GENERALIZED MODELS FOR INTUITIONISTIC AND CLASSICAL PREDICATE CALCULI

By

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The paper announces many future publications in those important and actually developed topics.

After my Visiting Professorship in a research fellowship of Delft Technological University, Holland, I gave about 20 lectures in 14 towns of England (3), United States (14), Germany (2) and Greece (1) indicating the following stages of development of Mathematical Sciences:

1. Primitive one with statistical data—the earliest level of various computational processes of a pure mechanical nature.
2. Partition problems with the gravitation to finite mathematics.
3. My Generalized Models created in 1954 year—the best tool in finite mathematics —containing wholly all other ones, e.g. "Forcing" with international prizes.

These stages supply a device in the historical achievements of our predecessor with mottos of my publications . . .

"Es ist etwas in Menschen, das sich keiner Gewalt beugt und fürchtet und durch keine Gewalt überwältigt werden kann. Es bleibt unbeschädigt und frei, wie auch die Sachen gehen, und spricht der Gewalt Hohn; und ist doch zugleich mild und rät zum Guten und Frieden."

And according to my published papers, [21]—[39], and lectures—showing the modelling of the reality—my generalized models provide a clear unity of Mathematics with classical and non-classical calculi. Thus, e.g. omitting my invariance relation of cited papers we shall obtain generalized models of intuitionistic calculi —a conclusion of maximal properties of presented constructions.

My generalized models for non-classical calculi[1] was also announced in 1954 year but my very bad conditions in Israel unabled me to publish this paper many years ago; so it is indicated that e.g. "Forcing" of cited bibliography (it is a kind of regarded generalized models) was created many years after my results and it is an additional remark in the discussion of a crisis of mathematical sciences in

[1] with values in algebras.

USA, . . ., see Notices of American Mathematical Society.

The paper has a closed character. Therefore it will be seen that it only suffices the intuitionistic calculus to construct all classical prime filters or ideals, it will be given a short discussion of the main constructive property of intuitionistic theories including Kolmogoroff and Lukasiewicz's absorption relations and so it will be shown an origin of constructive theories.

The important reduction to the suitable set of all models of a given power is also given.

The lecture indicates a classification of different predicate calculi; for instance, it is seen that $F'+F''$ holds in all generalized models Q of 2 elements and it does not hold in Q with 3 elements.

The exposition provides practical 0-1-verification of all formulas of regarded predicate calculi with the decidability of intuitionistic propositional calculi and classes of predicate calculi (e.g. $\Sigma\Pi$—formulas in realization); the important reduction of arbitrary formulas to a constructive proof of a formula of propositional calculus is also presented, i.e. the analogy of Gödel-Herbrand's reduction to propositional calculus but in intuitionistic one.

An important result are asymptoticly finite generalized models for a new predicate calculus containing wholly the classical propositional one but unclosed under substitutions. The very great importance of generalized models suggests an introduction of generalized quantifiers with famous problems of mathematical sciences: Great Fermat problem. Goldbach problem with Vinogradov's solutions, Waring's problem, Gauss and *Hilbert's* results with *Erdös* solutions and many others . . ..

Of course, according to the above the paper cannot be finished and many future ways are also seen, e.g. in the theory of natural numbers, my generalized algorithms, . . . The paper is closed by remarks about future directions; other results in context.

Closing the short introduction we complete the bibliography with names of last investigations: *Aczel, Beth, Bourbaki, Cohen, Dantzing, Destouches, Dijkman, Fefarman, Freudental, Fitting, Goodstein, Mc-Kinsey, Kripke, Kuroda, Porte, Rosenbloom, Ryll-Nardzewski, Scott, Skolem, Schröter, Spector, Vorobew*, ...

We present Generalized Models for intuitionistic predicate calculus according to my resulte of 1954 year and first of all we indicate the following denotations:

1. Logical signs: $'$ (negation), $+$ (alternative), $\cdot$ (conjuction), $\supset$ (implication), Π (general quantifier), Σ (existential quantifier).

2. Variables: (1°) individual: $x_1, x_2, \cdots$ (simply: x);
   (2°) apparent: $a_1, a_2, \cdots$ (simply: a);
   (3°) relation signs of m-arguments: $f_1^m, f_2^m, \cdots, f_q^m$ with $m=1, 2, \cdots, t$; x_0 is a constant.[1]
3. $w(E)$—the number of different individual variables occurring in the expression E plus 1;
4. $p(E)$—the number of different apparent variables occurring in the expression E;
5. $\{i_m\}$—different indices $i_1, \cdots, i_m$; $\{i_{w(E)}\}$—all indices of free variables in E and 0; $\{F_q^t\}$—the sequence of form: $F_1^1, \cdots, F_q^1, \cdots, F_1^t, \cdots, F_q^t$;
6. $i(E)=\max\{i_{w(E)}\}$; $n(E)=\max\{w(E)+p(E), i(E)\}$; (The reader should write a smaller number $n(E)$.)
7. $E(u/z)$—substitution of u for z in E with known restrictions;
8. $h \in E$—h belongs to signs of the expression E;
9. $M, M_1, \cdots$—models;
10. Q—a non-empty set of models with domains $D^1 \subset D^2 \subset \cdots$—called: Generalized model; $Q(k)$—Q is a generalized model whose elements are of power $\leqslant k+1$. We do not assume that k is finite; however in D. 4. the reader should generally suppose: k is finite for the functional W;
11. General and existential metaquantifiers: (K), $(\exists K)$, $(\{K_r\})$, $(\exists\{K_r\})$;
12. Dots separate stronger than other signs.

A model is a pair $<B, \{F_j^i\}>$, where B is a non-empty domain and $\{F_j^i\}$ is an arbitrary doubly finite sequence of relations such that F_j^m is a m-ary relation on B—a set of ordinary numbers, $i=1, \cdots, q$ and $j=1, \cdots, t$.

$M/s_0, \cdots, s_k/$—simply: $M/\{s_k\}$—is a truncated model of the power $k+1$ with respect to $\{s_k\}$, i.e. there exist models $<B, \{F_j^i\}>$ and $<B_k, \{\phi_j^i\}>$ such that B_k is of the power $k+1$ and:

$$\phi_j^m(r_1, \cdots, r_m) \text{ .iff. } F_j^m(s_{r_1}, \cdots, s_{r_m}),\ 1 \leqslant m \leqslant t,\ 1 \leqslant j \leqslant q.$$

Note that $M/\{s_k\}=<B_k, \{\phi_j^i\}>$ is a submodel of M in the sense of homomorphism and therefore instead of "truncations" we can speak about homomorphisms.

Of course, [11],:

L. 1'. $M/\{s_k\}/\{t_q\}=M/\{s_{t_q}\}$

Regarding descriptions of models only for atoms we obtain analogical notions for sets of atomic formulas and the reader can easy formulate L. 1'. for families of atoms; let him write the new formulation.

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[1] The introduction of the constant x_0 is only an editorial formulation.

Hence it is obviously:

L. 2. If T_1 is a description of M_1 and T_2 is a description of M_2, then:

$$T_1/\{j_m\}=T_2/\{j_m\} \text{ .iff. } M_1/\{j_m\}=M_2/\{j_m\}$$

The inclusion $M_1 \geq M_2$ means: if $R(s_1, \cdots, s_m)$ in M_2, then $R(s_1, \cdots, s_m)$ in M_1, where M_1, M_2 belong to some Q and R is an arbitrary relation of both models; so the inclusion $(M_1 \geq M_2)/\{i_l\}$ means: $M_1/\{i_l\} \geq M_2/\{i_l\}$.

From L. 1'. follows immediately:

L. 1. If $M_1 \geq M_2$, then: $M_1/\{s_k\} \geq M_2/\{s_k\}$.

Introducing my intuitionistic evaluation functional, [21]—[39], we suppose the following abbreviation:

$$V\{k, Q, M, \{i_l\}, E\}=0 \text{ .iff. } \sim V\{k, Q, M, \{i_l\}, E\}=1$$

For an arbitrary generalized model $Q(k)$, for an arbitrary $M=<B_k, \{F_q^t\}> \in Q$, for every formula E and each non-empty $\{i_l\} \supset \{i_{w(E)}\}$, $l+p(E) \leq k$ (this condition may be here omitted but it is used in the next evaluation functional W and it is written for the historical homogenity of my lectures), we introduce the following inductive definition of the evaluation functional V:

(1d) $V\{k, Q, M, \{i_l\}, f_j^m(x_{r_1}, \cdots, x_{r_m})\}=1.$ iff.[1] $F_j^m(r_1, \cdots, r_m)$,

(2d) $V\{k, Q, M, \{i_l\}, F'\}=1$ .iff. $(M_1)\{(M_1 \geq M)/\{i_l\} \rightarrow V\{k, Q, M_1, \{i_l\}, F\}=0\}$,

(3d) $V\{k, Q, M, \{i_l\}, F \supset G\}=1$ .iff. $(M_1)\{(M_1 \geq M)/\{i_l\} \rightarrow$
$\rightarrow (V\{k, Q, M_1, \{i_l\}, F\}=0 \vee V\{k, Q, M_1, \{i_l\}, G\}=1)\}$,

(4d) $V\{k, Q, M, \{i_l\}, FG\}=1$ .iff. $V\{k, Q, M, \{i_l\}, F\}=1 \wedge V\{k, Q, M, \{i_l\}, G\}=1$,

(5d) $V\{k, Q, M, \{i_l\}, F+G\}=1$ .iff. $V\{k, Q, M, \{i_l\}, F\}=1 \vee V\{k, Q, M, \{i_l\}, G\}=1$,

(6d) $V\{k, Q, M, \{i_l\}, \prod aF\}=1$ .iff. $(M_1)(i)\{(M_1 \geq M)/\{i_l\} \wedge$
$\wedge (i \in \{i_l\}) \rightarrow V\{k, Q, M_1, \{i_l\}, i, F(x_i/a)\}=1\}$,

(7d) $V\{k, Q, M, \{i_l\}, F(x_i/a)\}=1$ .iff. $(\exists i)\{(i \in \{i_l\}) \wedge V\{k, Q, M, \{i_l\}, F(x_i/a)\}=1\}$.

