ON LINKS WITH PROPERTY P^*

By

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0. Introduction

We are interested in the following problem in piecewise-linear 3-dimensional topology.

Problem *Is it possible to construct the counterexample to the Poincaré conjecture by removing a finite number of mutually disjoint solid tori from* S^3 *and sewing them back in a different way?*

To the purpose above, we will consider a problem as follows;

Let $C(l)$ be the closure of the complement in S^3 of a regular neighborhood of a link l. If every homotopy 3-sphere Σ^3 obtained by refilling $C(l)$ by solid tori with suitable identification of the boundary surface is a 3-sphere, then we say that l has *Property* P^*.

Conjecture. *Every link has Property* P^*.

It has been shown [12] that every closed, connected, orientable 3-manifold can be constructed by removing a finite number of mutually disjoint solid tori from S^3 and sewing them back in a different way. In paticular, every homotopy 3-sphere can be obtained by this way; thus this conjecture is equivalent to the Poincaré conjecture.

In [1][5][8] and [11], the problem above is discussed for a knot and it is obtained that some knots have Property P (stronger than Property P^*). We can show that there are many links without property P. So, considering a link with Property P^* will be meaningful.

In this paper we will prove the following theorem;

Theorem 1. *If links* l *and* l' *have Property* P^**, then* $l \cdot l'$ *is a link with Property* P^**, where* $l \cdot l'$ *means any product of links* l *and* l'*, see* [6].

As an immediate consequence, we have;

Corollary 2. *Every link has Property* P^* *if every prime link* [6] *has Property* P^*.

This implies that it is enough to decide whether the conjecture above is true for only prime links.

By Theorem 1, we will obtain that;

Theorem 3. *Every torus link has Prorerty P^*.*

In 1, we will show that some elementary links have Property P^*. In 2, Lemma 3 which plays a important role, and Theorem 1 will be obtained. In 3, some corollaries of Theorem 1 will be given, and some links with Property P^* will be obtained. The author is indebted to Professor *T. Homma*, *F. Hosokawa* and *F. Gonzáles-Acuña* for their kind suggestions.

1. Some elementary links with Property P*

Throughout this paper, let us denote the boundary, the interior and the closure of a manifold M by ∂M, $\operatorname{int} M$ and $\operatorname{cl} M$ respectively. A regular neighborhood of a submanifold A in a manifold M will be denoted by $N(A; M)$. For two loops f and g on a surface, $S(f, g)$ denotes the absolute value of the homological intersection number of an oriented chains f and g.

Let l be a link $k_1 \cup k_2 \cup \cdots \cup k_\mu$ in S^3, and $N(k_i; S^3)$ be a regular neighborhood of k_i in S^3 such that $N(k_i; S^3) \cap (l-k_i)=\phi$. Let m_i be a simple closed curve on $\partial N(k_i; S^3)$ which bounds 2-cell in $N(k_i; S^3)$ and l be a simple closed curve on $\partial N(k_i; S^3)$ which is homologous to 0 in $S^3 - \operatorname{int} N(k_i; S^3)$. We call m_i and l_i a meridan and a longitude of $N(k_i; S^3)$, respectively.

Let $T_1, T_2, \cdots, T_s$ be mutually disjoint solid tori in the interior of a connected, orientable 3-manifold M. We may then construct the 3-manifold

$$M' = \operatorname{cl}\{M-(T_1 \cup T_2 \cup \cdots \cup T_s)\} \underset{h}{\cup} \{T_1 \cup T_2 \cup \cdots \cup T_s\}$$

where h is a union of homeomorphisms h_i: $\partial T_i \to \partial T_i$. The manifold M' is said to be the result of *a surgery on* $\{T_1, T_2, \cdots, T_s\}$ *in* M, and h is said to be *a surgery homeomorphism.* When $T_1 \cup T_2 \cup \cdots \cup T_\mu$ is a regular neighborhood of a link l of μ components in $\operatorname{int} M$, the manifold M is said to be the result of a *surgery on a link* l *and* $\{T_{\mu+1}, \cdots, T_s\}$ *in* M; $1 \le \mu \le s$.

As a consequence of definition, we have;

Proposition 1. *If a link $l = k_1 \cup k_2 \cdots \cup k_\mu$ has Property P^*, then every sublink $l' = k_{i_1} \cup k_{i_2} \cup \cdots \cup k_{i_\nu}$ of l has Property P^*, where $\{i_1, i_2, \cdots, i_\nu\} \subset \{1, 2, \cdots, \mu\}$ and $i_k \neq i_l (k \neq l)$.*

Suppose that for a link l, there is a 3-cell B^3 such that $\partial B^3 \cap l = \emptyset$. Let l_1 be a link $l \cap B^3$ and l_2 be a link $l \cap \operatorname{cl}(S^3 - B^3)$. Then, we easily have;

Proposition 2. *If l_1 and l_2 have Property P^*, then a link l has Property P^*.*

We will show that the following links O_1, O_2 and O_3, described in Fig. 1, have Property P^*. For the components of O_i, we write k_j as described in Fig. 1.

Lemma 1. *The links O_1 and O_2 have Property P^*.*

Proof. By Proposition 1, if the link O_2 has Property P^*, then the link O_1 has Property P^*. So it is enough to prove that the link O_2 has Property P^*.

Let Σ be a homotopy 3-sphere obtained by doing surgery on the link O_2 in S^3, F be the boundary of a regular neighborhood of a component k_1 of the link O_2 in S^3, and M, N be two components of $\Sigma - F$, see Fig. 2. Both $\mathrm{cl}\, M$ and $\mathrm{cl}\, N$ are solid tori. Since $\Sigma = \mathrm{cl}\, M \cup \mathrm{cl}\, N$, Σ is homeomorphic to one of S^3, $S^2 \times S^1$ and lens space. Hence Σ is homeomorphic to S^3, for $\pi_1(\Sigma) = \{1\}$.

