HYPOCONTINUOUS MULTIPLICATION IN WEAKLY TOPOLOGIZED ALGEBRAS

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Abstract: *We prove some results on hypocontinuous multiplication in weakly topologized algebras and give their applications in function spaces.*

Let E be a complex (or real) algebra, E' be a total subspace of the algebraic dual E^* of E and $w(E, E')$ be the weak topology defined on E by E'. An algebra with a locally convex linear Hausdorff topology, for which multiplication is separately continuous, will be called a *locally convex algebra*. Multiplication in a locally convex algebra is said to be *hypocontinuous* if given a neighbourhood U of o and a bounded subset B there exists a neighbourhood V of o satisfying $(VB)\cup(BV)\subset U$. A locally convex algebra is said to have *jointly continuous multiplication* (or to be a *topological algebra* ([9], [10]) if given a neighbourhood U of o there exists a neighbourhood V of o satisfying $V^2=VV\subset U$. A *locally multiplicatively-convex* (*locally m-convex*, in short) algebra is a locally convex algebra with a base of neighbourhoods U of o satisfying $U^2\subset U$ ([2], [6]). By a locally convex *self-adjoint* algebra we mean a locally convex algebra with an involution '$*$' which satisfies $f(x^*)=\overline{f(x)}$ for all x in E and for all continuous nonzero multiplicative linear functionals f on E ($\bar{\alpha}$ denotes the complex conjugate of α). A real locally convex algebra is self-adjoint if we take $x^*=x$ for all x.

Following [5], we call a locally convex topological vector space *boundedly generated* (in short, BG) if it is the closed linear hull of a bounded subset of itself. Let N denote the set of natural numbers.

Warner ([10], Theorem 2) has proved that if E be a commutative, semi-simple Banach algebra over the complex numbers, then $(E, w(E, E'))$ is a topological algebra if and only if E is finite dimensional. It has been proved in ([3], Theorem (3.2)) that if E is a BG space then $(E, w(E, E'))$ has hypocontinuous multiplication if and only if it has jointly continuous multiplication. Combining these two results we have

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Theorem 1. *Let E be a commutative, semi-simple, self-adjoint Banach algebra over the complex numbers. Then $(E, w(E, E'))$ has hypocontinuous multiplication if and only if it is finite dimensional.*

In fact, we have more general results.

Theorem 2. *Let E be a self-adioint locally convex algebra such that its strong dual E'_b is sequentially complete. Let $M(E)$ be the set of continuous non-zero multiplicative linear functionals on E.*

If (i) *E is a BG space, and*

(ii) *there exists an infinite subset $\{f_n : n \in N\}$ of $M(E)$ and scalars $\alpha_n > 0$ such that $\{\alpha_n f_n : n \in N\}$ is a bounded subset of E'_b;*
then $(E, w(E, E'))$ does not have hypocontinuous multiplication.

Proof. Suppose that $(E, w(E, E'))$ has hypocontinuous multiplication. Because (i) is satisfied, Theorem (3.2) in [3] gives that $(E, w(E, E'))$ has jointly continuous multiplication. Since E is self-adjoint and (ii) is satisfied, the proof of Theorem 2 in [10] can be modified to give that E is finite dimensional. But E' contains an infinite linearly independent set $\{f_n : n \in N\}$. This contradiction completes the proof.

Example 1. Let T be a completely regular Hausdorff space and E be the locally m-convex algebra $C(T)$ of real continuous functions on T with pointwise algebraic operations equipped with the topology of uniform convergence on compact subsets of T. Then $M(E)$ can be identified with $\hat{T} = \{\hat{t} : t \in T\}$, where $\hat{t}(x) = x(t)$ for all x in E.

Suppose that T is a non-discrete locally compact Hausdorff space which is such that for every closed non-compact subset S of T, there exists a real lower semicontinuous function on T which is bounded on every compact subset of T but is unbounded on S ([11], Theorem 8). For instance, T can be a non-discrete locally compact Hausdorff space which is either a Q-space or is such that for every closed non-compact subset S of T, there exists a real continuous function on T that is unbounded on S (so that E is either bornological or barrelled) ([7], [8]). Examples of such spaces include any infinite compact Hausdorff space and any non-discrete closed subspace of a finite product of reals. Then by Theorem 8 in [11], E is infrabarrelled and therefore, by ([4], 8.4.13) E'_b is quasi-complete and hence sequentially complete. E' can be identified with the space $M_c(T)$ of real Radon measures with compact support ([4], 4.10.1). Because T is non-discrete, there exists a t in T such that $\{t\}$ is not open. T is locally compact and therefore, t has a compact neighbourhood K. If K were finite, $\{t\}=K\cap(\cap\{T\backslash\{s\}: s\in K\backslash\{t\})$ would be open, which is not so. Thus T has an infinite compact set K. $\hat{K}\subset M(E)$ is an infinite bounded subset of E'_b and thus (ii) in Theorem 2 is satisfied. The set $B=\{x\in E: |x(t)|\leq 1, t\in T\}$ is a bounded subset of E and because T is locally compact, its closed linear hull coincides with E. Hence E is a BG space and thus (i) is satisfied. An application of the above theorem now gives that $(C(T), w(C(T), M_c(T)))$ does not have hypocontinuous multiplication.

