r-QUICK CONVERGENCE FOR ABSOLUTELY REGULAR PROCESSES

By

KEN-ICHI YOSHIHARA

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1. Introduction. Let $\{\xi_j\}$ be a strictly stationary sequence of random variables which are defined on a probability space $(\Omega, \mathscr{A}, P)$. For $a \leqq b$, let $\mathscr{M}_a^b$ denote the σ-algebra of events generated by $\xi_a, \cdots, \xi_b$. As in [4], [5] and [6], we shall say that the sequence satisfies the absolute regularity condition if

(1.1) $$\beta(n) = E\{\sup_{A \in \mathscr{M}_n^\infty} |P(A|\mathscr{M}_{-\infty}^0) - P(A)|\} \downarrow 0 \quad (n \to \infty) .$$

For any positive integer n, let $C_n = C[0, n]$ be the space of all continuous functions on $[0, n]$. We give the uniform topology by defining the distance between two points x and y in C_n as

(1.2) $$\rho_n(x, y) = \sup_{0 \leqq t \leqq n} |x(t) - y(t)| .$$

Let $S_k = \sum_{j=1}^k \xi_j$ and $S_0 = 0$.

In [3], Strassen introduced the notion of r-quick limit points of a sequence of real-valued random variables θ_n. For any real number c, define the random variable

(1.3) $$T_c = \sup\{n \geqq 1 : \theta_n \geqq c\} \quad (\sup \phi = 0)$$

Definition 1. Let $r > 0$ and y be a real constant. Then

(1.4) $$\limsup_{n \to \infty} \theta_n = y \quad (r\text{-quickly})$$

if and only if the following two conditions are satisfied:

(1.5) $$ET_c^r < \infty \quad \text{for} \quad c > y$$

(1.6) $$ET_c^r = \infty \quad \text{for} \quad c < y .$$

We shall also say that if (1.5) holds

(1.7) $$\limsup_{n \to \infty} \theta_n \leqq y \quad (r\text{-quickly})$$

and if (1.6) holds

(1.8) $$\limsup_{n \leftarrow \infty} \theta_n \geqq y \quad (r\text{-quickly}) .$$

Therefore we say that $\limsup_{n\to\infty} \theta_n < \infty$ (r-quickly) if there exists a real constant c for which $ET_c^r < \infty$, and $\limsup_{n\to\infty} \theta_n = \infty$ if otherwise. Likewise if $ET_c^r < \infty$ for all real constants c, then we write $\lim_{n\to\infty} \theta_n = -\infty$ (r-quickly). (cf. [3] and [2]).

Definition 2. Let M be a metric space, endowed with its σ-algebra of Borel sets. Let $\{X_n\}$ be a sequence of random elements taking values in M. Then, $\{X_n\}$ is said to be r-quickly relatively compact in M if for every $\varepsilon > 0$, there is a finite union U of ε-spheres in M such that

$$E(\sup\{n: X_n \notin U\})^r < \infty . \tag{1.9}$$

An element X_0 of M is called an r-quick limit points of $\{X_n\}$ in M if for any open neighborhood V of X_0,

$$E(\sup\{n: X_n \in V\})^r = \infty . \tag{1.10}$$

As in [2], we shall prove Strassen's conjecture for absolutely regular process.

Theorem. *Let $r > 0$. Let $\{\xi_i\}$ be a strictly stationary, absolutely regular process satisfying the following conditions;*

$$\text{(i)} \quad E\xi_i = 0 \ \textit{and} \ E|\xi_i|^a < \infty \ \textit{for some} \ a > 2(2+r) \tag{1.11}$$

$$\text{(ii)} \quad \beta(n) = O(e^{-\gamma n}) \ \textit{for some} \ \gamma > 0 , \tag{1.12}$$

$$\text{(iii)} \quad \sigma^2 = E\xi_0^2 + 2\sum_{j=1}^{\infty} E\xi_0\xi_j > 0. \tag{1.13}$$

Let $X_n = \{X_n(t): 0 \leqq t \leqq 1\}$ $(n = 1, 2, \cdots)$ be random elements in $C[0, 1]$ defined by

$$X_n(t) = \begin{cases} (2\sigma^2 n \log n)^{-1/2} S_j \ \textit{for} \ t = \dfrac{j}{n}, \ j = 0, 1, \cdots, n \\ \textit{linearly interpolated for} \ t \in \left[\dfrac{j-1}{n}, \dfrac{j}{n}\right], \ j = 1, \cdots, n \end{cases} \tag{1.14}$$