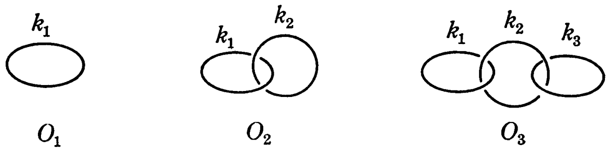

Fig. 1.

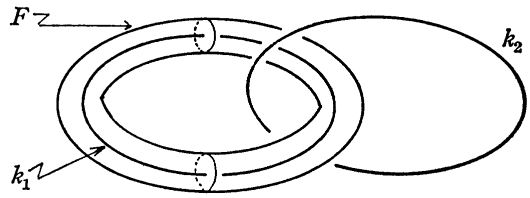

Fig. 2.

Lemma 2. *The link O_3 has Property P^*.*

Proof. Let C be the closure of the complement of a regular neighborhood of the link O_3 in S^3. Let m_i be a meridian of $N(k_i;\, S^3)$, and l_i be a longitude of $N(k_i;\, S^3)$, $i = 1, 2, 3$, see Fig. 3.

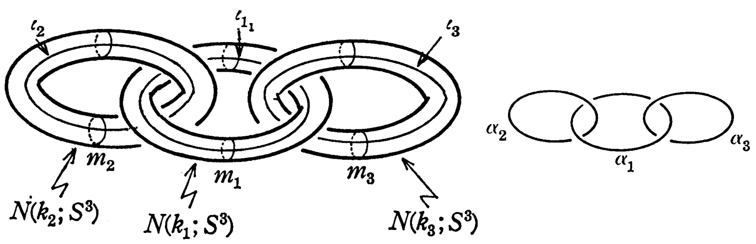

Fig. 3.

Suppose that Σ^3 is a homotopy 3-sphere obtained by doing surgery on the link O_3 in S^3 and h is a surgery homeomorphism $\bigcup_{i=1}^{3} \{h_i \colon \partial(D^2 \times S^1)_i \to \partial N(k_i;\, S^3)\}$. Let overpasses a_i represent generators and crossingpoints give the relators, see Fig. 3. There is a presentation of $\pi_1(C)$;

$$\{a_1, a_2, a_3;\ a_2 a_1 = a_1 a_2,\ a_1 a_3 = a_3 a_1\}$$

Since $h_i(\partial D_i^2) = h_i(\partial(D^2 \times \{p\})_i)$, where p is a point in S^1, is a simple closed curve on $\partial N(k_i;\, S^3)$, $h_i(\partial D_i^2)$ is represented by m_i and l_i on $\partial N(k_i;\, S^3)$, $i = 1, 2, 3$.

So $h_i(\partial D_i^2)$ is represented as an element having the form $m_i^{p_i} l_i^{q_i}$, $i=1,2,3$. Let x_i be an arc joining a base point of C to arbitrary one point in $h_i(\partial D_i^2)$ and γ_i be a closed curve represented as $x_i m_i^{p_i} l_i^{q_i} x_i^{-1}$. $\pi_1(\Sigma^3)$ is obtained from $\pi_1(C)$ by adding relations $\gamma_1=a_1^{\varepsilon_1 p_1}(a_2a_3)^{\varepsilon'_1 q_1}=1$, $\gamma_2=a_2^{\varepsilon_2 p_2}a_1^{\varepsilon'_2 q_2}=1$ and $\gamma_3=a_3^{\varepsilon_3\, p_3}a_1^{\varepsilon'_3\, q_3}=1$, where ε_i, $\varepsilon'_i=\pm 1$. This yields the following;

$$\begin{aligned}\pi_1(\Sigma^3)=\{a_1, a_2, a_3;\ a_2a_1=a_1a_2,\ a_3a_1=a_1a_3,\ & a_1^{\varepsilon_1 p_1}(a_2a_3)^{\varepsilon'_1 q_1}\\ =a_2^{\varepsilon_2 p_2}a_1^{\varepsilon'_2 q_2}=a_3^{\varepsilon_3\, p_3}a_1^{\varepsilon'_3\, q_3}=1\}\end{aligned}$$

Consider the group $G=\{R, S;\ R^{\varepsilon_2 p_2}=S^{\varepsilon_3\, p_3}=(SR)^{-\varepsilon'_1 q_1}=1\}$. If $p_2, p_3, q_1\neq\pm 1$, this group is nontrivial [2]. A nontrivial representation η of $\pi_1(\Sigma^3)$ onto G is given by $\eta(a_1)=1$, $\eta(a_2)=R$, $\eta(a_3)=S$. Note that $\eta(a_2a_1)=R=\eta(a_1a_2)$, $\eta(a_3a_1)=S=\eta(a_1a_3)$, $\eta(a_1^{\varepsilon_1 p_1}(a_2a_3)^{\varepsilon'_1 q_1})=(SR)^{-\varepsilon'_1 q_1}=1$, $\eta(a_2^{\varepsilon_2 p_2}a_1^{\varepsilon'_2 q_2})=R^{\varepsilon_2 p_1}=1$ and $\eta(a_3^{\varepsilon_3\, p_3}a_1^{\varepsilon'_3\, q_3})=S^{\varepsilon_3\, p_3}=1$. Hence η is a homomorphism. This gives the contradiction that $\pi_1(\Sigma^3)$ is trivial. We have that $p_2=\pm 1$, $p_3=\pm 1$ or $q_1=\pm 1$. We will prove Lemma 2 in respective cases.

