BILLINGSLEY'S THEOREMS ON EMPIRICAL PROCESSES OF STRONG MIXING SEQUENCES

By

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(Received December 27, 1973)

0. Summary. Let $\{x_j, -\infty<j<\infty\}$ be a strictly stationary sequence of random variables satisfying some mixing condition with mixing coefficient $\phi(n)$ or $\alpha(n)$. Let $F_n(t)$ be the empirical distribution function of $x_1,\cdots,x_n$ and $Y_n(,\omega)=n^{1/2}(F_n(t,\omega)-F(t))$. In [1], Billingsley proved the weak convergence theorem on $\{Y_n\}$ under the condition $\Sigma n^2\phi^{1/2}(n)<\infty$. (cf. Theorem 22.1 in [1]). Recently, in [5], Sen proved the result under the condition $\Sigma n\phi^{1/2}(n)<\infty$ and in [6] Yokoyama proved it under the condition $\Sigma n\alpha^{\beta}(n)<\infty (0<\beta<1/2)$. In this note, we shall show that Billingsley's theorem remains true under a less restrictive condition $\alpha(n)=O(n^{-3-\delta})$ $(\delta>0)$. A theorem corresponding to Theorem 22.2 in [1] is also proved (Section 4).

1. The main result. Let $\{x_j, -\infty<j<\infty\}$ be a strictly stationary sequence of random variables defined on a probability space $(\Omega, \mathfrak{B}, P)$. Suppose that the process satisfies one of the following conditions:
for all $B\in\mathfrak{M}_{k+n}^{\infty}$ with probability one

$$|P(B|\mathfrak{M}_{-\infty}^{k})-P(B)|\leqq\phi(n)\downarrow 0 \quad (n\to\infty) \tag{1}$$

(the ϕ-mixing condition) and

$$\sup|P(A\cap B)-P(A)P(B)|\leqq\alpha(n)\downarrow 0 \quad (n\to\infty) \tag{2}$$

(the strong mixing (s.m.) condition). Here the supremum is taken over all $A\in\mathfrak{M}_{-\infty}^{k}$ and $B\in\mathfrak{M}_{k+n}^{\infty}$, $\mathfrak{M}_a^b$ denotes the σ-algebra generated by events of the form

$$\{(x_{i_1},\cdots,x_{i_k})\in E\}$$

where $a\leqq i_1<i_2<\cdots<i_k\leqq b$ and E is a k-dimensional Borel set. The difference between the s.m. and ϕ-mixing conditions is explained in [4].

Let

$$c(u)=\begin{cases}1 & \text{if } u\geqq 0\\ 0 & \text{if } u<0\,.\end{cases} \tag{3}$$

Suppose that x_i has a continuous distribution function $F(u)$. Put $x_i^*=F(x_i)$ for any i and define the empirical distribution function by

$$F_n(t)=n^{-1}\sum_{i=1}^{n}c(t-x_i^*)\,,\quad 0\leqq t\leqq 1\,. \tag{4}$$

In [1], Billingsley proved that if $\{x_i\}$ is a strictly stationary ϕ-mixing sequence of random variables, then the sequence $\{Y_n\}$ of random elements in $D[0,1]$ defined by

$$Y_n(t)=n^{1/2}\{F_n(t)-t\}\,,\quad 0\leqq t\leqq 1 \tag{5}$$

converges weakly to a Gaussian random function under the condition $\Sigma n^2\phi^{1/2}(n)<\infty$ (cf. Theorem 22.1 in [1]). In [5], Sen proved the same result under the condition $\Sigma n\phi^{1/2}(n)<\infty$. On the other hand, in [6], Yokoyama showed that the theorem holds under the condition $\Sigma n\alpha^{\beta}(n)<\infty$ $(0<\beta<1/2)$, which is extensions of Billingsley's and Deo's results. The following theorem is a generalization of the results which are obtained by Billingsley, Sen, Deo and Yokoyama, respectively.

