where α: $[0, \infty)\to[0, \infty)$ is nonnegative and continuous such that

$$\lim_{t\to\infty} B(t)=\infty, \tag{3.3$'$}$$

$$\lim_{t\to\infty} A(2t)/B(t) \quad \text{exists} \quad \left(A(t)\equiv\int_{t_0}^{t}\alpha(u)du\right) \tag{3.3$''$}$$

and β: $[0, \infty)\to[0, \infty)$ is a monotone increasing, concave function of u which is twice continuously differentiable in $u>0$ such that

$$\lim_{u\to\infty} u\beta'(u^2)=0. \tag{3.4}$$

Then, there exist some constants $L_1>0$ and $L_2>0$ such that

$$\sup_{t_0 \leqq u \leqq t} E|X(u;\, t_0,\, x_0)|^2 \leqq L_1 + L_2 B(t)^2$$

for all $t \in [t_0, \infty)$ **and** $x_0 \in R^d$.

Proof. For notational simplicity we write as $X(t)=X(t;\, t_0,\, x_0)$ and $e_n=e_n(t_0, x_0)$, omitting t_0 and x_0. By $u \wedge v$ we mean the smaller of u and v in the following. Since $X(t)=(X_i(t))$ satisfies (1.1)′, by (3.1) and (3.2), we see, for any $t \geqq t_0$,

$$\begin{aligned} E|X(t \wedge e_n)|^2 &\leqq 3\Big[|x_0|^2 + D^2 \sum_{i=1}^d E\Big(\int_{t_0}^{t \wedge e_n} |c_i(u)|\, du\Big)^2 \\ &\quad + \sum_{i=1}^d E\Big\{\sum_{j=1}^d \int_{t_0}^{t \wedge e_n} \sigma_{ij}(u,\, X(u))\, dw_j(u)\Big\}^2\Big] \\ &\leqq 3\Big[|x_0|^2 + D^2 B(t)^2 + \int_{t_0}^t E|\sigma(u,\, X(u \wedge e_n))|^2 du\Big] \\ &\leqq 3\Big[|x_0|^2 + D^2 B(t)^2 + \int_{t_0}^t \alpha(u) E\beta(|X(u \wedge e_n)|^2)\, du\Big]. \end{aligned}$$

By the assumption, since β is concave, we have, by Jensen's inequality and the fact that β is monotone increasing,

$$E\beta(|X(u \wedge e_n)|^2) \leqq \beta(E|X(u \wedge e_n)|^2) \leqq \beta(\sup_{t_0 \leqq u \leqq t} E|X(u \wedge e_n)|^2)\ (t_0 \leqq u \leqq t).$$

Set $q_n(t) = \sup_{t_0 \leqq u \leqq t} E|X(u \wedge e_n)|^2$. Then above inequalities imply that

$$q_n(t) \leqq 3[|x_0|^2 + D^2 B(t)^2 + A(t)\beta(q_n(t))]\,. \tag{3.5}$$

Let $\delta > 0$ be arbitrary, and set $p_{n,\delta}(t) = q_n(t)/3 + B(t)^2 + \delta$. By (3.3)′, there exists some $t_1 > 0$ such that $3B(t)^2 > 1$ for all $t > t_1$. If $t > t_1$, then, by (3.3)″ and (3.4), we have,

$$0 \leqq A(t)\beta'(3B(t)^2) \leqq (A(2t)/B(t))B(t)\beta'(3B(t)^2) \longrightarrow 0 \quad (t \to \infty)\,.$$

Thus we find some $t_2 > t_1$ such that

$$1 - 3A(t)\beta'(3B(t)^2) > 0$$

for all $t > t_2$. In the following let t be arbitrary such that $t > \max\{t_0, t_2\}$. Then we take care of that

$$3p_{n,\delta}(t) > q_n(t) \quad \text{and} \quad 3p_{n,\delta}(t) > 3B(t)^2\,.$$

Therefore, since β is monotone increasing, by (3.5), we get,

$$q_n(t) \leqq 3[|x_0|^2 + D^2 B(t)^2 + A(t)\beta(3p_{n,\delta}(t))]\,,$$

which implies,

(3.6) $$p_{n,\delta}(t) \leqq \delta + |x_0|^2 + (D^2+1)B(t)^2 + A(t)\beta(3p_{n,\delta}(t)).$$