Then, for every $\varepsilon > 0$, letting U denote the open ε-neighborhood of $r^{1/2}\mathscr{K}$,

$$E(\sup\{n: X_n \notin U\})^r < \infty \tag{1.15}$$

and so the sequence $\{X_n: n \geqq 2\}$ is r-quickly relatively compact in C_1. The set of its r-quick limit points in C_1 is $r^{1/2}\mathscr{K}$. Here,

$$r^{1/2}\mathscr{K} = \left\{ h \in C_1: h(0) = 0, \ h \ \textit{is absolutely continuous and} \ \int_0^1 (h'(t))^2 dt \leqq r \right\} . \tag{1.16}$$

2. Some auxiliary results. In this and following sections, we shall denote by the letter K, with or without indices, various positive constants.

The followings were proved in [5] and [6] under the conditions of Theorem; for all $n \geqq 1$

$$(2.1) \qquad \text{(I)} \qquad |\operatorname{Var}(S_n) - n\sigma^2| \leqq K\,;$$

$$(2.2) \qquad \text{(II)} \qquad P(\max |S_j| \geqq \lambda) \leqq 2P\left(|S_n| > \frac{\lambda}{2}\right) + 2nk^{-1}\beta(k)$$
$$+2([nk^{-1}]+1)P\left(|\xi_1| + \cdots + |\xi_{2k}| \geqq \frac{\lambda}{4}\right)$$

where k is an arbitrary positive integer such that $k<n$ (cf. [6]);

$$(2.3) \qquad \text{(III)} \qquad \left(\left|\frac{S_n}{n}\right| \geqq \varepsilon\right) = o(n^{-a+1}(\log n)^{a+2})$$

where $\varepsilon>0$ is arbitrary. (cf. Theorem 4.2 in [5]).

Lemma 2.1. *Assume that the conditions of Theorem are satisfied. Then for any* $n(\geqq 1)$

$$(2.4) \qquad E|S_n|^{2k} \leqq Kn^k\{E|\xi_1|^a\}^{2k/a}$$

where k is an arbitrary positive integer such that $2k<a$.

Proof. Let $i_1 \leqq \cdots \leqq i_{j-1} < i_j \leqq \cdots \leqq i_{2k}$ be arbitrary integers. Then from Lemma 1 [4] and Hölder's inequality that

$$E|\xi_{i_1} \cdots \xi_{i_{j-1}} \xi_{i_j} \cdots \xi_{i_{2k}}|^{a/2k} \leqq E|\xi_1|^a \leqq K$$

and

$$|E\xi_{i_1} \cdots \xi_{i_{j-1}} \xi_{i_j} \cdots \xi_{i_{2k}} - E\xi_{j_1} \cdots \xi_{i_{j-1}} E\xi_{i_j} \cdots \xi_{i_{2k}}|$$
$$\leqq K\{\beta(i_j - i_{j-1})\}^{(a-2k)/a}\,.$$

So, using the method of the proof of Lemma 3 in [4], we have the desired inequality.

3. Proof of Theorem. To prove Theorem, we need some lemmas. Throughout this section, we put $b=b_n=(n \log n)^{1/2}$, $p=p_n=[(n \log n)^{1/2}]$, $k=k_n=[np^{-1}]$, $d=(1/3)\{a-2(2+r)\}>0$ and $q=q_n=[c \log n]$ (c being a positive number such that $c\gamma > r+d+2$). Here, $[s]$ denotes the largest integer m such that $m \leqq s$.