D. 1. $E \in P(k, Q, \{i_l\})$ .iff. $(M)\{(M \in Q) \rightarrow V\{k, Q, M, \{i_l\}, E\}=1\}$.

D. 2. $E \in P(k, Q)$ .iff. $(\{i_l\}(\{(w(E) \leq l \leq k-p(E)) \rightarrow (E \in P(k, Q, \{i_l\}))\}$.

D. 3. $E \in P(k)$ .iff. $(Q)\{Q(k) \rightarrow (E \in P(k, Q))\}$.

D. 4. $E \in P$ .iff. $(\exists k)\{(k \geq n(E)) \wedge (E \in P(k))\}$.

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[1] In (1d) variables are interpreted by their indices; as an exercise the reader may write the last definitions interpreting variables by means of arbitrary elements of B_k and not by their indices.

The condition $l+p(E) \leq k$ may be here omitted but it is used in the next evaluation functional W and it is written for the historical homogenity of my lectures in Generalized Models; $\aleph_0+p(E)=\aleph_0$.

Let us explain:

$V\{k, Q, M, \{i_l\}, E\}=1$ is read: the model M evaluates E with respect to Q.

If Q is one elementing, then (1d)—(7d) give the classical evaluation functional in the domain of ordinary numbers $\in \{i_l\}$.

If we choose such maximal $M \in Q$ that if $(M_1 \geq M)/\{i_l\}$, then $(M_1 = M)/\{i_l\}$ then (1d)—(7d) also give the classical evaluation functional in the domain of ordinary numbers $\in \{i_l\}$.

We may restrict ourselves to $\{i_l\}=\{l\}$ and to reformulate (1d)—(7d) without the sequence $\{i_l\}$; then $Q(k)$ is replaced by $Q(l)$ with $V\{Q, M, E\}$. But my papers also replace $i \in \{i_l\}$ in (6d) by $i \leq k$; then the reader may introduce my invariance rala-tion and obtain adequate characterizations of the intuitionistic predicate calculi.

But the paper does not deal with invariance relations though suitable in-variance of formulas means their decidability and certain ones are given in my cited papers; so we deal here with extensions of models, see Los' papers.

D. 2. is the intuitionistic truth definition in a generalized model $Q(k)$.

D. 3. is the intuitionistic truth definition for all generalized models $Q(k)$ with k—constant.

We shall prove that P is the class of all intuitionistic theorems of the first order predicate calculus.

D. 5. $M_1 \in M[k]$ .iff. $(\exists\{s_k\})\{M_1 = M/\{s_k\}\}$.

$M[k]$ is the set of all submodels of M of the power k and therefore:

If $M[k]=Q$, then the number i in (6d) is the name of an arbitrary element of the domain of M and so was explained the introduction of my generalized models. The reader should differ ones from other generalized models, where we only have a change in the algebra of values and not in the real notion of a model —we do not discuss derivative (i.e. artificial) replacements.

We shall write $V\{M, \{j_\tau\}, E\}$ or $V\{Q, M, \{j_\tau\}, E\}$ instead of $V\{k, Q, M, \{j_\tau\}, E\}$.

Of course:

(2d0) $V\{M, \{i_l\}, F'\}=0$ .iff. $(\exists M_1)\{(M_1 \geq M)/\{i_l\} \wedge V\{M_1, \{i_l\}, F\}=1\}$,

(3d0) $V\{M, \{i_l\}, F \supset G\}=0$ .iff. $(\exists M_1)\{(M_1 \geq M)/\{i_l\} \wedge$
$\wedge V\{M_1, \{i_l\}, F\}=1 \wedge V\{M_1, \{i_l\}, G\}=0\}$,

(4d0) $V\{M, \{i_l\}, FG\}=0$ .iff. $V\{M, \{i_l\}, F\}=0 \vee V\{M, \{i_l\}, G\}=0$,

(5d0) $V\{M, \{i_l\}, F+G\}=0$ .iff. $V\{M, \{i_l\}, F\}=0 \wedge V\{M, \{i_l\}, G\}=0$,

(6d0) $V\{M, \{i_l\}, \prod aF\}=0$ .iff. $(\exists M_1)(\exists i)\{(M_1 \geq M)/\{i_l\} \wedge (i \in \{i_l\}) \wedge$
$\wedge V\{M_1, \{i_l\}, i, F(x_i/a)\}=0\}$,

(7d0) $V\{M, \{i_l\}, \sum aF\}=0$ .iff. $(i)\{(i \in \{i_l\}) \rightarrow V\{M, \{i_l\}, F(x_i/a)\}=0\}$;

(D40) $E \in P$ .iff. $(k)((k \geqslant n(E)) \rightarrow (\exists Q)(\exists M)\{Q(k) \wedge (M \in Q) \wedge$
$\wedge (\exists \{i_t\})\{(w(E) \leqslant l \leqslant k - p(E)) \wedge V\{k, Q, M, \{i_t\}, E\} = 0\}\})$

The reader should prove by induction with respect to the length of a formula the following basic lemma:

L. 3. If $M^\circ, M \in Q$, $l + p(E) \leqslant k$ and[1] $(M \geqslant M^\circ)/\{i_t\}$, then:

If $V\{M^\circ, \{i_t\}, E\} = 1$, then $V\{M, \{i_t\}, E\} = 1$.

If additionally $(M = M^\circ)/\{i_t\}$, then:

$$V\{M, \{i_t\}, E\} = 1 \text{ .iff. } V\{M^\circ, \{i_t\}, E\} = 1 .$$

Hence, if E has no free variables and all models belonging to Q are identical on 0—the value of x_0—then:

Either $(M)\{V\{M, \{0\}, E\} = 1\}$ or $(M)\{V\{M, \{0\}, E\} = 0\}$.

The whole proof is similar to L. 3. of my publications.

Emhasize the smallest indice in L. 3. is $l = w(E)$; this basic lemma indicates other constructive evaluations of formulas, e.g. by suitable quantification in (1d)—(7d) with respect to L. 3. and my cited papers give several other intuitionistic functionals (e.g. A. Robinson's potential truth follows from [22], p. 211, footnote 1, and it can be analogously done for intuitionistic predicate calculi).

And in my lecture at Bonn Mathematics I regarded a special case, see p. 32, of the following evaluation functional W:

For brevity put $\{i_t\} = \{l\}$ and let:

(0d) $W = V$, for conjuction, alternative and existential quantifier,

(0d2) $W\{k, Q, M, \{l\}, F'\} = 1$ .ff. $(t)(M_1)\{(l \leq t \leq k - p(F)) \wedge$
$\wedge (M_1 \geq M)/\{l\} \rightarrow W\{k, Q, M_1, \{t\}, F\} = 0\}$,

(0d3) $W\{k, Q, M, \{l\}, F \supset G\} = 1$ .iff. $(t)(M_1)\{(l \leq t \leq k - p(F \supset G)) \wedge (M_1 \geq M)/\{l\} \rightarrow$
$\rightarrow (W\{k, Q, M_1, \{t\}, F\} = 0 \vee W\{k, Q, M_1, \{t\}, G\} = 1)\}$,

(0d6) $W\{k, Q, M, \{l\}, \prod a F\} = 1$ .iff. $(M_1)(i)\{(i \leq k) \wedge$
$\wedge (t = \max(i, l) \wedge (M_1 \geq M)/\{l\} \rightarrow W\{k, Q, M_1, \{t\}, F(x_i/a)\} = 1\}$.

And analogously it holds a stronger basic lemma:

L. 30. If $t + p(E) \leq k$, $l \leq t$, $(M \geq M^\circ)/\{l\}$, then

If $W\{M^\circ, \{l\}, E\} = 1$, then $W\{M, \{t\}, E\} = 1$.

If additionally $(M = M^\circ)/\{l\}$, then:

$$W\{M^\circ, \{l\}, E\} = 1 \text{ .iff. } W\{M, \{l\}, E\} = 1 .$$

The last conclusion of L. 3. also holds for the functional W.

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[1] L. 3. also holds, if $i \in \{i_t\}$ is replaced by $i \leq k$ in (6d).

D. 1.—D. 4. are changed in the second definition for the functional W and namely:

D. 20. $E \in P(k, Q)$ .iff. $E \in P(k, Q, \{i_{w(E)}\})$.

Having replaced the inequality by the equality in (0d)—(0d6) we obtain generalized models for the new predicate calculus containing the classical propositional one but unclosed under substitutions. So we continue the lecture for both functions V and W; small editorial remarks with complementions remain for readers.