Remember that $\beta(u)$ is concave and twice continuously differentiable in $u>0$. Then, for any $u_2>u_1>0$, $\beta(u_2) \leqq \beta(u_1)+(u_2-u_1)\beta'(u_1)$. Set $r(t)=3B(t)^2$ and note that

$$\begin{aligned}\beta(3p_{n,\delta}(t)) &\leqq \beta(r(t)) + (3p_{n,\delta}(t)-r(t))\beta'(r(t)) \\ &= \beta(1) + (r(t)-1)\beta'(r_\theta(t)) \\ &\quad + (3p_{n,\delta}(t)-r(t))\beta'(r(t)),\end{aligned}$$

where $r_\theta(t)=1+\theta(r(t)-1)$ $(0<\theta<1)$. Substituting the above to the right-hand side of (3.6), we obtain the following;

$$\begin{aligned}&p_{n,\delta}(t)[1-3A(t)\beta'(r(t))] \\ &\leqq \delta + |x_0|^2 + (D^2+1)B(t)^2 + A(t)[\beta(1)+(r(t)-1)\beta'(r_\theta(t)) - r(t)\beta'(r(t))],\end{aligned}$$

which implies,

$$\begin{aligned}&[q_n(t)/3 + B(t)^2]\,[1-3A(t)\beta'(r(t))] \\ &\leqq |x_0|^2 + (D^2+1)B(t)^2 + A(t)[\beta(1)+(r(t)-1)\beta'(r_\theta(t)) - r(t)\beta'(r(t))],\end{aligned}$$

since we can let δ tend to zero. Accordingly,

(3.7) $$[q_n(t)/3 + B(t)^2]/B(t)^2 \leqq I(t)/[1-3A(t)\beta'(r(t))],$$

where

$$I(t) = |x_0|^2/B(t)^2 + D^2 + 1 + (A(t)/B(t))J(t)$$

and

$$J(t) = \beta(1)/B(t) + \left[\frac{3(r(t)-1)}{r(t)r_\theta(t)}\right]^{1/2} r_\theta(t)^{1/2}\beta'(r_\theta(t)) - (3r(t))^{1/2}\beta'(r(t)).$$

Remember again that $\beta(u)$ is concave and twice continuously differentiable in $u>0$. Then we have that $\beta(u) \leqq \beta(\varepsilon) + (u-\varepsilon)\beta'(\varepsilon)$ $(u>\varepsilon)$. Thus, by (3.1) and (3.2), we see,

$$|b(t, x)|^2 + |\sigma(t, x)|^2 \leqq D^2 \sum_{i=1}^{d} c_i(t)^2 + \alpha(t)\beta(|x|^2) \leqq \gamma(t)(1+|x|^2)$$

with some continuous function $\gamma(t)$. Hence, by Theorem 2.1, we have,

(3.8) $$P(e(t_0, x_0)=\infty)=1.$$

Let n tend to infinity in (3.7). Then, (3.8) and Fatou's lemma imply that

(3.9) $$\left[\frac{1}{3}\sup_{t_0 \leqq u \leqq t} E|X(u)|^2 + B(t)^2\right]\Big/ B(t)^2 \leqq I(t)/[1-3A(t)\beta'(r(t))],$$

for all $t > \max\{t_0, t_2\}$. By (3.3)′, (3.3)″ and (3.4), note that

$$0 \leqq (A(t)/B(t))|J(t)| \leqq (A(2t)/B(t))|J(t)| \longrightarrow 0 \quad (t \to \infty),$$

and

$$0 \leqq A(t)\beta'(r(t)) \leqq (A(2t)/B(t))(r(t)/3)^{1/2}\beta'(r(t)) \longrightarrow 0 \quad (t \to \infty).$$

Then, we get,

$$I(t)/[1-3A(t)\beta'(r(t))] \longrightarrow D^2+1 \quad (t \to \infty). \tag{3.10}$$

Hence, letting t tend to infinity in (3.9), we obtain, by (3.10),

$$\limsup_{t\to\infty} \frac{1}{B(t)^2}\,[\sup_{t_0 \leqq u \leqq t} E|X(u)|^2] \leqq 3D^2.$$

This implies that, for some constants $L_1 > 0$ and $L_2 > 0$,

$$\sup_{t_0 \leqq u \leqq t} E|X(u)|^2 \leqq L_1 + L_2 B(t)^2,$$

and the proof is complete.