Lemma 3.1. *Assume that the conditions of Theorem hold. Then, for any* $\varepsilon>0$

$$(3.1) \qquad P(\max_{1 \leqq j \leqq p} |S_j| \geqq \varepsilon b) = O(n^{-(r+d+3/2)})$$

Proof. From (II)

$$P(\max_{1\leq j\leq p}|S_j|\geqq \varepsilon b)\leqq 2P\left(|S_p|\geqq \frac{\varepsilon}{2}b\right)+2n^{1/2}q^{-1}\beta(q)$$
$$+2([n^{1/2}q^{-1}]+1)P\left(|\xi_1|+\cdots+|\xi_{2q}|\geqq \frac{\varepsilon}{2}b\right). \tag{3.2}$$

By the definition of q

$$n^{1/2}q^{-1}\beta(q)=o(n^{-(r+d+3/2)}). \tag{3.3}$$

Now, we prove that

$$I=P\left(|\xi_1|+\cdots+|\xi_{2q}|\geqq \frac{\varepsilon}{2}b\right)=o(n^{-(r+d+3/2)}) \tag{3.4}$$

Let $N=(\varepsilon/10)bq^{-1}$ and for each i, put

$$\bar{\xi}_i=\xi_i^{(N)}=\begin{cases}\xi_i, & (|\xi_i|\leqq N),\\ 0, & \text{otherwise}.\end{cases}$$

and $\bar{\bar{\xi}}_i=\xi_i-\bar{\xi}_i$. Then

$$\sum_{j=1}^{2q}|\bar{\xi}_j|\leqq(\varepsilon/5)b=O(n^{1/2}\log^{1/2}n)$$

and

$$E|\bar{\bar{\xi}}_i|^2\leqq N^{-(a-2)}E|\xi_1|^a=O\left(\left(\frac{n}{\log n}\right)^{-(a-2)/2}\right).$$

Thus, for all n sufficiently large

$$I\leqq P\left(\sum_{j=1}^{2q}|\bar{\bar{\xi}}_j|\geqq\frac{1}{4}\varepsilon b\right)\leqq K\frac{1}{\varepsilon^2b^2}E\left(\sum_{j=1}^{2q}|\bar{\bar{\xi}}_j|\right)^2\leqq K\frac{q^2}{b^2}E|\bar{\bar{\xi}}_1|^2=o(n^{-(r+d+3/2)})$$

which implies (3.4).

By (III)

$$P\left(|S_p|\geqq\frac{\varepsilon}{2}b\right)=o(p^{-a+1}(\log p)^{a+2})$$

holds for any $\varepsilon>0$ and so for all n sufficiently large

$$P\left(|S_p|\geqq\frac{\varepsilon}{2}b\right)=o(n^{-(r+d+3/2)}). \tag{3.5}$$

Hence, from (3.3)–(3.5) we have the desired conclusion.

Define random elements $\bar{S}_n=\{S_n(t): 0\leqq t\leqq n\}$ and $\tilde{S}_n=\{S_n(t): 0\leqq t\leqq n\}$ in C_n, respectively, by

$$\bar{S}_n(t)=\begin{cases} S_i \text{ for } t=i,\ i=0,1,\cdots,n \\ \text{linearly interpolated for } t\in[i-1,i],\ i=1,\cdots,n \end{cases} \tag{3.6}$$

and

$$\tilde{S}_n(t)=\begin{cases} S_{jp} \text{ for } t=jp,\ j=1,\cdots,k \\ S_{kp} \text{ for } kp\leqq t\leqq n, \\ \text{linearly interpolated for } t\in[(j-1)p,\ jp),\ j=0,1,\cdots,k\ . \end{cases} \tag{3.7}$$

Lemma 3.2. *Assume that the conditions of Theorem are satisfied. Then, for any* $\varepsilon>0$

$$P(\rho_n(\bar{S}_n,\tilde{S}_n)\geqq\varepsilon b)=O(n^{-(r+d+1)}) \tag{3.8}$$

Proof. Since

$$\begin{aligned}\rho_n(\bar{S}_n,\tilde{S}_n)&=\sup_{0\leqq t\leqq n}|\bar{S}_n(t)-\tilde{S}_n(t)|\\ &\leqq 2\max_{1\leqq j\leqq k}\{\max_{(j-1)p\leqq i\leqq jp}|S_i-S_{(j-1)p}|\}+\max_{kp\leqq i\leqq n}|S_i-S_{kp}|\ ,\end{aligned}$$

so, by Lemma 3.1

$$\begin{aligned}&P(\rho_n(\bar{S}_n,\tilde{S}_n)\geqq\varepsilon b)\\ &\leqq\sum_{j=1}^{k}P\left(\max_{(j-1)p\leqq i\leqq jp}|S_i-S_{(j-1)p}|\geqq\frac{1}{4}\varepsilon b_n\right)+P\left(\max_{kp\leqq i\leqq n}|S_i-S_{kp}|\geqq\frac{1}{2}\varepsilon b_n\right)\\ &\leqq(k+1)P\left(\max_{1\leqq i\leqq p}|S_j|\geqq\frac{1}{4}\varepsilon b_n\right)=O(k\cdot n^{-(r+d+3/2)})=o(n^{-(r+d+1)})\ .\end{aligned}$$

Hence, the proof is completed.