Case 1 $p_2=\pm 1$. Since $\partial N(k_1;\ S^3)$ is a surface of genus 1, there is an embedding f of $S^1\times S^1\times I$ in S^3 such that $f(S^1\times S^1\times I)\cap N(k_1;\ S^3)=f(S^1\times S^1\times I)\cap \partial N(k_1; S^3)=f(S^1\times S^1\times\{0\})$ and $f(S^1\times S^1\times I)\cap N(k_2; S^3)=\emptyset=f(S^1\times S^1\times I)\cap N(k_3; S^3)$. Let M be a solid torus $\mathrm{cl}\,[S^3-\{N(k_1;\ S^3)\cup f(S^1\times S^1\times I)\}]$ and M' be the result of a surgery on a link $k_2\cup k_3$ in M by surgery homeomorphism $h_2\cup h_3$, see Fig. 4.

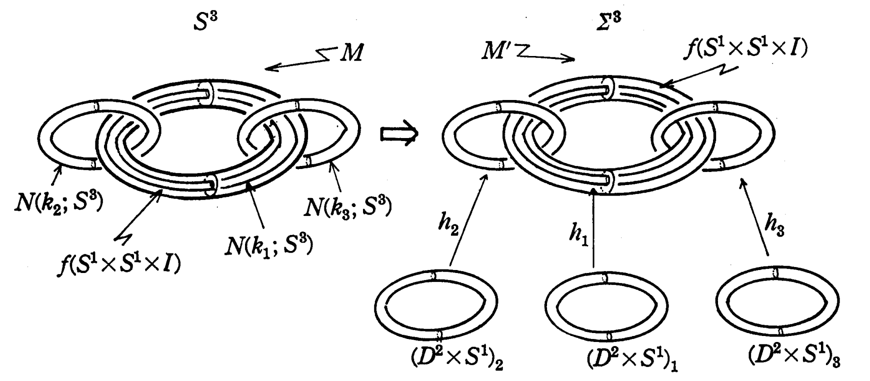

Fig. 4.

If M' is a solid torus, then Σ^3 is regarded as a homotopy 3-sphere obtained by removing solid tori $N(k_1;\ S^3)$ and M from S^3, and refilling solid tori $(D^2\times S^1)_1$ and M' with suitable identification of boundary surface. Hence Σ^3 is the result of a surgery on the link O_2 in S^3. Since the link O_2 has Property P^* by Lemma 1, Σ^3 is homeomorphic to 3-sphere.

We will show that M' is a solid torus. Let A be an annulus properly embedded in M such that A separates $N(k_1;\ S^3)$ and $N(k_2;\ S^3)$ in M. Since an annulus A is properly embedded in M', A divides M' into two parts, say V and W, see Fig. 5. Note that cl V and cl W are solid tori. There are simple closed curves v on ∂V, and w on ∂W, respectively, such that M' is homeomorphic to a 3-manifold obtained by pasting cl V and cl W along $N(v;\partial V)$ and $N(w;\ \partial W)$. Since $\mathrm{cl}\{V-(D^2\times S^1)_2\}$ is homeomorphic to $S^1\times S^1\times I$ and there are level preserving isotopies H_i: $S^1\times I\to S^1\times S^1\times I$, $i=1,2$; such that $H_1(S^1\times\{0\})=g(v)$, $H_1(S^1\times\{1\})=g(l_2)$, $H_2(S^1\times\{0\})=g(\mu)$ and $H_2(S^1\times\{1\})=gh_2(\partial D_2^2)$, then $S(v,\mu)=S(l_2, h_2(\partial D_2^2))=|p_2|=1$, where μ is a meridian of cl V, and g is a homeomorphism of $\mathrm{cl}\{V-(D^2\times S^1)_2\}$ onto $S^1\times S^1\times I$. Hence M' is a solid torus, see Fig. 6.

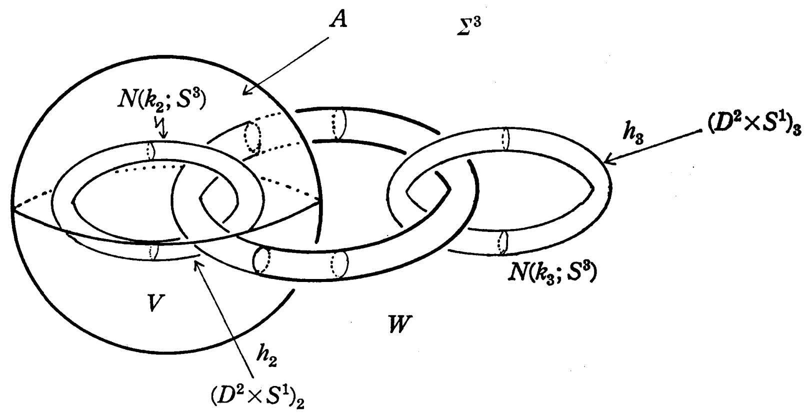

Fig. 5.

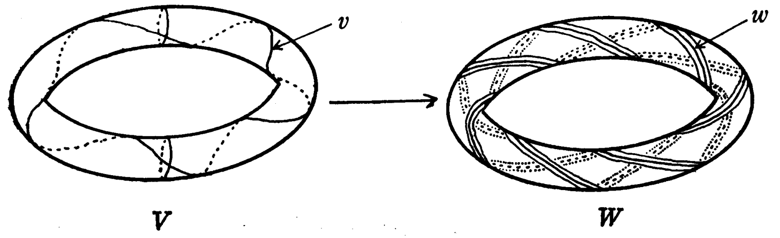

Fig. 6.

Case 2 $p_3=\pm 1$. In this case, Lemma 2 is obtained by the same way as those in the case 1.

Case 3 $q=\pm 1$. We convert k_1 into k_2, and apply the same argument for $N(k_2;\ S^3)$ as those in the case 1. Then, we show that Σ^3 is homeomorphic to 3-sphere see Fig. 7.