First of all note that my two evaluation functionals of Delft Technological University, 1970/1, omit the quantifier (t) in (0d2) and (0d3) introducing new characteristic numbers likely to $i(E)$ or $n(E)$; but the paper creates a complemention of my first publications and therefore Delft constructive functionals will be published later.

Let now $k=2$, i.e. we regard B_2. Let $M=\langle B_2, > \rangle$ and $M_1=\langle B_2, F_1^1(1), > \rangle$;[1] thus $F_1^1(1) \bar{\in} M$ and $F_1^1(1) \in M_1$. Let Q have only 2 elements M and M_1; then $V\{M, \{0,1\}, f_1^1(x_1)\}=0$ and $V\{M_1, \{0,1\}, f_1^1(x_1)\}=1$. But $(M_1 \geq M)/\{0,1\}$; hence by (2d0): $V\{M, \{0,1\}, (f_1^1(x_1))'\}=0$.

And in view of (5d0):

$$V\{2, Q, M, \{0,1\}, (f_1^1(x_1))'+f_1^1(x_1)\}=0$$

The above proves:

L. 4. $F'+F \bar{\in} P$ and maybe:

$$V\{k, Q, M, \{i_t\}, F'\}=0 \quad \text{and} \quad V\{k, Q, M, \{i_t\}, F\}=0 \,.$$

From L. 4. follows immediately:

L. 5. The following Heyting's formulas do not belong to P:

$$F'' \supset F \bar{\in} P, \quad \{\prod a(F+F')\}'' \bar{\in} P$$

And L. 3. easy implies:

L. 6. If $E'' \bar{\in} P$, then $E \bar{\in} P$.

L. 7. Kleene-Nelson's formulas[2]:

$$\{\prod a(F+G) \supset (F+\prod aG)'' \in P, \quad a \bar{\in} F \,,$$
$$\{\prod a \prod b(F+G) \supset (\prod aF+\prod bG)\}'' \in P, \quad a \bar{\in} G, \ b \bar{\in} F$$

Note that L. 5.—L. 7. are simple exercises for readers; other ones with cited names are given on last pages of the paper.

We shall deal with constructive properties of the intuitionistic predicate calcu-

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[1] $B_2=\{0,1\}$ is a domain of 2 ordinary numbers 0, 1.

[2] L. 7. is not used in the paper and so the reader can prove it in the second reading.

lus also asserted in L. 8. and therefore though we do not use the following remark let us indicate without proof Kreisel-Putnam lemma:

$$\{E\supset(F+G)\}\supset\{(E\supset F)\vee(E\supset G)\}\bar{\in} P$$

But some special cases of the last formula belong to P and ones can be found in Kreisel's publications or follow from the continuation.

Let us emphasize the first constructive property in the following description:

Many proofs of allmost all cited scientists has an analogon to the following (see pages 24, 25):

L. 8. Let Q have the following property:

There exists $M_0\in Q$ such that for every $M\in Q$: $(M\geq M_0)/\{i_l\}$; then:

If $F+G\in P(k, Q, \{i_l\})$, then $F\in P(k, Q, \{i_l\})$ or $G\in P(k, Q, \{i_l\})$.

L. 8. is an immediate conclusion of L. 3.

Note that the set of all models of a given power has the property asserted in the last lemma.

Emphasize here my first embedding property of the generalized model written in 1954 year, [22]:

$$(\{t\})(M_1)(M_2)\{(M_1, M_2\in Q)\wedge(M_1=M_2)/\{t\}\rightarrow$$
$$\rightarrow(\exists M_3)((M_3\in Q)\wedge(M_3=M_1)/\{t+1\}\wedge(M_3=M_2)/\{t\}, t+2)\};$$

then the number t in (0d2), (0d3) and (0d6) is put $t=L$ and L. 30. holds for the last evaluation functional; so we obtain another adequate asymptoticly finite characterization of intuitionistic predicate calculi with a stronger constructive property than L. 8. and it remains for readers.

Of course, the last embedding is done in Q but it also was done outside Q; then the asymptoticly finite characterizations are weaker, [22], [23].

Though we shall not use the following lemma, let the reader prove an analogon of Harrop-Kleene-Kreisel theorems, see page 14,:

L. 9. If $a\bar{\in} E$, then $\sum a(F\supset G)\supset(F\supset\sum aG)\in P$ and not inverse,

Each intuitionistic evaluation determines classical ones according to Gödel-Kolmogeroff-Lukasiewicz-Stupecki-Sobocinski's absorption relations; we shall present two ones and others in [41] and my future papers.

Using Lukasiewicz's brackets free denotation the first absorption relation can be written: $C_{pq}=NK_pN_q$ and $A_{pq}=CN_{pq}=NKN_pN_q$; then $AN_{pp}=CNN_{pp}=NKNN_pN_p$ and we prove that the last formula belongs to P, i.e.

L. 10. $(F''F')'\in P$.

Indeed, if $V\{M, \{i_l\}, (F''F')'\}=0$, then by (2d0) there exists $M_1\in Q, (M_1\geq M)/$

$\{i_t\}$ such that $V\{M_1, \{i_t\}, F''F'\}=1$; and in view of (2d0) and (4d): $V\{M_1, \{i_t\}, F''\}=1$ and $V\{M_1, \{i_t\}, F''\}=0$, i.e. a contradiction.

Thus the first absorption relation gives the following classical evaluation functional[1] $W°$:

Let us omit (7d), i.e. we replace here: $\sum aF=(\prod aF')'$, and instead of (3d) and (5d) let us assume respectively:

(3D) $W°\{M, \{i_t\}, F\supset G\}=1$ .iff. $(M_1)\{(M_1\geq M)/\{i_t\}\to(W°\{M_1, \{i_t\}, F\}=0\vee$
$\vee W°\{M_1, \{i_t\}, G'\}=0)\}$

(5D) $W°\{M, \{i_t\}, F+G\}=1$ .iff. $(M_1)\{(M_1\geq M)/\{i_t\}\to(W°\{M_1, \{i_t\}, F'\}=0\vee$
$\vee W°\{M_1, \{i_t\}, G'\}=0)\}$.

Then (1D)—(6D) with D. 1.—D. 4. for the functional $W°$ give all classical valid formulas, i.e. if we write in D. 4. instead of P the classical denotation $PW°$ of the class of all true formulas, then we shall prove:

E is a classical thesis if and only if $E\in PW°$.

Kolmogoroff and others, [41], gave many right absorption relations for the intuitionistic arithmetic and it will be a topic of my future papers.

A second symmetric absorption relation is given by the classical theses (Lukasiewicz's denotation) $EC_{pq}AC_{pq}CN_qN_p$, $EA_{pq}ACN_{pq}CN_{qp}$; then instead of (3D) and (5D) we obtain respectively:

(3D1) $W_1\{M, \{i_t\}, F\supset G\}=1$ .iff. $(M_1)\{(M_1\geq M)/\{i_t\}\to(W_1\{M_1, \{i_t\}, F\}=0\vee$
$\vee W_1\{M, \{i_t\}, G\}=1)\}\vee(M_1)\{(M_1\geq M)/\{i_t\}\to(W_1\{M_1, \{i_t\}, G'\}=0\vee$
$\vee W_1\{M_1, \{i_t\}, F')=1)\}$,

(5d1) $W_1\{M, \{i_t\}, F+G\}=1$ .iff. $(M_1)\{(M_1\geq M)/\{i_t\}\to(W_1\{M_1, \{i_t\}, F'\}=0\vee$
$\vee W_1\{M_1, \{i_t\}, G\}=1)\}\vee(M_1)\{(M_1\geq M)\to(W_1\{M_1, \{i_t\}, G'\}=0\vee$
$\vee W_1\{M_1, \{i_t\}, F\}=1)\}$.

And (1D1)—(6D1) with D. 1.—D. 4. for the functional W_1 give analogously all classically volid formulas.

Kolmogoroff also replaces $\prod$ by $\sum$ and the reader should try to write suitable classical evaluation functionals defining $\prod$, but according to known absorption relations.

Simplifications of the last definitions also remains for readers. And a very important absorption relation is given in T. 3.

The reader should prove by induction respectively to the length of a formula (using L. 3.):

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[1] L. 3. holds for all evaluation functions.

L. 11. Let E° result from E by the replacement of implication, alternative and existential quantifier according to the first absorption relation; then:

$$V\{M, \{i_l\}, E^\circ\}=1 \text{ .iff. } W^\circ\{M, \{i_l\}, E\}=1 .$$

(Hence we obtain a simple conclusion in Kreisel's inverse theorem, see page 11.)

An analogical theorem holds for the second absorption relation (and others).

We shall prove the completeness of regarded predicate calculi and therefore let us introduce the notion of formal theorem or briefly: thesis:

Axioms of the intuitionistic calculus are:

E_1) $F \supset (G \supset F)$

E_2) $(F \supset G) \supset (\{F \supset (G \supset H)\} \supset (F \supset H))$

E_3) $F \supset (G \supset FG)$

E_4) $FG \supset F$ $\qquad$ E_5) $FG \supset G$

E_6) $F \supset (F+G)$ $\qquad$ E_7) $G \supset (F+G)$

E_8) $(F \supset G) \supset \{(H \supset G) \supset (\{F+H\} \supset G)\}$

E_9) $(F \supset G) \supset \{(F \supset G') \supset F'\}$

E_{10}) $F \supset (F' \supset G)$

E_{11}) $\prod a F \supset F(x_i/a)$ $\qquad$ E_{12}) $F \supset \sum a F(a/x_i)$

And proof rules are the following:

R_1) If F and $F \supset G$ are theses, then G is a thesis.