Let

$$\eta_j=\sum_{i=1}^{p-q}\xi_{(j-1)p+i}$$

and define a random element $\hat{S}_n=\{\hat{S}_n(t): 0\leqq t\leqq n\}$ in C_n by

$$\hat{S}_n(t)=\begin{cases} \sum_{i=1}^{j}\eta_i \text{ for } t=jp,\ j=0,1,\cdots,k\ , \\ \sum_{i=1}^{m}\eta_i \text{ for } kp\leqq t\leqq n \\ \text{linearly interpolated for } t\in[(j-1)p,\ jp],\ j=1,\cdots,k\ . \end{cases} \tag{3.9}$$

Lemm 3.3. *Assume that the the conditions of Theorem are satisfied. Then for any* $\varepsilon>0$

$$P(\rho_n(\tilde{S}_n,\hat{S}_n)\geqq\varepsilon b)=o(n^{-(r+d+1)}) \tag{3.10}$$

Proof. Let

$$\zeta_j=\sum_{i=1}^{q}\xi_{jp-i+1}(j=1, \cdots, k);\ \zeta_{k+1}=\sum_{i=1}^{n-kp}\xi_{kp+i}\ .$$

Since

$$\rho_n(\widetilde{S}_n, \hat{S}_n)\leqq\max_{1\leqq j\leqq k}|\sum_{i=1}^{j}|\zeta_i|+|\zeta_{k+1}|\ ,$$

so for any $\varepsilon>0$

$$P(\rho_n(\widetilde{S}_n, \hat{S}_n)\geqq\varepsilon b)\leqq P\Big(\max_{1\leqq j\leqq k}|\sum_{i=1}^{j}\zeta_i|\geqq\frac{\varepsilon}{2}b\Big)+P\Big(|\zeta_{k+1}|\geqq\frac{\varepsilon}{2}b\Big) \tag{3.11}$$

From Lemma 3.1

$$P\Big(|\zeta_{k+1}|\geqq\frac{\varepsilon}{2}b\Big)=o(n^{-(r+d+3/2)}) \tag{3.12}$$

On the other hand, as $\{\zeta_i\}$ is a strictly stationary, absolutely regular sequence and from Lemma 2.3 $E|\zeta_i|^{2(2+[r])}<\infty$, so using the method of the proof of Lemma 2.2.

$$P\Big(\max_{1\leqq j\leqq k}|\sum_{i=1}^{j}\zeta_i|\geqq\frac{\varepsilon}{2}b\Big)\leqq 2P\Big(|\sum_{i=1}^{k}\zeta_i|\geqq\frac{\varepsilon}{4}b\Big)+k\beta(p)\ . \tag{3.13}$$

It follows from (III) that for all n sufficiently large

$$\begin{aligned}P\Big(|\sum_{i=1}^{k}\zeta_i|\geqq\frac{\varepsilon}{4}b\Big)&=P\Big(|\sum_{j=1}^{q}\{\sum_{i=1}^{k}\xi_{ip-j+1}\}|\geqq\frac{\varepsilon}{4}b\Big)\\&\leqq\sum_{j=1}^{q}P\Big(|\sum_{i=1}^{k}\xi_{ip-j+1}|\geqq\frac{\varepsilon}{4}bq^{-1}\Big)\\&=qP\Big(|\sum_{i=1}^{k}\xi_{ip}|\geqq\frac{\varepsilon}{4}bq^{-1}\Big)=qP\Big(\Big|\frac{\sum_{i=1}^{k}\xi_{ip}}{k}\Big|\geqq\frac{\varepsilon}{5c}\Big)\\&=q\cdot o(k^{-a+1}(\log k)^{a+2})=o(n^{-(r+d+3/2)})\ .\end{aligned}$$

As $k\beta(p)=o(n^{-(r+d+3/2)})$, so (3.10) follows from (3.11)-(3.13).

