GENERALISED OPERATIONAL FUNCTIONS II

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In this paper, we give a new approach to the definition of a generalised operational function different from that given in [1]. This approach is quite similar to the approach that has been employed in Theorem of [2]. We then show that the space of generalised operational functions constructed in this way, is homeomorphic to the space of generalised operational functions constructed in [1] and to the space of operator distributions of J. Wloka [7].

We first recall a few definitions that are necessary in the sequel.

Definition 1 [1]. An operational function is a function f, which assigns an operator $f(\theta)$ to each non-negative real number θ.

Definition 2 [1]. An operational function f is said to be a parametric operational function if each value $f(\theta)$ is itself an operator of a special kind namely a function of the real variable, say t.

Definition 3 [1]. An operational function f is called continuous in $0 \leqq \theta < \infty$, if it can be represented in $[0, \infty)$ as a ratio $f_1(\theta)/a$†, of a parametric operational function $f_1(\theta) = \{f_1(\theta, t)\}$ and an operator 'a' equal to a continuous function $\{a(t)\}$, $0 \leqq t < \infty$, where $a(t)$ is not identically equal to zero, such that the function $f_1(\theta, t)$ is continuous in the domain $D(0 \leqq t < \infty, 0 \leqq \theta < \infty)$.

Definition 4 [1]. Two continuous operational functions f and g are said to be related—in symbols $f \sim g$—where $f(\theta) = \{f_1(\theta, t)\}/\{a(t)\}$ and $g(\theta) = \{g_1(\theta, t)\}/\{b(t)\}$†† if and only if $f(\theta, t) * b(t) = g(\theta, t) * d(t)$.

This relation $\sim$ can be checked to be an equivalence relation which divides the class of all continuous operational functions into mutually disjoint classes.

Hereafterwards, a continuous operational function means an equivalence class of elements of the form $f(\theta, t)/a$ representing the function.

Definition 5 [1]. A sequence f_n of continuous operational functions is said to converge to continuous operational function f in symbol $f_n \to f$, if there exist a sequence of parametric operational functions $f_n(\theta, t)$, a parametric operational

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† / stands for convolution quotient.

†† For the sake of typographical conveniences we omit the braces hereafterwards.

function $f(\theta, t)$ and continuous function $a(t)$ such that $f_n(\theta)=f_n(\theta, t)/a(t)$; $f(\theta)=f(\theta, t)/a(t)$ and $f_n(\theta, t)$ converges almost uniformly to $f(\theta, t)$; i.e. conveges uniformly over every bounded rectangle in the domain D.

Definition 6 [1]. A continuous operational function f has the continuous function g as a derivative if $(\tau_h f-f)/h$ tends to g as $h\to 0$ where $\tau_h f(\theta)=f(\theta+h)$.

It is easy to see that there exists continuous operational functions which are not differentiable. To meet this situation, we build a new class of entities called generalised operational functions, which contains the original class of continuous operational functions as a subclass.

Definition 7 [1]. A continuous operational function p is called a polynomial operational function of degree less than k, where $p(\theta)=p(\theta, t)/a(t)$, if $p(\theta, t)=\sum_{j=0}^{k-1} a_j(t)\theta^j$ where the coefficients $a_j(t)$ are continuous functions of the variable t, is a polynomial of degree less than k.

Consider all ordered pairs (f, k) where f is a continuous operational function and k a non-negative integer. We introduce the notation $\int_k f(\theta)d\theta$ for the k-th repeated integral of $f(\theta)$ defined as $\int_k f(\theta)\,d\theta=\underbrace{\int_0^\theta \cdots\cdots}_{k \text{ times}}\int_0^\theta f(\theta)d\theta$.

Definition 8. $(f, k)\sim(g, l)$ if and only if $\int_l f(\theta)\,d\theta-\int_k g(\theta)\,d\theta$ is a polynomial operational function of degree less than or equal to $k+l$.

This relation $\sim$ can be easily proved to be an equivalence relation. This equivalence relation divides the class of all ordered pairs (f, k) into mutually disjoint classes.

Definition 9. A generalised operational function is an equivalence class of ordered pairs (f, k).

For sake of convenience we denote a generalised operational function by (f, k) itself.

Remark 1. Every continuous operational function f can be viewed as a generalised operational function (f, o).

Definition 10. $(f, k)+(g, k)=(f+g, k)$, $(f, k)+(g, m)=\left(\int_m f+\int_k g,\ k+m\right)$, $\alpha(f,k)=(\alpha f, k)$ where α is an operator of Mikusinski. $(f, k)\cdot(g, m)=(f*g, k+m)$ where $(f*g)(\theta)=f(\theta, t)*g(\theta, t)/(a(t)*b(t))$.

Definition 11. By the translation $\tau_h(f, k)$ of a generalised operational function (f, k), we mean the generalised operational function $(\tau_h f, k)$ where $\tau_h f(\theta)=f(\theta+h)$.

Definition 12. A sequence (f_n, k_n) of generalised operational functions converges to a generalised operational function (f, k) and we write $(f_n, k_n)\to(f, k)$, if and only if $(f_n, k_n)\sim(F_n, m)$, $(f, k)\sim(F, m)$ and F_n converges to F (in the sense of Definition 5.)