R_2) If $F \supset G$ is a thesis and $x \bar{\in} F$, then $F \supset \prod a G(a/x_i)$ is a thesis.

R_3) If $F \supset G$ is a thesis and $x \bar{\in} G$, then $\sum a F(a/x_i) \supset G$ is a thesis.

Obviously E_1—E_{10} are axioms of the intuitionistic propositional calculus and adding the classical axiom

E_{13}) $F'' \supset F$

we obtain the classical predicate calculus; then R_3) is a derivative rule.

We assume the usual definitions of a formal proof of a formula and the length of a formal proof. A thesis is the last element of a formal proof.

It is easy to show with respect to the length of a formal proof:

L. 12. If the length of a formal proof of E is m, then the length of some formal proof of $E(y/x)$ also is m.[1)]

T. 1. If $F_1, \cdots, F_m$ is a formal proof of the formula F and $k \geqslant n(F_i)$, $i=1, \cdots, m$, then:

(I) $F \in P(k)$,

[1)] According to L. 12. in the propositional proof of E appear only $\{i_{W(E)}\}$.

(II) If the formal proof is a classical one, then:

(1) $F \in PW(k)$, where $PW(k)$ corresponds to D. 3. of the classical evaluation,

(2) and if the classical proof is without R_2, then:
$F'' \in P(k)$, and if $F=F_1'$, for some F_1, then $F \in P(k)$.
(Equalize with L. 5.!).

Proof:—The theorem is proved by induction with respect to the length m of the formal proof.

First of all we verify that formula $E_{13}'' \in P(k)$. Indeed supposing a contrary $V\{M, \{i_t\}, E_{13}''\}=0$ we have alternatively:

$$(\exists M_1)\{(M_1 \geq M)/\{i_t\} \wedge V\{M_1, \{i_t\}, E_{13}'\}=1\},$$
$$(M_2)\{(M_2 \geq M_1)/\{i_t\} \rightarrow V\{M_2, \{i_t\}, E_{13}\}=0\}, \text{ i.e.}$$

(0) $(M_2)\{(M_2 \geq M)/\{i_t\} \rightarrow V\{M_2, \{i_t\}, F'' \supset F\}=0\}$.

According to (0): $V\{M_1, \{i_t\}, F'' \supset F\}=0$; hence

$(\exists M_3)\{(M_3 \geq M_1)/\{i_t\} \wedge V\{M_3, \{i_t\}, F''\}=1 \wedge V\{M_3, \{i_t\}, F\}=0\}$
$(\exists M_4)\{(M_4 \geq M_3)/\{i_t\} \wedge V\{M_4, \{i_t\}, F\}=1\}$. But according to (0):
$V\{M_4, \{i_t\}, F'' \supset F\}=0$; hence $(\exists M_5)\{(M_5 \geq M_4)/\{i_t\} \wedge V\{M_5, \{i_t\}, F''\}=1 \wedge$
$\wedge V\{M_5, \{i_t\}, F\}=0\}$. By virtue[1] of L. 3.: $V\{M_5, \{i_t\}, F\}=1$ and we obtained a contradiction.

Verifying formulas of the intuitionistic calculus the reader should introduce his abbreviations; then it is easy to verify that axioms E_1—$E_{12} \in P(k)$ and we shall give the hardest cases in a-contrary proofs:

E_2) $V\{M, \{i_t\}, E_2\}=0$, $(\exists M_1)\{(M_1 \geq M)/\{i_t\} \wedge V\{M_1, \{i_t\}, F \supset G\}=1 \wedge$
$\wedge V\{M_1, \{i_t\}, \{F \supset (G \supset H)\} \supset (F \supset H)\}=0\}$,
$(\exists M_2)\{(M_2 \geq M_1)/\{i_t\} \wedge V\{M_2, \{i_t\}, F \supset (G \supset H)\}=1 \wedge V\{M_2, \{i_t\}, F \supset H\}=0\}$,
$(\exists M_3)\{(M_3 \geq M_2)/\{i_t\} \wedge V\{M_3, \{i_t\}, F\}=1 \wedge V\{M_3, \{i_t\}, H\}=0\}$ and by L. 3.:
$(V\{M_3, \{i_t\}, F \supset G\}=1 \wedge V\{M_3, \{i_t\}, F \supset (G \supset H)\}=1)$. Thus
$(M_4)\{(M_4 \geq M_3) \rightarrow (V\{M_4\ \{i_t\}, F\}=0 \vee V\{M_4, \{i_t\}, G \supset H\}=1)\}$. However:
$V\{M_3, \{i_t\}, F\}=1$; so the case $V\{M_3, \{i_t\}, F\}=0$ is impossible and it remains
$V\{M_3, \{i_t\}, G \supset H\}=1$. Therefore $(M_4)\{(M_4 \geq M_3) \rightarrow (V\{M_4, \{i_t\}, G\}=0 \vee$
$\vee V\{M_4, \{i_t\}, H\}=1)\}$; but $(M_4)\{(M_4 \geq M_3)/\{i_t\} \rightarrow (V\{M_4, \{i_t\}, F\}=0 \wedge$
$V\{M_4, \{i_t\}, G\}=1)\}$. And substituting $M_4=M_3$ we obtain: $(V\{M_3, \{i_t\}, G\}=0 \vee$
$\vee V\{M_3, \{i_t\}, H\}=1)$ and $(V\{M_3, \{i_t\}, F\}=0 \vee V\{M_3, \{i_t\}, G\}=1)$.

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[1] Point out, we do not cite the use of (2d)—(7d) and (2d0)—(7d0); we do not cite the use of the inductive assumption and writing $(\exists M^\circ)$ the model M° is used as a constant in the continuation.

All cases give easy a contradiction with the above.

E_8) $V\{M, \{i_l\}, E_8\}=0$, $(\exists M_1)\{(M_1 \geq M)/\{i_l\} \wedge V\{M_1, \{i_l\}, F \supset G\}=1 \wedge$
$\wedge V\{M_1, \{i_l\}, (H \supset G) \supset (\{F+H\} \supset G)\}=0\}$,
$(\exists M_2)\{(M_2 \geq M_1)/\{i_l\} \wedge V\{M_2, \{i_l\}, H \supset G\}=1 \wedge V\{M, \{i_l\}, (F+H) \supset G\}=0\}$,
$(\exists M_3)\{(M_3 \geq M_2)/\{i_l\} \wedge V\{M_3, \{i_l\}, F+H\}=1 \wedge V\{M_3, \{i_l\}, G\}=0\}$ and by L. 3. also
$V\{M_3, \{i_l\}, F \supset G\}=1 \wedge V\{M_3, \{i_l\}, H \supset G\}=1$. Hence:
$(M_4)\{(M_4 \geq M_3)/\{i_l\} \rightarrow (V\{M_4, \{i_l\}, F\}=0 \vee V\{M_4, \{i_l\}, G\}=1)\}$ and
$(M_4)\{(M_4 \geq M_3)/\{i_l\} \rightarrow (V\{M_4, \{i_l\}, H\}=0 \vee V\{M_4, \{i_l\}, G\}=1)\}$.
Putting $M_4=M_3$ the case $V\{M_3, \{i_l\}, G\}=1$ gives a contradiction with the above. The last cases $V\{M_3, \{i_l\}, F\}=0$ and $V\{M_3, \{i_l\}, H\}=0$ also give a contradiction with the above: $V\{M_3, \{i_l\}, F+H\}=1$.

E_9) $V\{M, \{i_l\}, E_9\}=0$, $(\exists M_1)\{(M_4 \geq M)/\{i_l\} \wedge V\{M_1, \{i_l\}, F \supset G\}=1$
$\wedge V\{M_1, \{i_l\}, (F \supset G') \supset F'\}=0\}$,
$(\exists M_2)\{(M_2 \geq M_1)/\{i_l\} \wedge V\{M_2, \{i_l\}, F \supset G'\}=1 \wedge V\{M_2, \{i_l\}, F'\}=0\}$,
$(\exists M_3)\{(M_3 \geq M_2)/\{i_l\} \wedge V\{M_3, \{i_l\}, F\}=1\}$ and by L. 3. $V\{M_3, \{i_l\}, F \supset G\}=1$ and
$V\{M_3, \{i_l\}, F \supset G'\}=1$. Hence;
$(M_4)\{(M_4 \geq M_3)/\{i_l\} \rightarrow (V\{M_4, \{i_l\}, F\}=0 \vee V\{M_4, \{i_l\}, G\}=1)\}$ and
$(M_4)\{(M_4 \geq M_3)/\{i_l\} \rightarrow (V\{M_4, \{i_l\}, F\}=0 \vee V\{M_4, \{i_l\}, G'\}=1)\}$.
The possible case in last two lines is only $V\{M_4, \{i_l\}, F\}=0$.
But putting $M_4=M_3$ we obtain a contradiction with the above:
$V\{M_3, \{i_l\}, F\}=1$.

