ON VARIATIONAL PROBLEMS WHICH ADMIT AN INFINITE CONTINUOUS GROUP

By

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1. Introduction

This article is concerned with variational problems which are invariant under a certain group of transformations, a so-called infinite continuous group, which depends upon r arbitrary functions and their derivatives up to some given order q. Such problems were initially studied by *Emmy Noether* [1] who, in her classic paper on the subject, proved the basic result: If a variational integral is invariant under such a group, then there exist r identity relations between the variational derivatives and their derivatives up to order q. This theorem, now known as the Second Noether Theorem, has been discussed by *Hilbert* [2] in connection with electrodynamics and gravitation (relativity), by *Drobot* and *Rybarski* [3] in connection with hydromechanics, and by *Funk* [4], *Komorowski* [5], *Logan* [6, 7], *Plybon* [8], and others, in general.

In this paper we present a new, direct proof of the Second Noether theorem which avoids the fundamental variational formula for the variation of the action integral. In the course of the proof, a new set of invariance identities is obtained for single and multiple integral problems. These invariance identities, which imply the classical Noether identities, can be computed directly from the Lagrangian and the defining quantities of the group. Moreover, by introducing the variational derivatives into the invariance identities, it is shown that conservation laws can be derived under the assumption of invariance under an infinite continuous group. This differs from the usual result (First Noether Theorem) where invariance under a finite continuous group implies the conservation theorems.

We begin in Section 2 by formulating the Noether theorems and, in doing so, establishing the notation used throughout. The classical proof of the Second Noether Theorem is outlined in this section since a portion of it will be referred to in subsequent sections. Sections 3 and 4 contain the new proofs, as well as the

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invariance identities, for single and multiple integral problems, respectively; the multiple integral case is somewhat complicated by the appearance of a functional determinant.

2. The Second Noether Theorem

We consider a curve C in R^n given parametrically in the form

$$x^k = x^k(t)\,, \qquad k=1, 2, \cdots, n \tag{2.1}$$

where the parameter t ranges over $t_0 \leq t \leq t_1$. We assume that these functions are of class C^2 and we denote their derivatives by

$$\dot{x}^k(t) = \frac{dx^k}{dt}\,, \qquad \ddot{x}^k(t) = \frac{d^2x^k}{dt^2}\,.$$

We further assume that there is given a function, the Lagrangian, $L: R^{2n+1} \to R^1$ which is of class C^2 in all of its arguments. Then, we construct the integral of L along the curve C, namely,

$$J(C) = \int_{t_1}^{t_0} L(t, x(t), \dot{x}(t))\,dt\,, \tag{2.2}$$

where $x(t) = (x^1(t), \cdots, x^n(t))$ and $\dot{x}(t) = (\dot{x}^1(t), \cdots, \dot{x}^n(t))$.

For comparison, and since we shall discuss conservation theorem with respect to the Second Noether Theorem, we shall begin with the First Noether Theorem on invariant variational problems. We consider an r parameter group of transformations of the form

$$\bar{t} = \bar{t}(t, x, \varepsilon^s), \quad \bar{x}^k = \bar{x}^k(t, x, \varepsilon^s)\,, \tag{2.3}$$

where ε^s, $s=1, 2, \cdots, r$ denote the r independent parameters of the group. It is assumed that these transformations are C^∞ and that to the values $\varepsilon^1 = \varepsilon^2 = \cdots = \varepsilon^r = 0$ of the parameters correspond the identity transformation $\bar{t} = t$ and $\bar{x}^k = x^k$. The corresponding infinitesimal transformation associated with the group (2.3) is given by

$$\left.\begin{aligned} \bar{t} &= t + \tau_s(t, x)\varepsilon^s + o(|\varepsilon|)\,, \\ \bar{x}^k &= x^k + \xi^k_s(t, x)\varepsilon^s + o(|\varepsilon|)\,, \end{aligned}\right\} \tag{2.4}$$

where

$$|\varepsilon| = (\sum_s (\varepsilon^s)^2)^{1/2}\,, \qquad \lim_{|\varepsilon| \to 0} \frac{o(|\varepsilon|)}{|\varepsilon|} = 0\,,$$

and

$$\tau_s = \left.\frac{\partial \bar{t}(t, x, \varepsilon^s)}{\partial \varepsilon^s}\right|_{\varepsilon=0}, \qquad \xi^k_s = \left.\frac{\partial \bar{x}^k(t, x, \varepsilon^s)}{\partial \varepsilon^s}\right|_{\varepsilon=0}.$$

We will denote the above finite continuous group by G_r. The following definition will enable us to state the First Noether Theorem in concise form.