Let $w=\{w(t): 0\leqq t<\infty\}$ be a standard Wiener process on the probability space $(\Omega, \mathscr{A}, P)$.

Lemma 3.4. *Assume that the conditions of Theorem are satisfied. Then for each $n(\geqq1)$ there exists a sequence of non-negative, i.i.d. random variables $\tau_1^{(n)}, \cdots, \tau_k^{(n)}$ with the following properties: For any $\varepsilon>0$*

$$E(\max\{n: 2^{-1/2}b_n^{-1}\max_{1\leqq j\leqq k}|w(T_j^{(n)})|>\varepsilon\})^r-K_1 \tag{3.14}$$

$$\leqq E(\max\{n: 2^{-1/2}\sigma^{-1}b_n^{-1}\sup_{0\leqq t\leqq n}|\hat{S}_n(t)|\geqq\varepsilon\})^r$$
$$\leqq E(\max\{n: 2^{-1/2}b_n^{-1}\max_{1\leqq j\leqq k}|w(T_j^{(n)})|>\varepsilon\})^r+K_2$$

where for each $n(\geqq 1)$ $T_j^{(n)}=\sum_{i=1}^{j}\tau_i^{(n)}(j=1,\cdots,k)$, $T_0^{(n)}=0$ *and*

$$E\tau_1^{(n)}=\sigma^{-2}E\eta_1^2\,, \tag{3.15}$$

$$E(\tau_1^{(n)})^j\leqq K_jE\eta_1^{2j}\quad(j=2,3,\cdots) \tag{3.16}$$

Proof. Since $\eta_1,\cdots,\eta_k$ constitute an absolutely regular sequence, so from Theorem 2.1 in [6] we can conclude that there exists a sequence of nonnegative, i. i. d. random variables $\tau_1,\cdots,\tau_k$ satisfying (3.15) and (3.16) for which

$$|P(\max_{1\leqq j\leqq k_n}|w(T_j^{(n)})|\geqq 2^{1/2}\varepsilon b_n)-P(\max_{0\leqq t\leqq n}\sigma^{-1}|\hat{S}_n(t)|\geqq 2^{1/2}\varepsilon b_n)|\leqq Kk_n\beta(q_n)\,. \tag{3.17}$$

(cf. the proof of Lemma 4.2 in [6]). By the definition of k and q

$$\sum_{n=1}^{\infty}n^{r-1}k_n\beta(q_n)<\infty\,.$$

Hence, (3.14) follows from (3.17) and the proof is completed.

Let $\hat{Y}_n=\{\hat{Y}_n(t): 0\leqq t\leqq 1\}$ and $\tilde{Y}_n=\{\tilde{Y}_n(t): 0\leqq t\leqq 1\}$ be random elements in C_∞ defined respectively, by

$$\hat{Y}_n(t)=\begin{cases} w(T_j^{(n)})/2^{1/2}b_n \text{ for } t=jp_n,\ j=0,1,\cdots,k_n \\ w(T_j^{(n)})/2^{1/2}b_n \text{ for } \dfrac{k_np_n}{n}\leqq t\leqq 1 \\ \text{linearly interpolated for } t\in\left[\dfrac{(j-1)p_n}{n},\dfrac{jp_n}{n}\right],\ j=1,\cdots,k_n \end{cases} \tag{3.18}$$

and

$$\tilde{Y}_n(t)=\begin{cases} w(jp_n)/2^{1/2}b_n \text{ for } t=\dfrac{jp_n}{n},\ j=0,1,\cdots,k_n \\ w(k_np_n)/2^{1/2}b_n \text{ for } \dfrac{k_np_n}{n}\leqq t\leqq 1 \\ \text{linearly interpolated for } t\in\left[\dfrac{(j-1)p_n}{n},\dfrac{jp_n}{n}\right],\ j=1,\cdots,k_n\,. \end{cases}$$

