FIXED POINTS FOR $U+C$ WHERE U IS LIPSCHITZ AND C IS COMPACT

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The aim of this note is to give an extension of some results of *Nashed* and *Wong* [3] and *Ishikawa* and *Fujita* [1], by permitting nonlinearity. These authors considered mappings $U+C$ where U was linear and C a compact map. The former let some power U^p be a contraction, while the latter let U^p be a k-set-contraction, $k<1$. A result in which $I-U$ has been replaced by a Fredholm operator has been given by *Mawhin* [2].

I am grateful to Professor Nashed, who pointed out to me that we want to let U be nonlinear.

Two simple fixed point theorems are given here, one following [1] and the other [3], together with an application as in [3].

Lemma 1: *Let X be a Banach space. Let $A: X\to X$ be Lipschitz. For $f\in X$, let $A_f: X\to X$ be defined by $A_f x=Ax+f$. Suppose there exists a positive integer N and a real number $\alpha<1$ such that for all f in X, $(A_f)^N$ has Lipschitz norm $\leq\alpha$. Then $I-A$ is bijective, and $(I-A)^{-1}$ has Lipschitz norm $\leq (1-\alpha)^{-1}(1+\cdots+||A||^{N-1})$, where $||A||$ is the Lipschitz norm of A.*

Proof: Given $f\in X$, to solve $(I-A)x=f$ we want a unique fixed point of A_f. A unique fixed point of $(A_f)^N$ exists by the contraction mapping principle. By uniqueness, this is a fixed point of A_f.

Let $x-Ax=f$ and $y-Ay=g$. Put $K=(1-\alpha)^{-1}(1+\cdots+||A||^{N-1})$. We want to show $||x-y||\leq K||f-g||$. Define $B: X\to X$ by $Bz=A(z+x)+f-x$. Then $B0=0$ and the two equations above may be written $0-B0=0$ and $(y-x)-B(y-x)=g-f$. Now B has Lipschitz norm $||A||$, and $(B_h)^N$ has Lipschitz norm $\leq\alpha$ for all $h\in X$. That is, we could have assumed $A0=0$ and $f=0$. Since $B_{g-f}(0)=g-f$ and B_{g-f} has Lipschitz norm $||A||$,

$$\begin{aligned}||(B_{g-f})^N(0)|| &\leq ||0-B_{g-f}(0)||+\sum_{i=1}^{N-1}||(B_{g-f})^i(0)-(B_{g-f})^{i+1}(0)||\\ &\leq ||g-f||(1+||A||+\cdots+||A||^{N-1})\,.\end{aligned}$$

Because $(B_{g-f})^N$ has Lipschitz constant $\leq\alpha$, for z in X, $||(B_{g-f})^N z||\leq\alpha||z||+||g-f||(1+\cdots+||A||^{N-1})$.

The right hand side is $<\|z\|$ if $\|z\|>K\|g-f\|$. Hence, the fixed point of $(B_{g-f})^N$ has norm$\leq K\|g-f\|$. That is, $\|y-x\|\leq K\|g-f\|$. q.e.d.

Theorem 1: *Let A satisfy the hypotheses of Lemma 1. Let B be a bounded nonempty closed convex subset of X. Let $C\colon B\to X$ be compact. That is, C is continuous and takes bounded sets to relatively compact sets. If $(I-A)^{-1}C(B)\subseteq B$ then $A+C$ has a fixed point in B.*

Proof: By Lemma 1, $(I-A)^{-1}C$ is continuous. By the Schauder fixed point theorem it has a fixed point.

Corollary 1; *Let A and B be as in Theorem 1. Let $C\colon B\to X$ be compact. If $Ax+Cy\in B$ for all x in B and y in B then $A+C$ has a fixed point in B.*

Proof: For $y\in B$, A_{Cy} takes B to B. Hence, $(A_{Cy})^N$, and also A_{Cy}, have a unique fixed point in B. Thus, $(I-A)^{-1}C(B)\subseteq B$.

Corollary 2: *Let A be as in Theorem 1. Let $C\colon X\to X$ be compact. If $\limsup_{\|x\|\to\infty}\|x\|^{-1}\|Cx\|<(1-\alpha)(1+\|A\|+\cdots+\|A\|^{N-1})^{-1}$ then $R(I-A-C)=X$.*

Lemma 2: *Let Y be a Banach space, and let $[a,b]$ be bounded interval in $\mathbf{R}$. Let $F\colon[a,b]\times[a,b]\times Y\to Y$ be a function such that for $y\in Y$, the function $(t,s)\to F(t,s,y)$ is strongly measurable. Suppose $F(t,s,0)$ is in $L^2([a,b]\times[a,b];Y)$.*

Let $V\colon[a,b]\times[a,b]\to\mathbf{R}$ be measurable and let $\sup_{a\leq t\leq b}\int_a^t|V(t,s)|^2\,ds=M^2<\infty$. Suppose that for t and s a.e. in $[a,b]$ and x and y in Y,

$$\|F(t,s,x)-F(t,s,y)\|\leq V(t,s)\|x-y\|\,.$$

Then we may define $A\colon L^2([a,b];Y)\to L^2([a,b)];Y)$ by

$$Ax(t)=\int_a^t F(t,s,x(s))ds\,.$$

Given n elements $g(i)(1\leq i\leq n)$ in $L^2([a,b];Y)$, the map $\prod_{i=1}^n A_{g(i)}$ has Lipschitz norm $M^n((b-a)^n/n!)^{1/2}$. In particular, given $\beta\in(0,1)$, there exists N such that for any N-tuple $g(i)$ $(1\leq i\leq N)$, $\prod_{i=1}^n A_{g(i)}$ has Lipschitz norm $\leq\beta$.