Theorem 3.2. *Under the same assumptions as in Lemma* 3.1, *let the following conditions hold*;

(I) *there exists some* i_0 $(1 \leqq i_0 \leqq d)$ *such that* $a_{i_0}(x) \equiv 1$ *and*

$$\lim_{t\to\infty} \frac{1}{B(t)} \int_{t_0}^{t} c_{i_0}(u)du = \tilde{c}_{i_0} \quad (|\tilde{c}_{i_0}| < \infty), \tag{3.11}$$

(II) *for any large number* N,

$$f(N) \equiv \sum_{k=0}^{\infty} \beta(B(N2^{k+1})^2)/B(N2^k) \quad \textit{exists and}$$

$$\lim_{N\to\infty} f(N) = 0. \tag{3.12}$$

Then, for the i_0*-th component* $X_{i_0}(t; t_0, x_0)$ of $X(t; t_0, x_0)$, *we have,* $P(\lim_{t\to\infty} X_{i_0}(t; t_0, x_0)/B(t) = \tilde{c}_{i_0}) = 1$.

Proof. For notational simplicity we write as $X(t) = X(t; t_0, x_0)$ omitting t_0 and x_0. Then, by (3.8), $X_{i_0}(t)$ satisfies

$$X_{i_0}(t) = x_{i_0} + \int_{t_0}^{t} c_{i_0}(u)du + M_{i_0}(t)$$

for all $t \geqq t_0$, where x_{i_0} is the i_0-th component of $x_0 = (x_1, \ldots, x_d)$ and

$$M_{i_0}(t)=\sum_{j=1}^{d}\int_{t_0}^{t}\sigma_{i_0j}(u,\ X(u))dw_j(u)\,.$$

Hence, by (3.11), we have only to show that

$$P(M_{i_0}(t)/B(t)\longrightarrow 0 \text{ as } t\longrightarrow\infty)=1\,. \tag{3.13}$$

Now set $e_n=e_n(t_0,\ x_0)$ and consider the process $|X(u\wedge e_n)|^2$. Then we observe the proof of Lemma 3.1. By (3.7), we have, for large t

$$\sup_n\,[\{q_n(t)/3+B(t)^2\}/B(t)^2]\leqq I(t)/[1-3A(t)\beta'(r(t))]\,.$$

Take the superior limit as t tends to infinity in the above and note (3.10). Then, it is easy to see that $\sup\limits_n q_n(t)\leqq L_2'+L_2'B(t)^2$ for some constants $L_1'>0$ and $L_2'>0$. It is no loss of generality to take $L_1'=L_1$ and $L_2'=L_2$, where L_1 and L_2 are constants appeared in Lemma 3.1. Therefore,

$$\sup_n\,[\sup_{t_0\leqq u\leqq t}E|X(u\wedge e_n)|^2]\leqq L_1+L_2B(t)^2 \tag{3.14}$$

for all $t\geqq t_0$. Accordingly, since (3.2) holds and since β is concave, monotone increasing, by Jensen's inequality and (3.14), we get,

$$\begin{aligned}
E\int_{t_0}^{t\wedge e_n}\sum_{j=1}^{d}\sigma_{i_0j}^2(u,\ X(u))du&=E\int_{t_0}^{t\wedge e_n}\sum_{j=1}^{d}\sigma_{i_0j}^2(u,\ X(u\wedge e_n))du\\
&\leqq E\int_{t_0}^{t\wedge e_n}\alpha(u)\beta(|X(u\wedge e_n)|^2)du\\
&\leqq\int_{t_0}^{t}\alpha(u)E[\beta(|X(u\wedge e_n)|^2)]du\\
&\leqq\int_{t_0}^{t}\alpha(u)\beta(E|X(u\wedge e_n)|^2)du\\
&\leqq A(t)\beta(\sup_{t_0\leqq u\leqq t}E|X(u\wedge e_n)|^2)\\
&\leqq A(t)\beta(L_1+L_2B(t)^2)\,.
\end{aligned}$$