The verification of other axioms is immediately.
We verify proof rules:

R_1) Let $F \in P(k)$ and $F \supset G \in P(k)$; then by L. 12. we can assume $\{i_l\} \supset \{i_{w(F \supset G)}=i_{w(G)}\}$ and $\{i_l\} \supset \{i_{w(F)}\}$. Let $V\{M, \{i_l\}, G\}=0$ but in view of the assumptions $V\{M, \{i_l\}, F\}=1$ and $V\{M, \{i_l\}, F \supset G\}=1$; hence $V\{M, \{i_l\}, G\}=1$, i.e. a contradiction.[2]

R_2) Let a contrary $V\{M, \{i_l\}, F \supset \prod a G(a/x)\}=0$; therefore
$(\exists M_1)\{(M_1 \geq M)/\{i_l\} \wedge V\{M_1, \{i_l\}, F\}=1 \wedge V\{M_1, \{i_l\}, \prod a G(a/x)=0\}$,
$(\exists i)(\exists M_2)\{(M_2 \geq M_1)/\{i_l\} \wedge V\{M_2, \{i_l\}, i, G(x_i/x)\}=0\}$ and by L. 3. also
$V\{M_2, \{i_l\}, i, F\}=1$; hence $V\{M_2, \{i_l\}, i, F \supset G(x_i/x)\}=0$. And by virtue of L. 12. we have a contradiction, for $x \bar{\in} F$.

R_3) Let a contrary $V\{M, \{i_l\}, \sum a F(a/x) \supset G\}=0$; therefore

[1] See footnote 1, page 15.
[2] In the case of (3D) we iterate the process.

$(\exists M_1)\{(M_1 \geq M)/\{i_l\} \wedge V\{M_1, \{i_l\}, \sum aF(a/x)\}=1 \wedge V\{M_1, \{i_l\}, G\}=0\}$.

First recall $\{i_l\}$ is non-empty; then alternatively:

$(\exists i)\{(i \in \{i_l\}) \wedge V\{M_1, \{i_l\}, F(x_i/x)\}=1$ and so $V\{M, \{i_l\}, F(x_i/x) \supset G\}=0$ and by virtue of L. 12. it contradicts with the inductive assumption, for $x \bar{\in} G$.

So the proof of (I) is closed.

The proof of (II) (1) is partly included in the above and it is analogical. The reader should write it and let him point out that $F_1''' \supset F_1' \in P(k)$. Hence the second part of (II) (2) follows from the first one; and it remains the following complemention:

The first part of (II) (2) for axioms follows from L. 6. and $E_{13}'' \in P(k)$.[1)]

Hence, let $F \in PW(k)$ and $F \supset G \in PW(k)$; hence by the inductive assumption $F'' \in P(k)$ and $(F \supset G)'' \in P(k)$. So it suffices to prove $F'' \supset G'' \in P(k)$.

Let a contrary: $V\{M, \{i_l\}, F'' \supset G''\}=0$; then alternatively:

$(\exists M_1)\{(M_1 \geq M)/\{i_l\} \wedge V\{M_1, \{i_l\}, F''\}=1 \wedge V\{M_1, \{i_l\}, G''\}=0\}$,

$(\exists M_2)\{(M_2 \geq M_1)/\{i_l\} \wedge V\{M_2, \{i_l\}, G'\}=1\}$, $(M_3)\{(M_3 \geq M_2)/\{i_l\} \rightarrow V\{M_3, \{i_l\}, G\}=0\}$.

Hence by L. 3. $V\{M_2, \{i_l\}, F''\}=1$, $V\{M_2, \{i_l\}, G\}=0$; and

$(M_4)\{(M_4 \geq M_2)/\{i_l\} \rightarrow V\{M_4, \{i_l\}, F'\}=0\}$, $(\exists M_5)\{(M_5 \geq M_4)/\{i_l\} \wedge V\{M_5, \{i_l\}, F\}=1$;

the last conclusions also follow from $F'' \in P(k)$.

But $V\{M_5, \{i_l\}, (F \supset G)''\}=1$; hence $(M_6)\{(M_6 \geq M_5)/\{i_l\} \rightarrow V\{M_6, \{i_l\}, (F \supset G)'\}=0\}$,

$(\exists M_7)\{(M_7 \geq M_6)/\{i_l\} \wedge V\{M_7, \{i_l\}, F \supset G\}=1\}$,

$(M_8)\{(M_8 \geq M_7)/\{i_l\} \rightarrow (V\{M_8, \{i_l\}, F\}=0 \vee V\{M_8, \{i_l\}, G\}=1)\}$.

From the above by L. 3. also $V\{M_8, \{i_l\}, F\}=1$; therefore the case $V\{M_8, \{i_l\}, F\}=0$ is impossible and it remains: $V\{M_8, \{i_l\}, G\}=1$.

Putting $M_3 = M_8$ in $(M_3)\{(M_3 \geq M_2)/\{i_l\} \rightarrow V\{M_3, \{i_l\}, G\}=0\}$ we obtain also $V\{M_8, \{i_l\}, G\}=0$, i.e. a contradiction.

And it proves T. 1.

Starting to the inverse implication of T. 1. (I) we recall:

A set U of formulas is H_0-consistent iff the formula H_0 is not a thesis of the calculus with added axioms belonging to U. And for brevity we shall restrict ourselves only to the rule R_1. (In the general case we regard free variables of U as constants.) H_0 is constant and: $H_0 = F'$, for some F.

The construction of a maximal consistent family of formulas is analogical to Thiele [49]; my method of the construction of a maximal filter in the intuitionistic predicate calculus is published in another paper, see [21].

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1) The veryfication of R_3 is almost identical with the above. ie $\{i_l\}$ in (6d) and so we do not use here L. 3.; but the proof for the evaluation functional W uses L. 30.

Thus let

$(U)\quad H_1, \cdots, H_m, \cdots$

be all formulas of the predicate calculus and let us define:

U_0=the set of all theses of the calculus with closed axioms U^0,

$$U_{m+1}=\begin{cases} U_m, \text{ if } U_m\dotplus\{H_{m+1}\} \text{ is inconsistent;} \\ U_m\dotplus\{H_{m+1}\}, \text{ if the last set is consistent and } H_{m+1}\neq\prod aH'\supset H_0 \text{ or } H_{m+1}\neq \\ \neq\sum aH, \text{ for some } H, \text{ and } H_0 \text{ is given above;} \\ U_m\dotplus\{H_{m+1},\ H'(x/a)\supset H_0\}, \text{ if } U\dotplus\{H_{m+1}\} \text{ is consistent and } H_{m+1}=\prod aH'\supset H_0, \\ H_0 \text{ is the formula given above and } x\in U_m\dotplus H_{m+1}-U_0 \text{ with the smallest} \\ \text{index;} \\ U_m\dotplus\{H_{m+1},\ H(x/a)\}, \text{ if } U_m\dotplus\{H_{m+1}\} \text{ is consistent and } H_{m+1}=\sum aH \text{ and} \\ x\in U_m\dotplus H_{m+1}-U_0 \text{ with the smallest index;} \end{cases}$$

$U=\sum_{m=0}^{\infty} U_m$[1]

The set U is H_0-maximal filter[2] with properties asserted in the following 6 lemmas of classical and intuitionistic calculi:

LU1. $E'\in U$ .iff. $E\bar{\in} U$.

LU2. $E\supset F\in U$ .iff. $E\bar{\in} U\vee F\in U$.

LU3. $E+F\in U$ .iff. $E\in U\vee F\in U$.

LU4. $EF\in U$ .iff. $E\in U\wedge F\in U$.

LU5. $\sum aH\in U$ .iff. $(\exists i)\{H(x_i/a)\in U\}$.

LU6. $\prod aH'\in U$ .iff. $(i)\{H'(x_i/a)\in U\}$.

If $\prod aH\in U$, then $(i)\{H(x_i/a)\in U\}$.

If $(i)\{H(x_i/a)\in U\}$, then $\prod aH''\in U$.

Proving alternatively the last lemmas we additionaly use (in LU1. and LU2.) the intuitionistic theses $(E\supset H_0)\supset\{(E'\supset H_0)\supset H_0\}$ and $(E\supset H_0)\supset(\{(E\supset F)\supset H_0\}\supset H_0)$ for $H_0=F'$, for some F.[3]

LU1.—LU6. give the reason of the appearance of the intuitionistic predicate calculus asserted in the following two theorems:

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[1] The proof of consistency of U remains for readers; we use here the formal theorems of the intuitionistic propositional calculus:
$\{(H'\supset H_0)\supset H_0\}\supset\{(H_0\supset H_0)\supset H'\}$, $\{(H_0\supset H_0)\supset H\}\supset\{\{H\supset H_0)\supset H_0\}$; but
$\{(H\supset H_0)\supset H_0\}\supset\{(H_0\supset H_0)\supset H\}$ is not a thosis of one. We also used the set denotations of sum, difference and infinite sum alternatively: $+$, $-$, $\sum_{m=0}^{\infty}$.

We use the deduction theorem without citing it.

[2] So U creates a saturated theory.

[3] If the conditions do not hold, then both formulas are not theses of the calculus.