We use the same notations and definitions in [1]. Let

$$g_t(x_i^*)=c(t-x_i^*)-t\,,\quad 0\leqq t\leqq 1\,,\quad i\geqq 0\,. \tag{6}$$

Theorem 1. *Suppose that $\{x_j\}$ is a strictly stationary s.m. sequence of random variables with mixing coefficient $\alpha(n)$ and suppose x_0 has a continuous distribution function F on* $[0,1]$. *If $\alpha(n)=O(n^{-3-\delta})$ for some $\delta>0$, then*

$$Y_n\xrightarrow{\mathscr{D}}Y \tag{7}$$

where Y_n is defined by (5) *and Y is the Gaussian random function specified by*

$$EY(t)=0 \tag{8}$$

and

$$\begin{aligned}EY(s)Y(t)&=Eg_s(x_0^*)g_t(x_0^*)\\&\quad+\sum_{k=1}^{\infty}Eg_s(x_0^*)g_t(x_k^*)+\sum_{k=1}^{\infty}Eg_s(x_k^*)g_t(x_0^*)\,.\end{aligned} \tag{9}$$

These series converge absolutely and $P(Y\in C)=1$. (cf. Theorem 22.1 in [1], Theorem 3.1 in [5], Theorem in [7].)

2. A lemma. In this section, we assume that $\{z_i\}$ is a strictly stationary sequence of Bernoullian variables, centered at expectation, satisfying the s.m.

condition with mixing coefficient $\alpha(n)$. Put $Ez_1^2=\tau$. Then $E|z_1|=2\tau$.

We shall use the following

Lemma (*Davydov*). *Let the process $\{x_n\}$ satisfy the s.m. condition, and let the random variables ξ and η, respectively, be measurable with respect to $\mathfrak{M}_{-\infty}^{k}$ and $\mathfrak{M}_{k+n}^{\infty}$; moreover, assume that $E|\xi|^p<\infty$ for $p>1$ and $|\eta|<C$ a.s. Then*

$$|E\xi\eta-E\xi E\eta|\leqq 6C\{E|\xi|^p\}^{1/p}\{\alpha(n)\}^{1-1/p} .$$

(cf. Lemma 2.1 in [3]).

In what follows, by the letter K, we shall denote any positive quantity (not always the same) which is bounded and does not depend on n.

Lemma. *If $\alpha(j)=O(j^{-3-\delta})$ for some $\delta>0$, then*

$$ES_n^4\leqq K(n^2\tau^{4/3}+\tau^{\delta/(3+\delta)}n\log n) , \tag{10}$$

where $S_n=z_1+\cdots+z_n$.

Proof. We follow the proof of Lemma 2.1 in [5], (cf. [7]). We denote by $\sum_n$ the summation over all $i, j, k\geqq 0$ for which $i+j+k\leqq n$, and let $\sum_n^{(1)}$, $\sum_n^{(2)}$ and $\sum_n^{(3)}$ be, respectively, the components of $\sum_n$ for which $i\geqq(j, k)$, $j\geqq(i, k)$ and $k\geqq(i, j)$. Then, we have

$$ES_n^4\leqq 24n\{\textstyle\sum_n^{(1)}+\sum_n^{(2)}+\sum_n^{(3)}\}|Ez_0z_iz_{i+j}z_{i+j+k}| . \tag{11}$$

Since $\alpha(j)=O(j^{-3-\delta})$,

$$\sum_{j=1}^{n}(j+1)^2\{\alpha(j)\}^{3/(3+\delta)}\leqq K\log n . \tag{12}$$

Hence, from (12), the assumption $P(|z_i|>1)=0$ and Davydov's lemma, we have the following inequalities:

$$\begin{aligned}
&\textstyle\sum_n^{(1)}|Ez_0z_iz_{i+j}z_{i+j+k}|\\
&\quad\leqq 6\textstyle\sum_n^{(1)}\{\alpha(i)\}^{3/(3+\delta)}\{E|z_0|^{(3+\delta)/\delta}\}^{\delta/(3+\delta)}\\
&\quad\leqq 6\{E|z_0|\}^{\delta/(3+\delta)}\textstyle\sum_n^{(1)}\{\alpha(i)\}^{3/(3+\delta)}\\
&\quad\leqq K\tau^{\delta/(3+\delta)}\sum_{i=1}^{n}(i+1)^2\{\alpha(i)\}^{3/(3+\delta)}\leqq K\tau^{\delta/(3+\delta)}\log n ;
\end{aligned} \tag{13}$$