Let n tend to infinity in the above and take care of (3.8). Then Fatou's lemma implies that

$$E\int_{t_0}^{t}\sum_{j=1}^{d}\sigma_{i_0j}^2(u,\ X(u))du\leqq A(t)\beta(L_1+L_2B(t)^2)\,.$$

Because $\beta(u)$ is monotone increasing, concave and twice continuously differentiable in $u>0$, we find some constant $L>\max\{L_1+L_2+1,\ \beta(L_1+L_2+1)/\beta(1)\}$ such that

$$\beta(L_1+L_2u)\leqq\beta((L_1+L_2+1)u)\leqq L\beta(u)\quad .$$

if $u \geqq 1$. Moreover, by (3.3)′ and (3.3)″, we find some $t'>0$ such that

$$B(t)^2 \geqq 1 \quad \text{and} \quad A(2t)/B(t) \leqq C$$

for all $t>t'$ with a constant $C>0$. Thus, we get,

$$E\int_{t_0}^{t}\sum_{j=1}^{d}\sigma_{i_0 j}^2(u, X(u))du \leqq LA(t)\beta(B(t)^2) \tag{3.15}$$

for all $t>t''=\max\{t_0, t'\}$. Let $\varepsilon>0$ be arbitrary and let $T_2>T_1>t''$. Then, by martingale inequality and (3.15), we have the following:

$$\begin{aligned}
&P\Big(\sup_{T_1\leqq t\leqq T_2}\Big|\frac{M_{i_0}(t)}{B(t)}\Big|>\varepsilon\Big)\\
&\leqq P\Big(\frac{1}{B(T_1)}\sup_{T_1\leqq t\leqq T_2}|M_{i_0}(t)|>\varepsilon\Big)\\
&\leqq \frac{1}{\varepsilon^2 B(T_1)^2}E\int_{t_0}^{T_2}\sum_{j=1}^{d}\sigma_{i_0 j}^2(u, X(u))du\\
&\leqq \frac{LA(T_2)}{\varepsilon^2 B(T_1)^2}\beta(B(T_2)^2).
\end{aligned}$$

Let N be arbitrary such that $N>t''$. Then, it follows from above inequalities that

$$\begin{aligned}
&P\Big(\sup_{N\leqq t}\Big|\frac{M_{i_0}(t)}{B(t)}\Big|>\varepsilon\Big)\\
&\leqq \sum_{k=0}^{\infty}P\Big(\sup_{N2^k\leqq t<N2^{k+1}}\Big|\frac{M_{i_0}(t)}{B(t)}\Big|>\varepsilon\Big)\\
&\leqq \frac{L}{\varepsilon^2}\sum_{k=0}^{\infty}\frac{A(N2^{k+1})}{B(N2^k)^2}\beta(B(N2^{k+1})^2)\\
&\leqq \frac{CL}{\varepsilon^2}f(N),
\end{aligned}$$

here we used the fact that $A(2t)/B(t)\leqq C$ for all $t>t''$.
Since $Z_N=\sup_{N\leqq t}\Big|\frac{M_{i_0}(t)}{B(t)}\Big|$ is monotone decreasing as N increases, the limit $Z=\lim_{N\to\infty}Z_N$ exists, and then (3.12) implies that $P(Z>\varepsilon)=\lim_{N\to\infty}P(Z_N>\varepsilon)=0$. Therefore, we obtain (3.13) and the proof is complete.

Theorem 3.2 is a generalization of the result of Gihman and Skorohod [2], which is introduced in section 1, as we can imagine from the following example.

Example 3.3. Suppose that $c_i(t)\equiv 1$, $a_i(x)\equiv 1$, $i=1,\dots, d$, and $\sigma(t, x)$ satisfies (1.2), and let