2. Proof of the main theorem

Let l be a link $k_1\cup k_2\cup\cdots\cup k_\mu$ in S^3, and $N(k_1;\ S^3)$ be a regular neighbor-

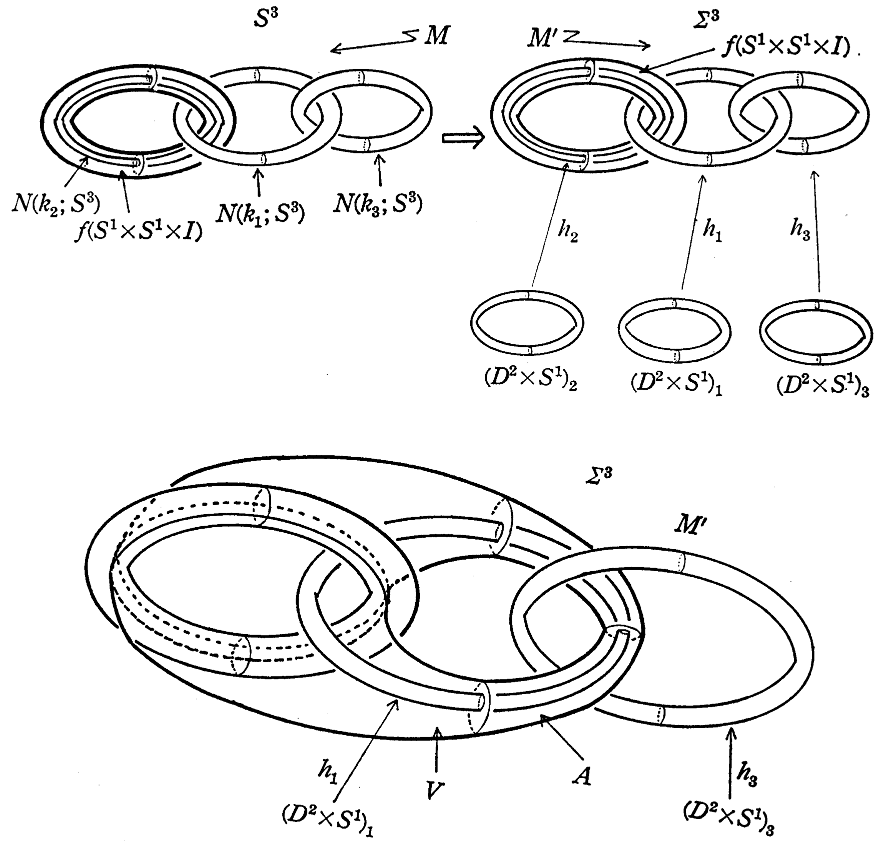

Fig. 7.

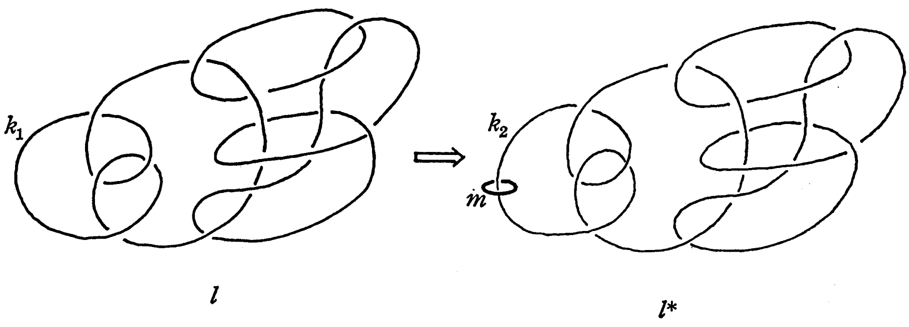

Fig. 8.

hood of k_1 in S^3 such that $N(k_1;\ S^3\cap(k_2\cup k_3\cup\cdots\cup k_\mu)=\emptyset$. Suppose that m is a meridian curve of a solid torus $N(k_1;\ S^3)$. We may then construct a new link $m\cup k_1\cup k_2\cup\cdots\cup k_\mu$, which is said to be a *$*$-link of l (in respect to k_1)* and denoted by l^*, see Fig. 8.

We will show the following lemma, which will play an important role in the proof of Theorem 1.

Lemma 3. *Let l^* be a *-link of a link l. If l has Property P^*, then l^* has Property P^*.*

Proof. Let Σ^3 be a homotopy 3-sphere obtained by doing surgery on a link l^* in S^3. Let N be a regular neighborhood of k_1 in S^3 such that $m\subset N$ and $N\cap\{N(k_2;\ S^3)\cup N(k_3;\ S^3)\cup\cdots\cup N(k_\mu;\ S^3)\}=\emptyset$, and F be the boundary of N, see Fig. 9. Since the intersection of F and $N(l^*;\ S^3)$ is empty, F may be embedded in Σ^3. Let M', N' be the closure of components of Σ^3-F. N' may be a 3-manifold obtained by doing surgery on $m\cup k_1$ in N, M' be the others. By [4][9][10], one of M' and N' is a homotopy solid torus. We will prove Lemma 3 in respective cases.

Case 1 Suppose that N' is a homotopy solid torus.

In respect of a homotopy solid torus N', we apply the following operation (Δ).

Operation (Δ) Since N' is a homotopy solid torus, there is a 2-cell $\tilde{D}^2$ in N' such that $\tilde{D}^2\cap\partial N'=\tilde{D}^2\cap F=\partial\tilde{D}^2$ is a simple closed curve which is not homolo-

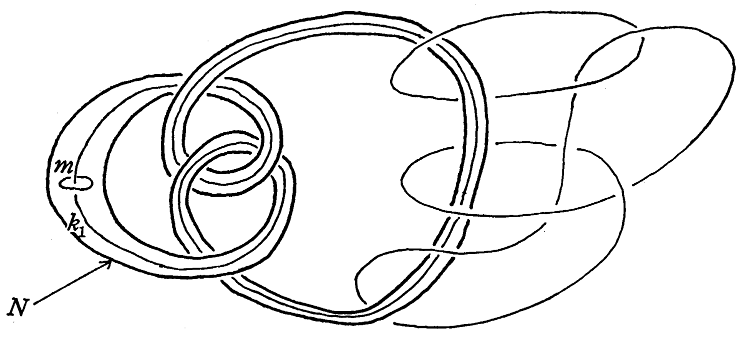

Fig. 9.