(1) This convergences in the class of all generalised operational functions is Hausdorff. In other words, if $(f_n, k_n)\to(f, k)$ and $(f_n, k_n)\to(g, m)$, then $(f, k)\sim(g, m)$. For, given $(f_n, k_n)\sim(F_n, r)$, $(f, k)\sim(F, r)$ and $F_n\to F$; also, $(f_n, k_n)\sim(G_n, s)$, $(g, m)\sim(G, s)$ and $G_n\to G$. Therefore, $(F_n, r)\sim(G_n, s)$ i.e. $\int_s F_n-\int_r G_n$ is a polynomial operational function of degree less than or equal to $s+r$. Since a sequence of polynomials $\sum_{j=0}^{k-1} a_{nj}(t)\theta^j$ of degree less than a positive integer k, converges almost uniformly to a polynomial $\sum_{j=0}^{k-1} a_j(t)\theta^j$ of degree less than k if and only if the sequence $\{a_{nj}(t)\}$ of continuous functions, converges almost uniformly to the continuous function $a_j(t)$, it follows that $\int_s F-\int_r G$ is a polynomial operational function of degree less than or equal to $s+r$. So we have $(F, r)\sim(G, s)$ and thus $(f, k)\sim(g, m)$.

(2) Also, this notion of convergence is compatible with the other basic operations available in the class of generalised operational functions. In other words, if $(f_n, k_n)\to(f, k)$, $(g_n, m_n)\to(g, m)$ then

(i) $(f_n, k_n)\pm(g_n, m_n)\to(f, k)\pm(g, m)$

(ii) $\alpha_n(f_n, k_n)\to\alpha(f, k)$ where the sequence (α_n) of operators of Mikusinkski converges to α in the operational sense [4].

(iii) $(f_n, k_n)\cdot(g_n, m_n)\to(f, k)\cdot(g, m)$.

(3) A sequence of continuous operational functions which is convergent in the sense of Definition 5, also converges in the generalised sense [Definition 12].

Theorem 1. *If (f, k) is generalised operational function, then*

$$(f, k)=(f, k+1)=\operatorname*{Lt}_{h\to 0}\frac{\tau_h(f, k)-(f, k)}{h}$$

exists.

Proof.

$$\operatorname*{Lt}_{h\to 0}\frac{\tau_h(f, k)-(f, k)}{h}=\operatorname*{Lt}_{h\to 0}\frac{(\tau_h F, k+1)-(F, k+1)}{h}$$

where

$$F=\int f=\operatorname{Lt}\left(\frac{\tau_h F-F}{h}, k+1\right)$$

and this limit exists and is equal to $(f, k+1)$.

We immediately have the following properties:

(i) $((f, k)+(g, m))'=(f, k)'+(g, m)'$

(ii) $\alpha(f, k)'=(\alpha f, k)'$ where α is an operator.

(iii) $((f, k)\cdot(g, m))'=((f\cdot g)', k+m)=(f\cdot g, k+m+1)$

Remark 2. If a continuous operational function has a continuous m-th derivative, then it coincides with its m-th derivative in the generalised sense.

Definition 13 [2]. A *CD* space X is a topologioal translation vector space in which every element has a derivative and whenever a sequence converges, its derived sequence also converges.

Theorem 2. *Let C be the space of all continuous operational functions. There exists an unique (upto isomorphism) space $\bar{C}$ which is a CD space such that there exists a* 1–1 *continuous linear differential mapping t_1 of C onto a dense subspace C_1 of $\bar{C}$ and further that if Σ is any other CD space with a dense subspace Σ_1 onto which C can be mapped in a* 1–1 *continuous linear differential way by a map t_2, then there exists a* 1–1 *continuous linear differential map t_3 of $\bar{C}$ onto Σ such that $t_3\cdot t_1=t_2$.*

The proof follows immediately for we may note that the scheme we have worked out in such an elaborate detail can be side-tackled by appeal to the process of embedding a primitive space into a *CD* space as in [2].

Theorem 3. *The class of generalised operational functions defined in* [1] *is linearly homeomorphic to the class of generalised operational functions defined here.*

Proof. Let f be a generalised operational function which is an equivalent class of fundamental sequences of continuous operational functions. (i. e) $f=[f_n]$ where (f_n) is a fundamental sequence of continuous operational functions. Hence, there exists a non-negative integer k, and a sequence (F_n) and F of continuous operational functions such that, $f_n=F_n^{(k)}$ and $F_n\rightarrow F$ (in the sense of Definition 5). Correspond to f, the generalised operational function (F, k). To this F, there exists a sequence of polynomial operational functions p_n such that $p_n\rightarrow F$. Correspond to (F, k), the generalised operational function defined by the fundamental sequence P_n where $P_n=p_n^{(k)}$. It is easy to verify that the fundamental sequence P_n is equivalent to the fundamental sequence f_n. This correspondence is 1–1 and onto. Also it is linear and continuous.

J. Wloka [7] has defined operator distributions starting from the class of continuous functions of two variables in the positive plane. *J. Mikusinski* raised the question whether the class of operator distributions defined by *J. Wloka* is the same as the one developed here. Here we answer this question in the affirmative, by proving

Theorem 4. *The space of generalised operational functions is linearly homeomorphic to the space of operator distributions of Wloka.*

Proof. Since an operator distribution is an equivalent class of elements of form $f(\theta, t)/a(t)$ where $f(\theta, t)$ is the k-th distributional derivative in the Schwartz sense [6] of a continuous function $F(\theta, t)$ in $0 \leqslant \theta < \infty$, $0 \leqslant t < \infty$ and $a(t)$ is a continuous function in $0 \leqslant t < \infty$, the proof follows immediately from the theorem [3] that there exists a 1–1 bicontinuous linear differential map from the space of Σ of Mikusinski-Sikorski distributions [5] onto the space D' of Schwartz distributions.

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