Lemma 3.5. *Assume that the conditions of Theorem are satisfied. Then for any* $\varepsilon>0$

$$\sum_{m=1}^{\infty}m^{r-1}P(\rho_1(\hat{Y}_n,\tilde{Y}_n)\geqq\varepsilon b_n \textit{ for some } n\geqq m)<\infty \tag{3.19}$$

Proof. We use the method in the proof of Theorem 5 in [2]. Since

$$\rho_1(\hat{Y}_n, \tilde{Y}_n)=\max_{1\leq i\leq k_n}|w_n(T_i)-w(ip_n)|/2^{1/2}b_n$$
$$=\max_{1\leq i\leq k_n}|w_n(T_ip_n^{-1})-w(i)|/2^{1/2}b_np_n^{-1/2}\ ,$$

so

(3.20) $P(\rho_1(\hat{Y}_n, \tilde{Y}_n)\geqq\varepsilon b_n$ *for some* $n\geqq m)$

$$\leqq P(\max_{1\leq i\leq k_n}\max_{|t-i|\leq k_n^\delta}|w(t)-w(i)|\geqq(\varepsilon/4)2^{1/2}b_np_n^{-1/2} \text{ for some } n\geqq m)$$
$$+P(\max_{1\leq i\leq k_n}|T_i^{(n)}p_n^{-1}-i|\geqq k_n^\delta \text{ for some } n\geqq m)$$
$$=P_m^{(1)}+P_m^{(2)},\ \text{(say)}$$

where δ is a positive number such that $4/5<\delta<1$.
We remark that from Lemma 2.1 and (3.15)

$$|E(\tau_1p_n^{-1})-1|=|\sigma^{-2}p_n^{-2}E\eta_1^2-1|=O(\log n)$$

and from Lemma 2.3 and (3.16)

$$\text{Var}\,(\tau_1p_n^{-1})\leqq p_n^{-2}E\tau_1^2\leqq Kp_n^{-2}E\eta_1^4\leqq K$$

and

$$p_n^{-2j_0}E|\tau_1-E\tau_1|^{2j_0}\leqq Kp_n^{-2j_0}\max_{1\leq i\leq 2j_0}[\{E\tau_1^i\}\{E\tau_1\}^{2j_0-i}]\leqq K$$

where $j_0=2+[r]$. Thus, it follows from Corollary 17.12 in [1] that for any $\varepsilon>0$

$$P(\max_{1\leq j\leq k_n}|T_jp_n^{-1}-j|\geqq\varepsilon k_n^\delta)$$
$$\leqq P(\max_{1\leq j\leq k_n}|p_n^{-1}\sum_{i=1}^{j}(\tau_i^{(n)}-E\tau_i^{(n)})|\geqq\varepsilon k_n^\delta-\sum_{j=1}^{k_n}p_n^{-1}|E\tau_i^{(n)}-p_n|)$$
$$\leqq P\left(\max_{1\leq j\leq k_n}|p_n^{-1}\sum_{i=1}^{j}(\tau_i^{(n)}-E\tau_i^{(n)})|\geqq\frac{\varepsilon}{2}k_n^\delta\right)$$
$$\leqq 2P\left(|\sum_{i=1}^{k_n}p_n^{-1}(\tau_i^{(n)}-E\tau_i^{(n)})|\geqq\frac{\varepsilon}{4}k_n^\delta\right)$$
$$\leqq K\cdot k_n^{-(2j_0-2)/2}k_n^{-2j_0(\delta-1)/2}=o(n^{-(r+6/5)})$$

and, so putting $n_i=2^im(i=0, 1, 2, \cdots)$ we have

$$P_m^{(2)}\leqq\sum_{i=1}^{\infty}P(\max_{n_{i-1}\leq n\leq n_i}\max_{1\leq j\leq k_n}|T_j^{(n)}p_n^{-1}-j|\geqq k_{n_{i-1}}^\delta)$$
$$\leqq\sum_{i=1}^{\infty}n_i\{\max_{n_{i-1}\leq n\leq n_i}P(\max_{1\leq j\leq k_{ni}}|T_n^{(n)}p_n^{-1}-j|\geqq k_{n_{i-1}}^\delta)\}$$
$$\leqq K\sum_{i=1}^{\infty}(2^im)^{-(r+1/5)}=O(m^{-(r+1/5)})\ .$$

Therefore, we have

$$\sum_{m=1}^{\infty} m^{r-1} P_m^{(2)} < \infty . \tag{3.21}$$

On the other hand,

$$P_m^{(1)} \leqq P\,(\max_{1\leqq i\leqq k_n} \max_{|t-i|\leqq k_n^{\delta}} |w(t)-w(i)| \geqq (\varepsilon/4)(2k_n \log\log k_n)^{1/2} \text{ for some } n\geqq m)$$

and so from (4.9) in [2]

$$\sum_{m=1}^{\infty} m^{r-1} P_m^{(1)} < \infty . \tag{3.22}$$

Hence, from (3.20)-(3.22), the desired conclusion follows.