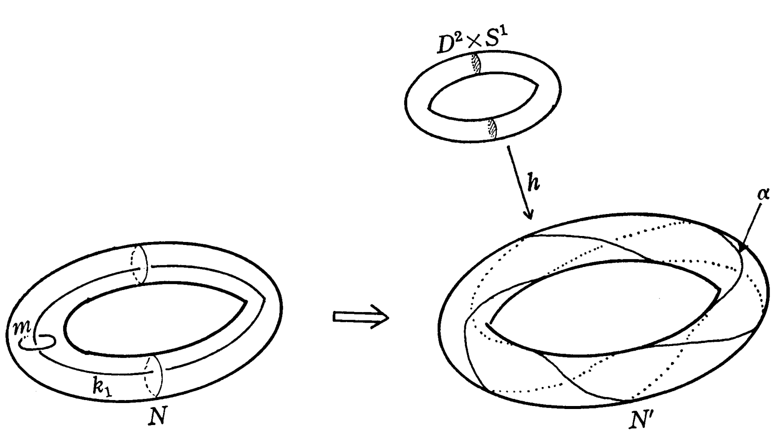

Fig. 10.

gous to 0 on F. Let a be a simple closed curve on F such that $a\cap\partial\tilde{D}^2$ is one point. Let h: $\partial D^2\times S^1\to F$ be a homeomorphism of the boundary of a solid torus $D^2\times S^1$ onto F, such that $h(\partial D^2\times\{p\})=a$, where p is a point in S^1. We may then construct the 3-manifold $\tilde{\Sigma}^3=N'\underset{h}{\cup}D^2\times S^1$, see Fig. 10.

Note $\tilde{\Sigma}^3$ is a homotopy 3-sphere obtained by doing surgery on the link O_3 in S^3. Since the link O_3 has Property P^*, $\tilde{\Sigma}^3$ is homeomorphic to 3-sphere. Hence N' is a solid torus. Σ^3 is regarded as a homotopy 3-sphere obtained by doing surgery on a link $k_2\cup k_3\cup\cdots\cup k_\mu$ and a solid torus N in S^3. Hence Σ^3 is the result of a surgery on a link l in S^3. Since a link l has Property P^*, Σ^3 is homeomorphic to 3-sphere.

Case 2 Suppose that M' is a homotopy solid torus.

In respect of a homotopy solid torus M', we apply the operation (Δ) and we may then construct a homotopy 3-sphere $\tilde{\tilde{\Sigma}}=M'\underset{h'}{\cup}D^2\times S^1$. Note $\tilde{\tilde{\Sigma}}$ is the result of a surgery on a link $k_2\cup k_3\cup\cdots\cup k_\mu$ and a solid torus N in S^3, hence a surgery on a link l in S^3, see Fig. 11. Since a link l has Property P^*, $\tilde{\tilde{\Sigma}}$ is homeomorphic to 3-sphere. Hence M' is a solid torus.

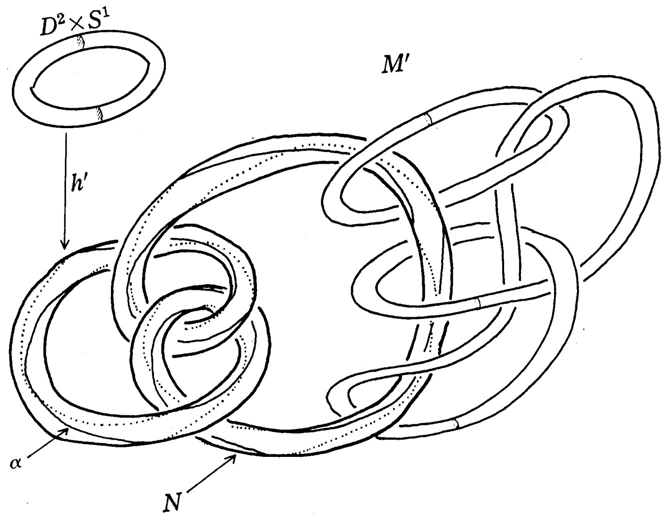

Fig. 11.

A homotopy 3-sphere Σ^3 is a union of N' and M', where N' is the result of a surgery on a link $m\cup k_1$ in N. Σ^3 is the result of a surgery on the link O_3 in S^3. Hence Σ^3 is homeomorphic to 3-sphere.

Let Q be a 3-cell in S^3 and $l=k_1\cup k_2\cup\cdots\cup k_\mu$ be a link which has an arc v of k_i in common with ∂Q, the remaining $l-v$ lying wholly within Q except for v. Similary, let Q' be a 3-cell in S^3 such that $Q\cap Q'=\emptyset$, and $l'=k_1'\cup k_2'\cup\cdots\cup$

k'_λ be a link which has an arc v' of k'_j in common with $\partial Q'$, the remaining $l'-v'$ lying wholly within Q' except for v'.

Let B be a 2-cell in $\mathrm{cl}(S^3-Q\cup Q')$ such that $B\cap\partial Q=\partial B\cap\partial Q=v$ and $B\cap\partial Q'=\partial B\cap\partial Q'=v'$. We may then construct a new link $\tilde{l}=(l-v)\cup(\partial B-v\cup v')\cup(l'-v')$ and $\tilde{l}$ is said to be a product of l and l' associated with (k_i, k'_j), see [6]. Since we take no notice of the locality of product in this paper, we say merely that $\tilde{l}$ is a product of l and l' and denote $\tilde{l}$ by $l\cdot l'$. Let us denote a component $(k_i-v)\cup(\partial B-v\cup v')\cup(k'_j-v')$ of a link $\tilde{l}$ by $k_i\# k'_j$.