On the other hand if T be such that every compact set is finite, then by Theorem 9 in [11], E has the weak topology and therefore, $(C(T), w(C(T), (C(T))'))$ is locally m-convex. A discrete space is trivially such a space. We give another example. Let T be an uncountable set and t a fixed element of T. A subset A of T is open if either $t\notin A$ or $t\in A$ and $T\backslash A$ is countable. If S be a non-compact closed subset of T then S is infinite and we can find a real continuous function on T that is unbounded on S. So E is barrelled ([7], [8]) and by ([4], 8.4.13) E'_b is quasi-complete and therefore, sequentially complete. The set $B=\{x\in E: |x(t)|\leq 1, t\in T\}$ is bounded and its linear hull is dense in E and E is thus a BG space. An application of the above theorem then gives that for no infinite subset $\{t_n; n\in N\}$ of T and scalars $\alpha_n>0$ the set $\{\alpha_n\hat{t}_n: n\in N\}$ is a bounded subset of E'_b.

Example 2. This example shows that condition (ii) in the above theorem is not necessary for the conclusion to be true. Let E be the algebra L^ω of (equivalence classes of) complex or real functions x on the interval $[0, 1]$ such that x^p is Lebesgue integrable for each natural number p with the topology given by norms $\{\|\ \|_p: p\in N\}$ defined by $\|x\|_p=\left(\int_0^1 |x|^p\right)^{1/p}$. Then E is a metrizable locally convex algebra which has jointly continuous multiplication and has no absolutely convex closed neighbourhood $U\neq E$ of o for which $U^2\subset U$ [1]. Because the polar of an f in $M(E)$ must be such a neighbourhood of o we have that $M(E)$ is empty. So (ii) is not satisfied. Since E is bornological, by ([4], 8.4.13) E'_b is complete.

Let L^∞ be the subspace of essentially bounded functions and for $x\in L^\infty$ let $\|x\|_\infty$ denote the essential upper bound of $|x|$. Then $B=\{x\in L^\infty: \|x\|_\infty\leq 1\}$ is a bounded subset of E and its linear hull L^∞ is dense in E. So E is a BG space.

We claim that $(E, w(E, E'))$ does not have hypocontinuous multiplication. Suppose it does. Then by Corollary (3.3) in [3], it is locally m-convex. If f be defined by $f(x)=\int_0^1 x$ for x in E then $f\in E'$. So by ([10], Theorem 1), the kernel $K(f)$ of f contains a closed ideal L of finite codimension in E. Let $x\in L$.

Then $x^* \in E$, where $x^*(t)=\overline{x(t)}$ for $t \in [0,1]$. So $xx^* \in L \subset K(f)$. Therefore, $0=f(xx^*)=\int_0^1 |x|^2$, which implies that $x=o$. Hence $L=\{o\}$, but E is infinite dimensional and L has finite codimension in E. Hence our claim is valid.

Example 3. Condition (ii) cannot, however, be left out altogether as can be seen from the second part of Example 1. We give another example to show the same. Let E be the algebra of all complex or real polynomials in one indeterminant without the constant term. E has a base $\{e_n: n \in N\}$ with multiplication table $e_n e_m = e_{n+m}$ [9]. Let A be a countable bounded subset of reals and let α be any number bigger than or equal to 1 such that $|\lambda| \leq \alpha$ for all $\lambda \in A$. For $\lambda \in A$ let f_λ be the linear functional on E given by $f_\lambda(e_n)=\lambda^n$ for all $n \in N$. Also for each $n \in N$ let g_n be the linear functional on E given by $g_n(e_m)=1$ if $n=m$ and 0 otherwise. Let E' be the linear hull of $\{f_\lambda: \lambda \in A\} \cup \{g_n: n \in N\}$. Let E have the topology $w(E, E')$. By Proposition 3 in [9] E is a metrizable locally m-convex algebra and therefore, has hypocontinuous multiplication. Also $M(E)=\{f_\lambda: 0 \neq \lambda \in A\}$ and E is semi-simple if and only if A is infinite. For $x=\sum_{j=1}^n \alpha_j e_j \in E$, let $x^*=\sum_{j=1}^n \bar{\alpha}_j e_j$. Because each $\lambda \in A$ is real, $f_\lambda(x^*)=\overline{f_\lambda(x)}$ and therefore, E is self-adjoint. E is bornological and so by ([4], 8.4.13) E'_b is complete. The set $B=\{x=\sum_{j=1}^n \alpha_j e_j \in E: \sum_{j=1}^n |\alpha_j| \alpha^j \leq 1\}$ is bounded in E and its linear hull is E. So E is a BG space. From Theorem 2 we conclude that there exists no infinite subset $\{\lambda_n: n \in N\}$ of A and scalars $\alpha_n > 0$ such that $\{\alpha_n f_{\lambda_n}: n \in N\}$ is a bounded subset of E'_b.