Finally, let

$$Y_n(t) = \frac{w(nt)}{2^{1/2} b_n}, \quad 0 \leqq t \leqq 1 . \tag{3.23}$$

Lemma 3.6. *For any* $\varepsilon > 0$

$$\sum_{m=1}^{\infty} m^{r-1} P(\rho_1(\tilde{Y}_n, Y_n) \geqq \varepsilon \textit{ for some } n \geqq m) < \infty . \tag{3.24}$$

Proof. We remark that for any $s>0$ and for any j

$$P(\sup_{0\leqq t\leqq s} |w(t)| \geqq \varepsilon) \leqq \frac{2^{j+1}\Gamma\left(\frac{2j+1}{2}\right)}{\sqrt{\pi}\varepsilon^{2j}} s^j . \tag{3.25}$$

Since

$$\rho_1(\hat{Y}_n, Y_n) = \max_{1\leqq i\leqq k_n} \sup_{(i-1)p_n \leqq nt \leqq ip_n} \left| \frac{w(nt)-w(ip_n)}{2^{1/2} b_n} \right| + \sup_{k_n p_n \leqq nt \leqq n} \left| \frac{w(nt)-w(k_n p_n)}{2^{1/2} b_n} \right|$$

so, from (3.24) it follows that

$$\begin{aligned} P(\rho_1(\tilde{Y}_n, Y_n) \geqq \varepsilon) &\leqq (k_n+1) P(\sup_{0\leqq nt\leqq p_n} |w(nt)| \geqq \varepsilon\sqrt{2}\, b_n) \\ &= (k_n+1) P(\sup_{0\leqq t\leqq n^{-1}p_n} |w(t)| \geqq \sqrt{2}\,\varepsilon \log^{1/2} n) \\ &\leqq K(k_n+1) \left(\frac{p_n}{n}\right)^j (\log n)^{-j} \\ &\leqq K k_n^{-(j-1)} (\log n)^{-j} \end{aligned} \tag{3.26}$$

for any $j(\geqq 1)$. Thus, (3.24) is easily obtained from (3.26).

Proof of Theorem. We now proceed to prove Theorem. Let

$$\tilde{V}_n=\{\tilde{V}_n(t):0\leqq t\leqq 1\} \quad \text{and} \quad \hat{V}_n=\{\hat{V}_n(t);\ 0\leqq t\leqq 1\}$$

be random elements in C_1 defined, respectively, by

$$\tilde{V}_n(t)=\frac{\tilde{S}_n(nt)}{2^{1/2}b_n}, \quad 0\leqq t\leqq 1, \tag{3.27}$$

and

$$\hat{V}_n(t)=\frac{\hat{S}_n(nt)}{2^{1/2}b_n}, \quad 0\leqq t\leqq 1. \tag{3.28}$$

Then, from Lemmas 3.2 and 3.3 the following inequalities easily follow:

$$\sum_{m=1}^{\infty} m^{r-1}P(\rho_1(X_n, \tilde{V}_n)\geqq\varepsilon \text{ for some } n\geqq m)<\infty \tag{3.29}$$

$$\sum_{m=1}^{\infty} m^{r-1}P(\rho_1(\hat{V}_n, \tilde{V}_n)\geqq\varepsilon \text{ for some } n\geqq m)<\infty \tag{3.30}$$

Thus, the desired conclusion follows from Corollary 1 in [2], Lemmas 3.4–3.6, (3.29) and (3.30) using the method of the proof of Theorem 2 in [2].

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Yokohama National University
Department of Mathematics
Faculty of Engineering
156, Tokiwadai, Hodogaya,
Yokohama, Japan