Proof: Given x in $L^2([a,b];Y)$, $\|F(t,s,x(s)\|\leq V(t,s)\|x(s)\|+\|F(t,s,0)\|$. Hence, Ax is in $L^2([a,b];Y)$. The proof about the Lipschitz norm of $\prod_{i=1}^n A_{g(i)}$ is by induction. q.e.d.

Theorem 2: *Let A be as in Lemma 2. Suppose $K\in L^2([a,b]\times[a,b];\mathbf{R})$. Suppose $g\colon[a,b]\times Y\to Y$ has the property that $g(s,u)$ is strongly measurable in s for u in Y and for s a.e. it is continuous in u. Suppose for s a.e. in $[a,b]$*

and u in Y,

$$||g(s, u)|| \leq \Sigma g_i(s) \, ||u||^{1-\beta(i)} + g_0(s)$$

where $g_0 \in L^2([a, b]; \mathbf{R})$ and $g_i \in L^{2/\beta(i)}$, where $0<\beta(i)<1$, for $1 \leq i \leq n$. Define $C: L^2([a, b]; Y) \to L^2([a, b]; Y)$ by

$$Cx(t) = \int_a^b K(s, t) g(s, x(s)) ds \ .$$

Then $I-A-C$ is surjective.

Proof: $C=HG$ where $Hx(t)=\int_a^b K(s, t)x(s)ds$ and $Gx(s)=g(s, x(s))$. H is compact, giving C compact. By Lemma 2, A satisfies the conditions of Lemma 1. Since $||Gx||/||x|| \to 0$ as $||x|| \to \infty$,

$$\limsup_{||x|| \to \infty} ||Cx||/||x|| < (1-\alpha)(1+\cdots+||A||^{N-1})^{-1} \ .$$

The result follows by Corollary 2. q.e.d.

We recall [4] that if (Y, d) is a bounded metric space, then the measure of noncompactness $\gamma(Y)$ equals $\inf \{d>0$: there exists a finite number of sets $S_1 \cdots S_n$ such that $Y=\bigcup_{i=1}^{n} S_i$ and diameter $(S_i) \leq d\}$. If Y_1 and Y_2 are metric spaces and $f: Y_1 \to Y_2$ is continuous, f is called a k-set-contraction if for every bounded subset S of Y, $f(S)$ is bounded and $\gamma_2 f(S) \leq k\gamma_1(S)$.

Lemma 3: *Let $A: X \to X$ be a Lipschitz mapping of a Banach space. Suppose $\alpha<1$ and N a positive integer, and for all $g(1) \cdots g(N)$ in X, $\prod_{i=1}^{N} A_{g(i)}$ has Lipschitz norm $\leq \alpha$. Let $C: X \to X$ be compact. Then $(A+C)^N$ is an α-set-contraction.*

Proof: Let S be bounded. Take R with $(A+C)^i(S) \subseteq B_R(0)$ for $1 \leq i \leq N$. Given $\varepsilon>0$, let N_ε be a finite ε net for $C(B_R(0))$. We claim that for x in S, and each positive integer $n \leq N$, there is an n-tuple $z(i)(1 \leq i \leq n)$ of elements of N_ε such that

$$||(A+C)^n x - \Pi A_{z(i)}(x)|| < \varepsilon(1+||A|| \cdots + ||A||^{n-1}) \ .$$

The proof is by induction. It follows that $(A+C)^N(S)$ is contained in an $\varepsilon(1+\cdots+||A||^{N-1})$ neighborhood of $\cup \{\prod_{i=1}^{N} A_{z(i)}(S)$: $z(i)$ $(1 \leq i \leq N)$ an N-tuple of elements of $N_\varepsilon\}$.

Since $\gamma \cup \prod_{i=1}^{N} A_{z(i)}(S) \leq \alpha\gamma(S)$, we have $\gamma(A+C)^N(S) \leq \alpha\gamma(S) + 2\varepsilon(1+\cdots+||A||^{N-1})$. The result follows because $\varepsilon>0$ was arbitrary. q.e.d.

Theorem 3: *Let B be a closed bounded convex subset of a Banach space X, having nonempty interior. Suppose $A: X \rightarrow X$ is C^1 and satisfies the conditions of Lemma 3. Let $C: X \rightarrow X$ be compact and C^1. Let the closure of $(A+C)B$ be contained in the interior of B.*

Then $A+C$ has a fixed point in B.

Proof: by [4, Corollary 10] we need only show $(A+C)^N$ is a k-set-contraction, $k<1$, for some N. This holds by Lemma 3. q.e.d.

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