Let W_0 be the classical evaluation functional in a usual model; then obviously:

T. 2. Let U be H_0-maximal filter and let us regard only such for formulas which have ≥ 1 negation after each general quantifier; let $F_j^m \in M$ and let: $F_j^m(r_1, \cdots, r_m)$ .iff. $f_j^m(x_{r_1}, \cdots, x_{r_m}) \in U$; then:

$$E \in U \text{ .iff. } W_0\{M, E\}=1 .$$

The easy inductive proof of T. 2. is published, for instance, in [21]; it also contains Skolem-Lövenheim's theorem.

T. 1. and T. 2. give a strong absorption relation asserted in the following completeness theorem:

T. 3. H_0 is classically valid iff H_0 is a thesis of the classical predicate calculus. Let now $H_0=F'$, for some F; then:

If H_0 has at least 1 negation after each general quantifier, then:

H_0 is classically valid iff H_0 is a thesis of the intuitionistic predicate calculus. (The structure of H_0 gives further generalizations of the last theorem.)

So the strong absorption relation means: write 2 negations after each general quantifier of a regarded formula and negate it 2-times, if it is not negated (it is a very important property of the calculus, e.g. defining $\prod$ by means of $\sum$).

Therefore we can determine the classically valid formulas according to the last strong absorption relation writing a suitable formula instead of E in D. 4. Try to reformulate D. 4. without the added negations?

It remains to complete the construction by maximal non-classical filters; ones have the following properties (H_0-arbitrary):

L1U. If $E' \in U$, then $E \bar{\in} U$. If $E \bar{\in} U$, $E' \bar{\in} U$, then $E'' \in U$.

L2U. If $E \supset F \in U$, then $E \bar{\in} U$ or $F \in U$.

If $F \in U$, then $E \supset F \in U$. If $E=G' \bar{\in} U$, for some G, then $E \supset F \in U$.

And LU3.—LU6. without changes.

So the difference between H_0-maximal filter, with arbitrary H_0, and H_0-maximal filter with $H_0=F'$, for some F, is in L1U and L2U; proving ones we additionally use the following intuitionistic theses: $E+E'+E''$ and $(E'+F)\supset(E\supset F)$, respectively.

The basic idea of the following proof is a construction of an intuitionistic chain family Q^0.

First of all regard an infinite sequence of regarded calculi: $L_0, L_1, \cdots, L_m, \cdots$, where L_{m+1} results from L_m by adding an infinite number of new free variables, $m=0, 1, \cdots$, and L_0 also has an infinite number of free variables.

Guess L_ω is the union of all calculi: $L_0, L_1, \cdots$

For an arbitrary maximal filter U in L_m introduce the following denotations —called: extension chain properties:

If $E \in U$, then put $U_E = U$.

If $F' \in U$, then $U+\{F\}$ is F'-consistent (by E_9); hence there is F'-maximal filter $U_1 \supset U \dotplus \{F\}$ in L_m with $F' \in U_1$; denote the last $U_1 = U_{F'}$.

If $F \supset G \in U$, then it is a G-maximal filter $U_2 \supset U \dotplus \{F\}$ in L_m with $G \in U_2$; denote it $U_2 = U_{(F \supset G)}$.

If $\prod aF \in U$, then for $x \in L_m$ there exists $F(x/a)$-maximal filter $U_3 \supset U$ in L_{m+1} with $F(x/a) \in U_3$, for some $x \in F$; denote it $U_3 = U_{\Pi aF}$.[1)]

If E is not of the last 4 forms, then also $U = U_E$.

Let now U_0 be an arbitrary H_0-maximal filter in L_0; so $H_0 \in U_0$ and H_0 is not a thesis in L_0. Guess

(1) $H_0, H_1, \cdots, H_n, \cdots$

is a sequence of all formulas of the calculus L_0 and define a multiple increasing chain of maximal filters:

$U_0 \subset U_{i1} \subset U_{i2} \subset \cdots$

1. $U_0 = U_{H_0}$, where H_0 is given above;
2. $U_{01} = U_{H_1}$, where U_{H_1} is defined above with $U = U_0$; obviously $U_0 \subset U_{01}$;
3. Guess we defined $U_0 \subset U_{01} \subset \cdots \subset U_{0m-1}$; then $U_{0m} = U_{H_m}$, where U_{H_m} is defined above with $U = U_{0m-1}$;
4. $U_{11} = U_{H_2}$, where U_{H_2} is defined above with $U = U_0$ and not U_{01} as in 2;
5. Guess we defined $U_0 \subset U_{11} \subset U_{12} \subset \cdots \subset U_{1m-1}$; then $U_{1m} = U_{Hm}$, where U_{H_m} is defined above with $U = U_{1m-1}$;
6. $U_{21} = U_{H_3}$, where U_{H_3} is defined above with $U = U_0$ and not U_{01} or U_{11} as in 2 or 4.
7. Guess we defined $U_0 \subset U_{21} \subset \cdots \subset U_{2m-1}$; then analogously $U_{2m} = U_{H_{m+2}}$, where $U_{H_{m+2}}$ is defined above with $U = U_{2m-1}$;
8. $U_{31} = U_{H_4}$, where U_{H_4} is defined above with $U = U_0$ and not U_{01}, U_{11} or U_{21} as as in 2, 4, 7;
9. And analogously to 7, 8 till infinity.

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1) If L_0 has a finite number of free variables, then in the definition of H_0-maximal filter, p. 13., omit the last two lines: " $U_m \dotplus \{H_{m+1}, H(x/a)\}$, if $U_m \dotplus H_{m+1}$ is consistent and $H_{m+1} = \cdots$".

Then guess also the following extension property:

If $\sum aF \in U$, then for $x \overline{\in} L_m$ there exists $(\sum aF)'$-maximal filter $U_4 \supset U \dotplus \{F(x/a)\}$ in L_{m+1} with $F(x/a) \in U_4$, for some $x \overline{\in} F$; denote it $U_4 = U_{\Sigma aF}$.

So $L_0, L_1, \cdots$ may have a finite number of free variables but L_ω must have an infinite number of ones.

Thus we obtained a sequence of maximal filters U_0, U_1^1, U_2^1, $\cdots$ and we go over to the calculus L_1 replacing (1) by the sequence of all formulas of L_2.

Perform operations 1—9 with the last maximal filters (i.e. replacing U_0 by alternatively: U_1^1, U_2^1, $\cdots$). Hence we obtain a new sequence of maximal filters U_0, U_1^2, U_2^2, $\cdots$ and we go over to the calculus L_3; and iterate the operations till infinity.

The results of those operations give the intuitionistic chain family.

Because L_0 has an infinite number of free variables, therefore it is a one-one correspondence g between formulas of L_ω and those free variables denoting: $g(x)$ $=E_{g(x)}$, $x \in L_0$.

Guess now S is a new sign of a monadic relation, i.e. $S \bar{\in} L_\omega$, and let U be a regarded maximal filter; then put:

(S) $$S(x) \in U \text{ .iff. } E_{g(x)} \in U .$$ [1]

D. 6. Two different definitions:

(1) Q^0 has the following elements: the maximal filter U_0 in L_0 with $H_0 \bar{\in} U_0$ and all filters of the generated intuitionistic chain with the described extension chain properties and (S);

(2) Q^0 is the set of all maximal filters (or with restrictions to parts of H_0) with all generated chains and extension properties described above. (So we start here not from U_0 but from an arbitrary maximal filter.)

D. 7. For D. 6., (1):

(1, 1) $(U_2 \geq U_1)/\{i_l\}$ iff $U_2/\{i_l\}$ contains all atoms belonging to U_1 with indices belonging to $\{i_l\}$. For D. 6., (2):

(1, 2) $(U_2 \geq U_1)/\{i_l\}$ iff $U_2/\{i_l\}$ contains all formulas belonging to $U_1/\{i_l\}$, where $U/\{i_l\}$ is the set of all formulas belonging to U with indices belonging to $\{i_l\}$. And it is reformulated analogously to (1, 1) introducing new signs of monadic relations $g_1, g_2, \cdots$

So let $E_1, E_2, \cdots$ be all formulas of the predicate calculus; then put: $g_i(0) \in U$ iff $E_i \in U$, $i=1, 2, \cdots$; then obviously (1, 2) has the sense of (1, 1) but for the generated set of maximal filters with new signs $g_1, g_2, \cdots$.

Of course the chains Q^0 generated by H_0 in D. 6. and D. 7. have the following propertes:

(1°) If $U_1, U_2 \in Q^0$, $\{i_l\}=0, 1, \cdots$, then:

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[1] If L_0 has a finite number of free variables, then S is replaced by a finite number of monadic relations: $S_1, S_2, \cdots, S_r$. (Then we have more formulas of L_0 than free variables; and analogously for $L_1, L_2, \cdots$)

$(U_2 \geq U_1)/\{i_l\}$ .iff. $(U_1, U_2$ are equal for atoms$) \vee (U_2 \supset U_1)$.

(2°) For subformulas of a given formula H_0 and an arbitrary finite sequence $\{i_l\}$ the chain $U_0/\{i_l\} \subset U_1/\{i_l\} \subset \cdots$ has a finite number of different elements, i.e. they are only finite chains composed of subformulas of H_0 with indices $\{i_l\}$ in the described form.