Theorem 1. *Suppose that l and l' are links with Property P^*. Then, a product $l\cdot l'$ of l and l' is a link with Property P^*.*

Proof.. By renumbering the k_i's and k'_j's, we may assume that $l\cdot l'$ is a product associated with (k_1, k'_1).

Let Σ^3 be a homotopy 3-sphere obtained by doing surgery on a link $l\cdot l'$ in S^3. Let C be a component of $N(l\cdot l'; S^3)$ containing $k_1\# k'_1$ and C' be a regular neighborhood of C such that $C'\cap N(l\cdot l'; S^3)=C$. $M=Q\cup C'$ is a solid torus and $F=\partial M$ is a closed surface of genus 1. F may be embedded in Σ^3. Let M', N' be the closure of components of Σ^3-F. M' may be a 3-manifold obtained by doing surgery on a link $(k_1\# k'_1)\cup k_2\cup\cdots\cup k_\mu$ in M and N' be the others, see Fig. 12. By [4][9][10], one of M' and N' is a homotopy solid torus. We will prove Theorem 1 in respective cases.

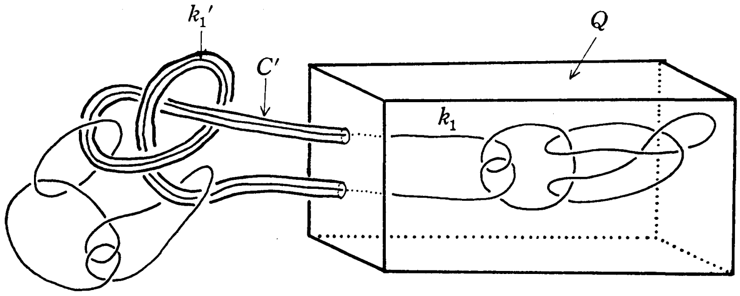

Fig. 12.

Case 1. Suppose that M' is a homotopy solid torus.

Apply an operation (Δ) in respect of a homotopy solid torus M', and let $\tilde{\Sigma}=M'\cup_h D^2\times S^1$ be the result. Note $\tilde{\Sigma}$ is a homotopy 3-sphere obtained by doing surgery on a link $(k_1\# k'_1)\cup k_2\cup\cdots\cup k_\mu$ and a solid torus in S^3, see Fig. 13. Hence there is a *-link l^* of l such that $\tilde{\Sigma}$ is the result of a surgery on a link l^* in S^3. By Lemma 3, a link l^* has Property P^*. Hence $\tilde{\Sigma}$ is homeomorphic to 3-sphere, and M' is a solid torus.

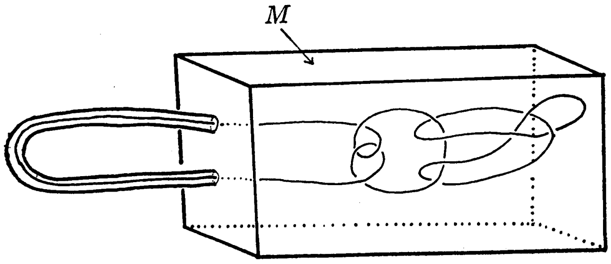

Fig. 13.

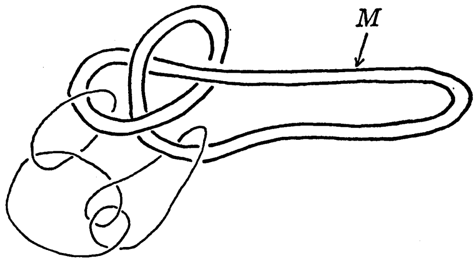

Fig. 14.

A homotopy 3-sphere Σ^3 may be a union of a solid torus M' and a 3-manifold N' obtained by doing surgery on a link $k'_2 \cup k'_3 \cup \cdots \cup k'_\lambda$ in $S^3 - M$. Hence Σ^3 is the result of a surgery on a link l' in S^3. Since a link l' has Property P^*, Σ^3 is homeomorphic to 3-sphere.

Case 2. Suppose that N' is a homotopy solid torus.

Apply an operation (Δ) in respect of a homotopy solid torus N', we construct a homotopy 3-sphere $\tilde{\tilde{\Sigma}} = N' \cup_{h'} D^2 \times S^1$. Note $\tilde{\tilde{\Sigma}}$ is the result of a surgery on a link $k'_2 \cup k'_3 \cup \cdots \cup k'_\lambda$ and a solid torus M in S^3, hence a surgery on a link l' in S^3, see Fig. 14. Since a link l' has Property P^*, $\tilde{\tilde{\Sigma}}$ is homeomorphic to 3-sphere. Hence N' is a solid torus.

A homotopy 3-sphere Σ^3 may be a union of a solid torus N' and a 3-manifold M' obtained by doing surgery on a link $(k_1 \# k'_1) \cup k_2 \cup \cdots \cup k_\mu$ in M. Hence, there is a *-link l^* of a link l such that Σ^3 is the result of a surgery on a link l^* in S^3. By Lemma 3, a link l^* has Property P^*. Hence Σ^3 is homeomorphic to 3-sphere.