$$\begin{aligned}
&\textstyle\sum_n^{(2)}|Ez_0z_iz_{i+j}z_{i+j+k}|\\
&\quad\leqq\textstyle\sum_n^{(2)}|Ez_0z_i||Ez_0z_k|+6\sum_n^{(2)}\{\alpha(j)\}^{3/(3+\delta)}\{E|z_0z_i|^{(3+\delta)/\delta}\}^{\delta/(3+\delta)}\\
&\quad\leqq 36\textstyle\sum_n^{(2)}\{\alpha(i)\}^{1/3}\{E|z_0|^{3/2}\}^{2/3}\{\alpha(k)\}^{1/3}\{E|z_0|^{3/2}\}^{2/3}\\
&\qquad+6\textstyle\sum_n^{(2)}\{\alpha(j)\}^{3/(3+\delta)}\{E|z_0|\}^{\delta/(3+\delta)}\\
&\quad\leqq K\tau^{4/3}\textstyle\sum_n^{(2)}\{\alpha(i)\}^{1/3}\{\alpha(k)\}^{1/3}+K\tau^{\delta/(3+\delta)}\sum_n^{(2)}\{\alpha(j)\}^{3/(3+\delta)}\\
&\quad\leqq Kn\tau^{4/3}[\sum_{i=1}^{n}\{\alpha(i)\}^{1/3}]^2+K\tau^{\delta/(3+\delta)}\sum_{j=1}^{n}(j+1)^2\{\alpha(j)\}^{3/(3+\delta)}
\end{aligned} \tag{14}$$

$$\leqq K(n\tau^{4/3}+\tau^{\delta/(3+\delta)}\log n)\ ;$$

$$\begin{aligned}&\textstyle\sum_n^{(3)}|Ez_0z_iz_{i+j}z_{i+j+k}|\\ &\qquad\leqq 6\textstyle\sum_n^{(3)}\{\alpha(k)\}^{3/(3+\delta)}\{E|z_0|^{(3+\delta)/\delta}\}^{\delta/(3+\delta)}\\ &\qquad\leqq K\tau^{\delta/(3+\delta)}\sum_{k=1}^{n}(k+1)^2\{\alpha(k)\}^{3/(3+\delta)}\leqq K\tau^{\delta/(3+\delta)}\log n\ .\end{aligned}\tag{15}$$

Thus, (10) follows from (11), (13), (14) and (15), and the proof is completed.

3. Proof of Theorem 1. Let

$$z_i=g_t(x_i^*)-g_s(x_i^*)\quad(0\leqq s<t\leqq 1)\ .$$

Then, the sequence $\{z_i\}$ satisfies the conditions of Lemma and

$$Ez_i^2=(t-s)(1-t+s)\leqq t-s\ .$$

Moreover,

$$Y_n(t)-Y_n(s)=n^{-1/2}\sum_{i=1}^{n}z_i\ .$$

Thus, if $\varepsilon(0<\varepsilon<1)$ is a fixed number such that

$$\frac{\varepsilon}{n}\leqq t-s\ ,$$

we have

$$\begin{aligned}E|Y_n(t)-Y_n(s)|^4&\leqq K\left\{(t-s)^{4/3}+\frac{\log n}{n}(t-s)^{\delta/(3+\delta)}\right\}\\ &\leqq K\{(t-s)^{4/3}+n^{-(1-2\delta/3(3+\delta))}(t-s)^{\delta/(3+\delta)}\}\\ &\leqq K\{(t-s)^{4/3}+\varepsilon^{-(1-2\delta/3(3+\delta))}(t-s)^{1+\delta/3(3+\delta)}\}\\ &\leqq K_0(t-s)^{1+\delta/3(3+\delta)}\end{aligned}$$

for all n sufficiently large. Hence, the method of the proof of Theorem 22.1 in [1] can be completely carried over to this case and the proof is obtained.

4. Functions of strong mixing processes. Let $\{x_n\}$ be a strictly stationary sequence of random variables satisfying the s.m. condition. Let f be a measurable mapping from the space of doubly infinite sequences $(\cdots,\alpha_{-1},\alpha_0,\alpha_1,\cdots)$ of real numbers into the real line. Define random variables

$$\eta_n=f(\cdots,x_{n-1},x_n,x_{n+1},\cdots)\ ,\quad n=0,\pm1,\pm2,\cdots\tag{16}$$

where x_n occupies the 0-th place in the argument of f.