$$\sup_{|x|\leqq 1}|\sigma(t, x)|^2\leqq K$$

for all $t\geqq 0$ and

$$|\sigma(t, x)|^2\leqq K(1+|x|^\gamma)$$

for all $t\geqq 0$ and $|x|\geqq 1$ with some constants $K>0$ and $\gamma<1$. Then, as we have already seen in Example 2.2, the explosion time of the solution $X(t; t_0, x_0)=(X_i(t; t_0, x_0))$ of (1.1) corresponding to such coefficients is infinite with probability one. Clearly, $\sigma(t, x)$ satisfies (3.2), where we take $\alpha(t)\equiv K;\beta(u)=1+u^{\gamma/2}(0\leqq\gamma<1)$; $\beta(u)=2$ $(\gamma<0)$. Also, $A(t)=K(t-t_0)$ and $B(t)=d^{1/2}(t-t_0)$ satisfy (3.3)′ and (3.3)″, and $\beta(u)$ satisfies (3.4). Therefore, Lemma 3.1 implies that

$$\sup_{t_0\leqq u\leqq t} E|X(u; t_0, x_0)|^2\leqq L_1+L_2 d(t-t_0)^2$$

for certain constants $L_1>0$ and $L_2>0$. Moreover, we see that

$$\frac{1}{B(t)}\int_{t_0}^{t} c_i(u)du=d^{-1/2}$$

for $i=1,\dots, d$. Furthermore, let $f(N)$ be defined in (II) of Theorem 3.2, where $\beta(u)$ is replaced by our example. Let N be so large that $N>\max\{2t_0, (t_0+d^{-1/2})/2\}$. Then, we see that $1\leqq B(N2^{k+1})=d^{1/2}(N2^{k+1}-t_0)\leqq d^{1/2}N2^{k+1}$ and $B(N2^k)=d^{1/2}(N2^k-t_0)>d^{1/2}N2^{k-1}$ for $k\geqq 0$. If $\beta(u)=1+u^{\gamma/2}$ $(0\leqq\gamma<1)$, then $\beta(u)\leqq 2u^{\gamma/2}$ for $u\geqq 1$, and then,

$$\begin{aligned} f(N) &\leqq \sum_{k=0}^{\infty}\frac{2[B(N2^{k+1})]^\gamma}{B(N2^k)} \\ &\leqq \sum_{k=0}^{\infty}\frac{2[d^{1/2}N2^{k+1}]^\gamma}{d^{1/2}N2^{k-1}} \\ &=2^{\gamma+2}d^{-(1-\gamma)/2}N^{-(1-\gamma)}\sum_{k=0}^{\infty}2^{-(1-\gamma)k}\,. \end{aligned}$$

If $\beta(u)=2$, then

$$f(N)=\sum_{k=0}^{\infty}\frac{2}{d^{1/2}(N2^k-t_0)}\leqq 2^2 d^{-1/2}N^{-1}\sum_{k=0}^{\infty}2^{-k}\,.$$

Thus, the condition (3.12) is satisfied and Theorem 3.2 implies that

$$P(X(t; t_0, x_0)/(t-t_0)\longrightarrow \tilde{c}\quad\text{as}\quad t\longrightarrow\infty)=1,$$

where $\tilde{c}=(1, 1,\dots, 1, 1)$ is a d-vector.

Remark 3.4. When the drift and diffusion coefficients are bounded, it is well known that the following holds (see [1. p. 194]): let $X(t; t_0, x_0)$ be the solution of (1.1) with the initial condition $X(t_0)=x_0\in R^d$ and assume that $|\sigma(t, x)|\leqq C$,

$|b(t, x)| \leqq C (C = \text{Const})$, $|b(t, x) - \tilde{b}| \to 0$ if $t \to \infty$, uniformly with respect to, x then, $P(X(t; t_0, x_0)/(t - t_0) \to \tilde{b}$ as $t \to \infty) = 1$.

References

[1] A. Friedman: *Stochastic differential equations and applications,* vol. 1, Academic Press, New York, 1975.

[2] I. I. Gihman and A. V. Skorohod: *Stochastic differential equations,* Springer, Berlin-Heidelberg-New York, 1972.

[3] R. Z. Hasminskii: *Stability of systems of differential equations in the presence of random disturbances* (In Russian), Nauka, Moscow, 1969.

[4] H. P., Jr. McKean: *Stochastic integrals,* Academic Press, New York, 1969.

[5] K. Narita: *Remarks on explosion of inhomogeneous diffusion processes,* Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **11** (1972) 235–244.

[6] K. Narita: *Sufficient conditions for no explosion of inhomogeneous diffusion processes,* Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **12** (1973) 95–100.

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