Since every link has a factorization into links called prime link [6], we obtain the following corollary of Theorem 1;

Corollary 2. *Every link has Property P^* if every prime link has Property P^*.*

3. Some links with Property P^*

By Theorem 1, we will obtain that;

Theorem 3. *Every torus link has Property P^**

Proof. Let l be a torus link of type (p, q), $p, q \geq 0$. If $pq = 0$, by Lemma 1 and Proposition 2, Theorem 3 is obvious. Suppose $pq > 0$. Let α be the greatest common divisor of p, q. Since l is a torus link, there is a unknotted solid torus R in S^3, such that l is contained on a boundary ∂R of R. Let a be a core of the solid torus R and b be a core of the solid torus $\mathrm{cl}\,(S^3 - R)$, see Fig. 15. We will show that a link $\tilde{l}=l\cup a\cup b$ has Property P^*. Clearly there is a torus link l_0 of type $(0, \alpha)$ or $(\alpha, 0)$ on ∂R such that the complement of $\tilde{l}$ is homeomorphic to the complement of a link $\tilde{l}_0=l_0\cup a\cup b$, see Fig. 15 and 16.

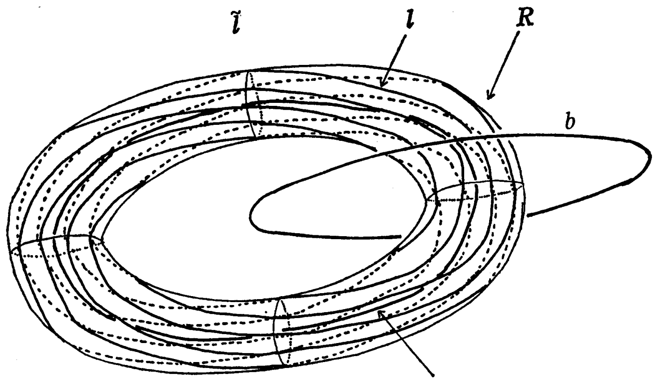

Fig. 15.

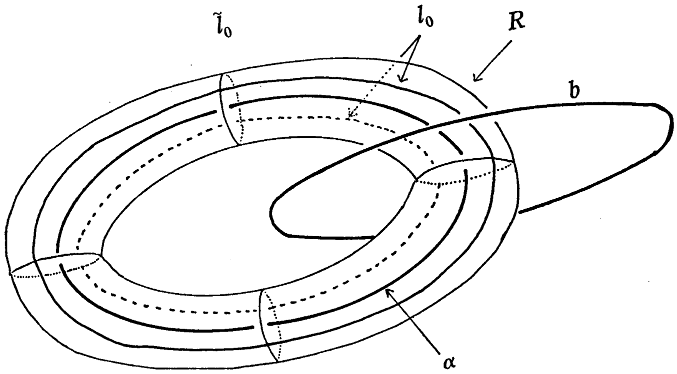

Fig. 16.

Let Σ^3 be the result of a surgery on a link $\tilde{l}$. Since $\mathrm{cl}\,(S^3-\tilde{l})$ is homeomorphic to $\mathrm{cl}\,(S^3-\tilde{l}_0)$, Σ^3 is the result of a surgery on a link $\tilde{l}_0$.

By induction on α, we will prove that a link $\tilde{l}_0$ has Property P^*. If $\alpha=1$, then $\tilde{l}_0$ is ambient isotopic to the link O_3. Hence $\tilde{l}_0$ has Property P^*. Suppose $\alpha>1$. Let $\tilde{l}_0'$ be a link $l_0'\cup a\cup b$, where l_0' is a torus link of type $(0, \alpha-1)$ or $(\alpha-1, 0)$ on ∂R. By induction, $\tilde{l}_0'$ has Property P^*. A link $\tilde{l}_0$ is a product of the link O_2 and a link $\tilde{l}_0'$. By Theorem 1, a link $\tilde{l}_0$ has Property P^*. Hence Σ^3 is homeomorphic to 3-sphere, and $\tilde{l}$ has Property P^*. By Proposition 1, a torus link l of type (p, q) has Property P^*.

Let $B_1^3, B_2^3, \cdots, B_n^3$ be mutually disjoint 3-cells in S^3 and $H_1=(D^2\times I)_1$, $H_2=(D^2\times I)_2, \cdots, H_m=(D^2\times I)_m$ be mutually disjoint 1-handles of $B_1^3\cup B_2^3\cup\cdots\cup B_n^3$ in

S^3 satisfying the following condition;

(*) For any i, $1\leq i\leq m$, there are exactly two numbers $p(i)$, $q(i)$ such that a 1-handle H_i joins $B^3_{p(i)}$ and $B^3_{q(i)}$.

Let M be a 3-manifold obtained by attaching 1-handles $H_1, H_2, \cdots, H_m$ to $B^3_1\cup B^3_2\cup\cdots\cup B^3_n$ in S^3; and for each 3-cell B^3_λ, $B^3_\lambda\cap\cup_{i=1}^m H_i$ are 2-cells on ∂B^3_λ, say $C_{\lambda,1}$, $C_{\lambda,2}$, $\cdots$, $C_{\lambda r(\lambda)}$.

Let l_λ be a link in B^3_λ which has arcs $v_{\lambda 1}$, $v_{\lambda 2}$, $\cdots$, $v_{\lambda,r(\lambda)}$ of l_λ in $C_{\lambda 1}$, $C_{\lambda 2}$, $\cdots$, $C_{\lambda,r(\lambda)}$, respectively, the remaining $l_\lambda-(v_{\lambda,1}\cup v_{\lambda,2}\cup\cdots\cup v_{\lambda,r(\lambda)})$ lying wholly within B^3_λ except for $v_{\lambda,1}\cup v_{\lambda,2}\cup\cdots\cup v_{\lambda,r(\lambda)}$. Let $C_{p,s}$, $C_{q,t}$ be 2-cells $H_i\cap(B^3_1\cup B^3_2\cup\cdots\cup B^3_n)$; and β_i and β'_i be disjoint arcs in H_i satisfying the following conditions;

(1) an arc β_i joins two points $\partial v_{p,s}$.