Lindenbaum's algebras of subformulas of H_0 or D. 7. (1, 2) indicate analogical asymptoticly finite adequate generalized models for other (constructive) calculi, e.g. restricting ourselves to Q with properties: $F'+F'' \in P(k, Q)$ or $(F \supset H) \supset \supset \{(F' \supset H) \supset E\} \in P(k, Q)$, for closed H; and in the last case we obtain LU1.—LU6. for closed H.

The reader should remember the generation of Q^0 and all properties of D. 6. and D. 7., for we do not cite their applications.

T. 4. 1. Let $\{i_l\}=0, 1, \cdots$ and let Q_0 be the set of all models M^u defined in the equivalence:

$$F_j^m(r_1, \cdots, r_m) \in M^u \text{ .iff. } f_j^m(x_{r_1}, \cdots, x_{r_m}) \in U;$$

Then:

$$V\{Q_0, M^u, \{i_l\}, E\}=1 \text{ .iff. } E \in U .$$

Proof:—The case D. 6. (1) and D. 6. (2) are similar; so we restrict ourselves to D. 6. (1) and D. 7. (1) and the appropriate reading of the following proof remains for readers.

Because $\{i_l\}=0, 1, \cdots$, $U_0 \subset U$, for all $U \in Q^0$ and (1°), therefore:

If $M^{u_1}, M^{u_2} \in Q_0$, then:

$$(M^{u_2} \geq M^{u_1})/\{i_l\} \text{ .iff. } U_2 \supset U_1 .$$

And we shall write $(M^{u_2} \supset M^{u_1})$ instead of $(M^{u_2} \geq M^{u_1})/\{i_l\}$.

Of course, T. 4. 1. holds for atoms.

Let it hold for formulas of the length $<m$; we shall prove it for formulas of the length m:

We regard different cases according to the structure of E:

1. $E=F'$, for some F; then by the inductive hypothesis:
$V\{M^u, F'\}=1. \rightarrow .(M^{u_1})\{(M^{u_1} \supset M^u) \rightarrow V\{M^{u_1}, F\}=0\}. \rightarrow .(U_1)\{(U_1 \supset U) \rightarrow (F \notin U_1)\}.$[1]
Putting $U_1=U_{F'}$ we obtain a contradiction; and the inverse implication:
$F' \in U. \rightarrow .(U_1)\{(U_1 \supset U) \rightarrow (F' \in U_1)\}. \rightarrow$
$(U_1)\{(U_1 \supset U) \rightarrow (F \notin U_1)\}. \rightarrow .(M^{u_1})\{(M^{u_1} \supset M^u) \rightarrow V\{M^{u_1}, F\}=0\}. \rightarrow . V\{M^u, F'\}=1$.

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[1] $V\{M, E\}$ is an abbreviation of $V\{Q_0, M, \{i_l\}, E\}$ for $\{i_l\}=0, 1, 2, \cdots$.

[2] And if Q° is one-elementing, then the reader should simultaneously verify the classical completeness proof.

2. $E=F\supset G$, for some F and G; then by the inductive hypotheis:
$V\{M^u, F\supset G\}=1.\rightarrow.(M^{u_1})\{(M^{u_1}\supset M^u)\rightarrow(V\{M^{u_1},F\}=0\vee V\{M^{u_1}, G\}=1)\}$
$\rightarrow(U_1)\{(U_1\supset U)\rightarrow(F\bar{\in} U_1\vee G\in U_1)\}$; putting $U_1=U_{(F\supset G)}$ we obtain a contradiction. And the inverse implication:
$F\supset G\in U.\rightarrow.(U_1)\{(U_1\supset U)\rightarrow(F\supset G\in U_1)\}.\rightarrow.(U_1)\{(U_1\supset U)\rightarrow(F\bar{\in} U_1\vee G\in U_1)\}.\rightarrow.$
$.\rightarrow.(M^{u_1})\{(M^{u_1}\supset M^u)\rightarrow(V\{M^{u_1}, F\}=0\vee V\{M^{u_1}, G\}=1)\}.\rightarrow. V\{M^u, F\supset G\}=1$.
3. The cases that E is an alternative or conjuction of two formulas are trivial and remain for readers.
4. Because $\{i_l\}=0, 1, 2,\cdots$, therefore the case $E=\sum aF$ is also immediately.
5. $E=\prod aF$, for some F; then by the inductive hypothesis:
$V\{M^u, \prod aF\}=1.\rightarrow.(M^{u_1})(i)\{(M^{u_1}\supset M)\rightarrow(V\{M^{u_1}, F(x_i/a)\}=1\}.\rightarrow.$
$.\rightarrow.(U_1)(i)\{(U_1\supset U)\rightarrow(F(x_i/a)\in U_1)\}$. Putting $U_1=U_{\Pi aF}$ we obtain a contradiction; the inverse implication:
$\prod aF\in U.\rightarrow.(U_1)(i)\{(U_1\supset U)\rightarrow(F(x_i/a)\in U_1)\}.\rightarrow.$
$.\rightarrow.(M^{u_1})(i)\{(M^{u_1}\supset M^u)\rightarrow V\{M^{u_1}, F(x_i/a)\}=1.\rightarrow. V\{M^u, \prod aF\}=1$, q.e.d.

T. 4. 1. with L. 30. and $W=V$ in the domain of all numbers $\geqslant 0$ imply:

T. 4. 2. Let $\{l\}\supset i(E)$ and let here hold other assumptions of T. 4. 1., then:

$$W\{\aleph_0, Q_0, M^u, \{l\}, E\}=1.\rightarrow E\in U/\{l\} .$$

It remains to replace $\aleph_0$ in T. 4. 2. by an arbitrary number $k\geq n(E)$:

First of all note that adding to each model a new sequence of monadic relations $h_1, h_2,\cdots$, with $h_i(j)$ iff $i=j$, we obtain the following property introduced in my cited papers:

$R(Q).\text{iff}.(M_1)(M_2)(i)(j)\{(M_1, M_2\in Q)\wedge(M_1/i/=M_2/j/)\rightarrow(i=j)\}$

So after the last extension of models in D. 6. (1) with D. 7. (1) we obtain a generalized model Q_0 with $R(Q_0)$; and it holds:

T. 5. Let Q be the smallest set of finite models of power k with $Q\supset M_1[k]$, for all $M_1\in Q_0$, $M\in Q_0$, $M^*\in Q$, $F\in E$, $k\geq n(E)$ and $l+p(F)\leq k$; then:

(1°) If $R(Q_0)$ and $M/\{i_l\}=M^*/\{l\}$, then:
$W\{\aleph_0, Q, M, \{\aleph_0\}, F\}=1$ .iff. $W\{k, Q, M^*, \{l\}, F\}=1$;

(2°) If $(M_1\geq M)/\{l\}$ is replaced by $(M_1=M)/\{l\}$ in (0d)—(0d6), i.e. we regard the new "classical" predicate calculus, with one-elementing $Q_0=\{M^0\}$ and E is an alternative of closed formulas $\sum a_1\cdots\sum a_i\prod a_{i+1}\cdots\prod a_m H$, where H has not quantifiers, and $W\{\aleph_0, Q_0, M^0, \{\aleph_0\}, E\}=0$;[1] then omitting the last relation R

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[1] It gives the evaluation function in the model M.

T. 5. (2°) may be generalized but it is formulated with respect to the classical predicate calculus.

we obtain: If $M^0/\{i_l\}=M^*/\{l\}$ and $W\{\aleph_0, Q, M^0, \{\aleph_0\}, F\}=0$, then $W\{k, Q, M^*, \{l\}, F\}=0$; so $E \in P^0$ in the new classical predicate calculus with E_{13} restricted to propositional formulas.

Proof:—We prove (1°) and (2°) by induction with respect to the length of the formula F. And the proof of (1°) is simple in our assumption having noted:

$$R(Q_0)\wedge(M/\{i_l\}=M^*/\{l\})\wedge(M_1/\{j_l\}=M_1^*/\{l\})\wedge((M_1/\{j_1\}\geq M)/\{i_l\}\vee$$
$$\vee(M_1^*/\{l\}\geq M^*)/\{l\}).\rightarrow(i_1=j_1)\wedge\cdots\wedge(i_l=j_l)$$

The whole proof is similar to my published one and so we restrict ourselves to 3 cases:

If F is an atom, then obviously (1°) holds.

Let (1°) hold for formulas of the length $<n$; we prove it for formulas of the length n:

Regard cases:

(1′) $F=G'$, for some G.

Then by (0d2) and the inductive assumption:

$$W\{\aleph_0, Q_0, M, \{\aleph_0\}, F'\}=1 \text{ .iff. } (t)(M_1)\{(t\geq l)\wedge(M_1\geq M)\rightarrow$$
$$\rightarrow W\{\aleph_0, Q, M_1, \{\aleph_0\}, F\}=0\} \text{ .iff. } (t)(M_1^*)\{(l\leq t\leq k-p(F))\wedge(M_1^*\geq M^*)/\{l\}\rightarrow$$
$$\rightarrow W\{k, Q, M_1^*, \{t\}, F\}=0\} \text{ .iff. } W\{k, Q, M_1^*, \{l\}, F'\}=1\ .$$

(2′) $F=\prod aG$, for some G.