Example 4. Let E be the algebra φ of complex or real sequences with only a finite number of non-zero elements and E' be the space ω of all complex or real sequences ([3], Example (3.7)). Then $M(E)=\{e^{(n)}: n \in N\}$ where $e^{(n)}_m=1$ if $n=m$ and 0 otherwise. For $x \in E$ let $x^* \in E$ be given by $x^*_n=\bar{x}_n$ for all $n \in N$. Then $E=(E, w(E, E'))$ is a self-adjoint locally convex algebra and has hypocontinuous multiplication. Its strong dual E'_b is the space ω with the topology of pointwise convergence which is a complete metrizable space. Also $\{e^{(n)}: n \in N\}$ is an infinite subset of $M(E)$ that is bounded in E'_b. But E is not a BG space. Hence E satisfies all conditions except (i) in the theorem and the conclusion of the theorem is not valid. This shows that (i) is not insignificant.

All bounded subsets of E are finite dimensional and therefore, all self-adjoint BG subalgebras of E are finite dimensional. This motivates our next result.

Theorem 3. *Let E, E'_b and $M(E)$ be as in Theorem 2.*

If (i) *E is semi-simple,*

(ii) *for any sequence $\{f_n: n \in N\}$ of distinct elements of $M(E)$ there exists a subsequence $\{g_n: n \in N\}$ and positive numbers $\alpha_n > 0$ such that $\{\alpha_n g_n: n \in N\}$ is a bounded subset of E'_b, and*

(iii) *$(E, w(E, E'))$ has hypocontinuous multiplication;*

then all self-adjoint BG subalgebras of E are finite dimensional.

Proof. Let F be a self-adjoint BG subalgebra of E. Let F^0 be the polar of F in E' and let F' be the quotient E'/F^0 of E' by F^0. For $f \in E'$, let $\bar{f}$ be the corresponding element of E'/F^0. We can idenitify $\bar{f}$ with the restriction of f to F. Then by Proposition 8.1.2 [4] $w(F, F')$ is the restriction of $w(E, E')$ to F and thus $(F, w(F, F'))$ has hypocontinuous multiplication. By Corollary (3.3) in [3] it is a locally m-convex algebra. Suppose F is infinite dimensional. Since E is semi-simple, $\{o\} = \cap \{\bar{f}^{-1}\{o\}: \bar{f} \in M(E)/F^0\}$. So $M(E)/F^0$ is infinite. Let $\{\bar{f}_n: n \in N\}$ be an infinite subset of $M(E)/F^0$. For each $n \in N$, choose $f_n \in M(E)$ such that $\bar{f}_n$ corresponds to f_n. Then $\{f_n: n \in N\}$ is a sequence of distinct elements in $M(E)$ and by (ii) we have a subsequence $\{g_n: n \in N\}$ and scalars $\alpha_n > 0$ such that $\{\alpha_n g_n: n \in N\}$ is a bounded subset of E'_b. Then $\{\sum_{k=1}^{n} 2^{-k} \alpha_k g_k\}_{n \in N}$ is a Cauchy sequence in E'_b and because E'_b is sequentially complete, $g = \sum_{k=1}^{\infty} 2^{-k} \alpha_k g_k$ exists as an element of E'_b. Then $\bar{g} \in F'$. Since $(F, w(F, F'))$ is locally m-convex, by Theorem 1 [9], we have a closed ideal L of finite codimension in F contained in the kernel $K(\bar{g})$ of $\bar{g}$. As argued in the proof of Theorem 2 in [10] we obtain that the dual $(F/L)'$ of F/L is infinite dimensional. But F/L is finite dimensional and so is $(F/L)'$. This contradiction shows that F itself must be finite dimensional.

REFERENCES

[1] R. F. Arens, *The space L^{ω} and convex topological rings*, Bull. Amer. Math. Soc. 52 (1946), 931-935.

[2] R. F. Arens, *A generalization of normed rings*, Pacific J. Math. 2 (1952), 455-471, MR 14, 482.

[3] A. K. Chilana, *Topological algebras with a given dual*, Proc. Amer. Math. Soc. 42 (1974), 192-197.

[4] R. E. Edwards, *Functional Analysis, Theory and Applications*, Holt, Rinehart and Winston 1965.

[5] T. Ito and T. Seidman, *Bounded generators of linear spaces*, Pacific J. Math. 26 (1968), 283-287, MR 39, 746.

[6] E. A. Michael, *Locally multiplicatively-convex topological algebras*, Mem. Amer. Math. Soc. No. 11 (1952), MR 14, 482.

[7] L. Nachbin, *Topological vector spaces of continuous functions*, Proc. Natl. Acad.

Sci. U.S.A. 40 (1954), 471-474.
[8] T. Shirota, *On locally convex vector spaces of continuous functions*, Proc. Japan Acad. 30 (1954), 294-299.
[9] S. Warner, *Weak locally multiplicatively-convex algebras*, Pacific J. Math. 5 (1955), 1025-1032, MR 17, 876.
[10] S. Warner, *Weakly topologized algebras*, Proc, Amer. Math. Soc. 8 (1957), 314-316, MR 18, 911.
[11] S. Warner, *The topology of compact convergence on continuous function spaces*, Duke Math. J. 25 (1958), 265-282.

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