(2) an arc β'_i joins two points $\partial v_{q,t}$.

(3) $v_{p,s}\cup\beta_i$ and $v_{q,t}\cup\beta'_i$ make together a torus link of type $(2, p_i)$ where p_i is even positive number for $i=1, 2, \cdots, m$.

We may construct a new link $l=\cup_{\lambda=1}^n\{l_\lambda-(v_{\lambda,1}\cup v_{\lambda,2}\cup\cdots\cup v_{\lambda,r(\lambda)})\}\cup\cup_{i=1}^m(\beta_i\cup\beta'_i)$ and l is said to be *a union of* l_λ *winded along* $v_{p,s}$ *and* $v_{q,t}$ *with the winding number* p_i. We will consider the following graph G;

(a) Take a point corresponding to *a* 3-cell B^3_λ, say b_λ, $\lambda=1, 2, \cdots n$. Let $\{b_\lambda\}$ be the set of vertices of G.

(b) Take a line corresponding to a 1-handle H_i, say a_i, $i=1, 2, \cdots, m$, such

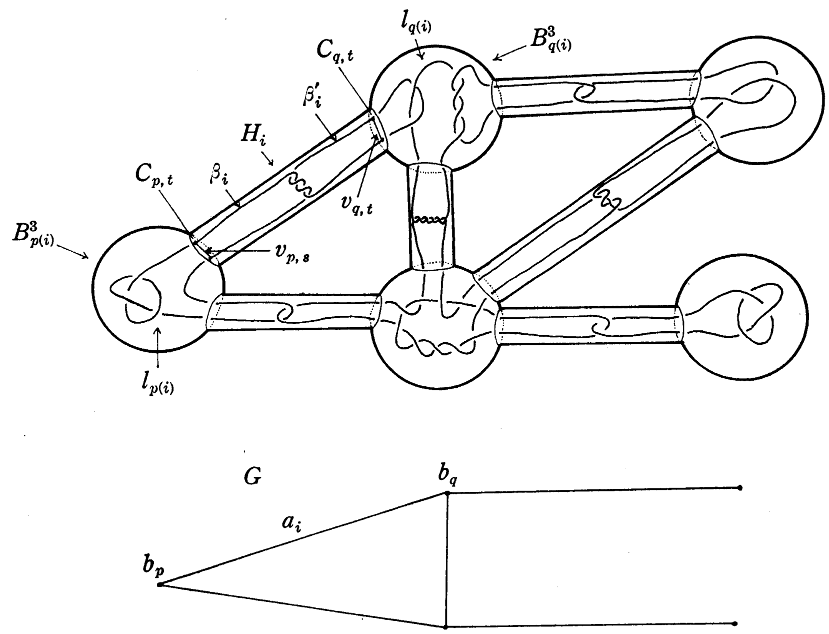

Fig. 17.

that a_i joins $b_{p(i)}$ and $b_{q(i)}$, where $p(i)$ and $q(i)$ are two numbers for i by a condition (*) above. Let $\{a_i\}$ be the set of lines of G, see Fig. 17.

A graph G is said to be *a corresponding graph of a union of a link l.*

Corollary 4. *If every link l_λ has Property P^* for $\lambda=1, 2, \cdots, n$, and the corresponding graph of a union of a link l_λ is tree, then a union l of l_λ is a link with Property P^*.*

Proof.. By induction on n, the number of the vertices of G. If $n=1$, this consequence is obvious. So we assume that $n\geq 2$. Suppose Corollary 4 is true for $n\leq k$. Then we will prove Corollary 4 for $n=k+1$.

Since the corresponding graph G is tree, there are a 3-cell B^3_λ and a 1-handle H_i such that $B^3_\lambda \cap \cup_{i=1}^m H_i = B^3_\lambda \cap H_i$. By renumbering the B^3_λ's, the H_i's and $C_{\lambda,\mu}$'s, we may assume $i=1$, $\lambda=p=1$, $s=1$, $q=2$ and $t=1$. Let D^3 be a 3-cell such that $D^3 \cap M = D^3 \cap (B^3_\lambda \cup H_i) = B^3_\lambda \cup H_i$. Then l is a product link $[l_1 \cdot (\{v_{1,1} \cup \beta_1\} \cup \{v_{2,1} \cup \beta_1\})] \cdot \tilde{l}$, where $\tilde{l}$ is a sublink $l \cap (S^3 - D^3)$ of l, see Fig. 18. By induction, $\tilde{l}$ is a link with Property P^*.

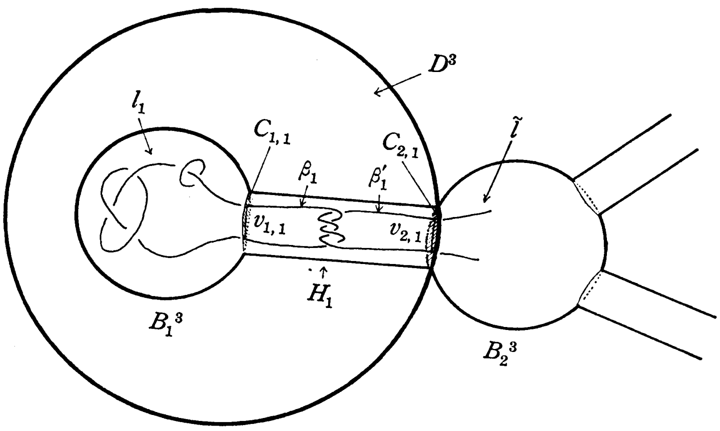

Fig. 18.

$v_{1,1} \cup \beta_1$ and $v_{2,1} \cup \beta_1'$ make together a torus link in H_1, hence, by Theorem 3, this is a link with Property P^*. By assumption, a link l_1 has Property P^*. Hence by Theorem 1, a link l has Property P^*.

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