Then by (0d6) and the inductive assumption:

$$W\{\aleph_0, Q_0, M, \{\aleph_0\}, \textstyle\prod aF\}=1 \text{ .iff. } (i)(M_1)\{(M_1\geq M)\rightarrow$$
$$\rightarrow W\{\aleph_0, Q_0, M_1, \{\aleph_0\}, F(x_i/a)\}=1\} \text{ .iff. } (i)(M_1^*)\{i\leq k)\wedge(t=\max(i, l))\wedge$$
$$\wedge(M_1^*\geq M^*)/\{l\}\rightarrow W\{k, Q, M_1^*, \{t\}, F(x_i/a)\}=1\} \text{ .iff. } W\{k, Q, M, \{l\}, \textstyle\prod aF\}=1\ .$$

The proof of (2°) is almost identical with the published one but in (2′) we have only an implication and starting the induction from H we point out that each substitution of $\prod a_{t+1}\cdots\prod a_m H$ is always 0 in the model M^0.

The whole proof remains for readers.

The generalized model Q defined in T. 5. (1) has the embedding property, page 12, and my invariance relation holds for Q.

From T. 4. 1. and T. 5. follow:

T. 6. 1. Classical predicate calculus:—Replace $(M_1\geq M)/\{l\}$ by $(M_1=M)/\{l\}$ and Let E be in a normal form; then:

E is a thesis of the classical predicate calculus iff $E\in P$.

T. 6. 2. Intuitionistic predicate calculus:—E is a thesis of the calculus iff

[1] We regard here the functional W and Skolem's normal forms remain true.
If we define $\prod$ by $\sum$, the assumption of normal forms may be omitted.

$E \in P$.

And T. 6. 2. for Heyting's predicate calculus is proven in the following cases of definitions (0d)—(0d6):

1. $k=\aleph_0$ and Q_0 multiple increasing chains, i.e. a tree of models with a defined model included in all other ones.
2. The last Q_0 enlarged by means of monadic relations $h_1, h_2, \cdots$ with $h_i(j)$ iff $i=j$ and truncated to arbitrary $k \geqslant n(E)$.
3. $k=\aleph_0$ and Q the set of all models of power k according to D. 6. (2) with D. 7. (1, 2). The enlargement of Q by means of new relations and truncation to arbitrary $k \geq n(E)$ remain here for readers.

And in view of my bad existence in Israel it was open, if the infinity $\aleph_0$ can be replaced by an arbitrary number $k \geq n(E)$ in T. 4. 2. of Heyting's calculus in the general case without an infinite sequence of monadic relations. So let us note that it can be reduced to a given finite number of monadic relation in the following manner:

Look to definitions (0d2)—(0d6)!—

For a given finite k the operators of conjuction, alternative and $\sum$ correspond only one model to formulas but the operators of negation, implication and $\prod$ correspond a finite set of models to the formulas. The same picture we have in a domain of power $\aleph_0$ if we restrict it for a moment to a given finite k; and the last correspondence may be preserved for finite k, if we add a finite number of new monadic relations likely to T. 5. (1°). But the last number of those monadic relations tends to infinity for $k \to \infty$.

Such is a temporary reply to the last question and it is not so very important in this paper, for we determided intuitionistic predicate calculi with algorithmic approximations according to T. 6. and D. 4. (e.g. defining also $\prod$ by $\sum$); so formulas of the intuitionistic calculi can be verified in many ways and let us indicate three ones:

1. The evaluation functional W of (0d)—(0d6);
2. The functional W with a finite number of added monadic relations;
3. Replacing the inequality of (0d)—(0d6) by the equality: $(M_1 = M)/\{l\}$.

The determined predicate calculi are constructive by virtue of the above and L. 8. or additionally:

L. 8. 0. Let for each $M_1, M_2 \in Q$:

Either $(M_1 \geq M_2)/\{0\}$ or $\{M_2 \geq M_1)/\{0\}$.

(It may be said that it is not a restriction for Q, if we define all relations by means of one relation of $n \geqslant 2$ arguments; and it is an exercise for readers.)

Let F and G be closed and $F+G \in P(k, Q)$; then $F \in P(k, Q)$ or $G \in P(k, Q)$.

An immediate proof is a contrary. Let L. 8. 0. do not hold. Then there exist $M_1, M_2 \in Q$ such that: $W\{M_1, \{0\}, F\}=0$ and $W\{M_2, \{0\}, G\}=0$. By the assumption we may guess: $(M_1 \geq M_2)/\{0\}$; hence by virtue of L. 30.: $W\{M_1, \{0\}, G\}=0$; hence $W\{M_1, \{0\}, F+G\}=0$, i.e. a contradiction with the assumptions.

D. 4. and T. 6. give different generalizations of Herbrand's theorems in the constructive calculi.[1]—

Regarding here an arbitrary formula E we may consider even and odd quantifiers of E (e.g. $\prod$ is in the scope of even or odd number of negations of E, respectively), and the complete description of Herbrand's theorems remain for readers (with different forms of their formulation).

The reader should note that (3d) may be replaced by:

(3d′) $V\{k, Q, M, \{i_l\}, F \supset G\}=1$ .iff. $(M_1)\{(M_1 \geq M)/\{i_l\} \rightarrow$
$\rightarrow (V\{k, Q, M_1, \{i_l\}, F'\}=1 \vee V\{k, Q, M_1, \{i_l\}, G\}=1)\}$.

Then it is possible to regard the structure of F in (2d) with further generalizations.

My lectures in the Annual Meetings of Dutch Mathematicians and Bonn Mathematical Seminar of 1972 replace the quantifier (t) in (0d2) and (0d3) by new indices according to the following example:

(d′2) $V\{k, Q, M, \{i_l\}, F'\}=1$ .iff. $(M_1)(i)\{(i \leqslant k) \wedge (1+n(E) \leq k) \wedge$
$\wedge (M_1 = M)/\{i_l\} \rightarrow V\{k, M, \{i_l\}, i, F\}=0\}$

And analogously for implication and general quantifier, i.e. the inequality is replaced by the equality with a longer sequence $\{i_l\}, i$ instead of $\{i_l\}$ but then $F+F'$ is falsed only, if F contains the general quantifier; of course, the last replacement is a special kind of an inclusion.

According to L. 8. and L. 8. 0. we obtain different constructive properties of the intuitionistic predicate calculi and for the functional W and closed formulas:

L. 13. $F+G \in P$ .iff. $(F \in P) \vee (G \in P)$

Properties of evaluation functionals in Generalized Models closed under permutations of numbers likely to my published papers remain for readers.

The paper gives new ultraproducts; and adequate characterizations of constructive theories by means of generalized ultraproducts will be published . . .

T. 3. and T. 6. 1. with T. 6. 2. give simple deductions of formulas of regarded calculi:

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[1] For we start from U_0, see footnotes 1, p. 20.

So as exercises—assuming that free variables of E are constants prove immediately:

$\prod aE\supset(\sum aE')'\in P$, $\sum aE\supset(\prod aE')'\in P$, $(F\supset G)\supset(G'\supset(G'\supset F')\in P$, and Glivenko's theorems: $\{E'\supset(F\supset G')\}\supset\{E'\supset(F''\supset G')\}\in P$, $(E\supset F)\supset(E\supset F'')\in P$.

The above considerations imply Generalized Models for Modal Predicate Calculi and they will be published in another paper, [1], [4]—[21].

Introducing the Theory of Natural Numbers we suppose that all elements of a Generalized Model are identical on signs of natural numbers; then we obtain all analogical theorems with both constructive properties of $+$ and $\sum$, [41]—[49].

Partition calculi in generalized models will be published in an other paper, [37], [39], [2], [3].

Forcing Theory has a simpler picture in Generalized Models, [32], [38], [40], with stronger theorems.

Of course, the regarded predicate calculi are decidable with probability 1 and decidable in partitions, [30], [31], [33], [34], [36], [37], [39], [50].

I owe the occasion to reassume:

The exposition deals with the oldest problem of sciences—the decidability problem—solved first in primitive computations, afterwards in partition theorems and according to this lecture and we presented three ways of algorithmization of mathematical sciences:

1. Evaluation in generalized models.
2. Evaluation in usual models with my truncated general quantifier.
3. Evaluation in usual models with the usual interpretation of quantifiers but with a negation of an intuitionistic property or with a classical negation and with a modal operator or topology. Having received the results, it appears a problem:

Is it the best algorithmization?

And analogously to the lecture, about a half year ago, I obtained generalized models for the classical predicate calculus composed only of 2 finite models $M_0\subset M^0$ with stronger complementions of important results.

Hence, taking the Cartesian product of both models M_0 and M^0 and identifying two non-designated values $(0,1)$ and $(1,0)$ we obtain Lukasiewicz's epoch (citation) result of 3-valued calculus.

Thus was created a general approach in algorithmization of mathematical sciences (with many valued calculi) and my assertion:

Finite mathematical sciences are a generalization of infinite ones.

And it is an obvious conclusion from the lecture with indicated differences in both ways: finite and infinite ones.

So is sketched the unification of mathematical achievements—the only one right direction of knowledge in compulsory studies of mathematical sciences including statistics and especially—computers; and many new-old problems with automatical predictions of artificial inteligence are seen ...

Generalized Models demand monographies.

I believe that this approrch gives the reader a feeling in the epoc ideas with their strength and influence at forms of mathematical outlooks of all generations.

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