Suppose now that

$$0 \leqq \eta_n(\omega) \leqq 1 \tag{17}$$

and let $F_n(t, \omega)$ be the empirical distribution function of $\eta_1(\omega), \cdots, \eta_n(\omega)$ and define Z_n by

$$Z_n(t, \omega) = n^{1/2}(F_n(t, \omega) - F(t)) \tag{18}$$

where F is the distribution function for η_0. Let f_k be a measurable mapping from R^{2k+1} into R^1. Moreover, let

$$\eta_{kn} = f_k(x_{n-k}, \cdots, x_n, \cdots, x_{n+k}) \tag{19}$$

for which

$$0 \leqq \eta_{kn}(\omega) \leqq 1 .$$

Finally, we shall suppose that there exist sets H_k in $[0, 1]$ with the following properties;

(i) If $t \in H_k$, then

$$I_{[0,t]}(\eta_0) = I_{[0,t]}(\eta_{k,0})$$

with probability one, where $I_E(.)$ is the indicator of the set E.

(ii) If $J_k = \{F(t): t \in H_k\}$, then J_k is a ρ_k-net in $[0, 1]$, where ρ_k goes to zero exponentially.

(iii) We have $H_k \subset H_{k+1}$.

Define g_t by (6) as before.

Theorem 2. *Suppose that* $\{x_n\}$ *is a strictly stationary s.m. sequence with mixing coefficient* $\alpha(n)$, *that* η_0 *has a continuous distribution function* F *on* $[0, 1]$, *and that there exist sets* H_k *with the three properties just described. If* $\alpha(n) = O(n^{-3-\delta})$ *for some* $\delta > 0$, *then*

$$Z_n \xrightarrow{\mathscr{D}} Z$$

where Z *is the Gaussian random function specified by*

$$EZ(t) = 0$$

and

$$EZ(s)Z(t) = Eg_s(\eta_0)g_t(\eta_0) + \sum_{k=1}^{\infty} Eg_s(\eta_0)g_t(\eta_k) + \sum_{k-1}^{\infty} Eg_s(\eta_k)g_t(\eta_0) .$$

The series converge absolutely and $P(Z \in C) = 1$. (cf. Theorem 22.2 in [1]).

Proof. As in the proof of Theorem 22.2 in [1], we can show that it suffices to consider the case in which η_0 is uniformly distributed. So, we assume that η_0 is uniformly distributed. If s and t both lie in H_k, then the process

$$g_t(\eta_n)-g_s(\eta_n)=g_t(\eta_{kn})-g_s(\eta_{kn}) , \quad n=0, \pm 1, \pm 2, \cdots$$

is strong mixing with mixing coefficient $\alpha^{(k)}(n)$ where

$$\alpha^{(k)}(n)=\begin{cases}1 & \text{if } n\leqq 2k \\ \alpha(n-2k) & \text{if } n>2k .\end{cases}$$

Let n be arbitrarily fixed. Since η_0 is uniformly distributed and since

$$\sum_{j=0}^{n}\{\alpha^{(k)}(j)\}^{1/8}\leqq Kk$$

and

$$\sum_{j=0}^{n}(j+1)^2\{\alpha^{(k)}(j)\}^{\delta/(8+\delta)}\leqq Kk^8 \log n ,$$

so by the analogous method of the proof of Lemma we can prove that

$$\begin{aligned}&E\left|\sum_{i=1}^{n}(g_t(\eta_i)-g_s(\eta_i)\right|^4 \\ &\leqq Kk^8(n^2|t-s|^{4/8}+|t-s|^{\delta/(8+\delta)}n \log n)\end{aligned}$$

where K depends on α alone. Therefore

$$s, t\in H_k , \quad \frac{\varepsilon}{n}\leqq t-s \quad (0<\varepsilon<1)$$

imply

$$P(|Z_n(t)-Z_n(s)|\geqq\lambda)\leqq K_1\frac{k^8}{\lambda^4}(t-s)^{1+\beta}$$

for some $\beta>0$ where K_1 depends only on α and ε. The rest of the proof is identical to that of Theorem 22.2 in [1] and so is omitted.

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