Definition. $J(C)$ is an invariant of the group G_r if

$$\int_{\bar{t}_0}^{\bar{t}_1} L\left(\bar{t}, \bar{x}, \frac{d\bar{x}}{d\bar{t}}\right) d\bar{t} = \int_{t_0}^{t_1} L(t, x, \dot{x}) dt . \tag{2.5}$$

Theorem 1. (Noether) If $J(C)$ is an invariant of the group G_r then there exist r identities of the form

$$-E_k(L)(\xi^k_s - \dot{x}^k \tau_s) = \frac{d}{dt}\left(\left(L - \dot{x}^k \frac{\partial L}{\partial \dot{x}^k}\right)\tau_s + \frac{\partial L}{\partial \dot{x}^k}\xi^k_s\right), \tag{2.6}$$

where $E_k(L)$ denotes the variational derivatives

$$E_k(L) \equiv \frac{\partial L}{\partial x^k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^k} . \tag{2.7}$$

Under the assumption that Euler-Lagrange equations are satisfied, i.e., $E_k(L)=0$ for all k, then there result r conservation laws of the form

$$\left(L - \dot{x}^k \frac{\partial L}{\partial \dot{x}^k}\right)\tau_s + \frac{\partial L}{\partial \dot{x}^k}\xi^k_s = \text{constant} . \tag{2.8}$$

We shall show in Section 3 how similar first integrals of the Euler-Lagrange Equations can be obtained from invariance under an infinite continuous group.

Rather than dependence on parameters, the Second Noether Theorem requires that the group of transformations depend upon arbitrary functions and their derivatives. In particular, we consider the transformation group

$$\left.\begin{aligned} \bar{t} &= \bar{t}(t, x, \phi_s(t), \phi_s^{(1)}(t), \cdots, \phi_s^{(q)}(t)) , \\ \bar{x}^k &= \bar{x}^k(t, x, \phi_s(t), \phi_s^{(1)}(t), \cdots, \phi_s^{(q)}(t)) , \end{aligned}\right\} \tag{2.9}$$

where $\phi_s(t)$, $s=1, \cdots, r$ are the r arbitrary, independent functions defined on $t_0 \leq t \leq t_1$, and $\phi_s^{(p)}$ denote their p^{th} derivatives. We are using the standard convention that $\phi_s(t) = \phi_s^{(0)}(t)$. The index p then ranges over $p=0, 1, \cdots, q$. It is assumed that to the values $\phi_s(t) = \phi_s^{(1)}(t) = \cdots = \phi_s^{(q)}(t) = 0$ correspond the identity transformation. The infinitesimal transformation associated with (2.9) is

$$\left.\begin{aligned} \bar{t} &= t + a_p^s(t, x)\phi_s^{(p)} + \cdots , \\ \bar{x}^k &= x^k + b_p^{ks}(t, x)\phi_s^{(p)} + \cdots , \end{aligned}\right\} \tag{2.10}$$

where the "dots" denote higher order terms involving products of the $\phi_s^{(p)}$, and

$$a_p^s = \frac{\partial \bar{t}}{\partial \phi_s^{(p)}}\bigg|_{\phi_s^{(p)}=0,\ \forall p \forall s},$$

and

$$b_p^{ks} = \frac{\partial \bar{x}^k}{\partial \phi_s^{(p)}}\bigg|_{\phi_s^{(p)}=0,\ \forall p \forall s}.$$

We denote the above infinite continuous group by $G_{r\infty}$ and, for conciseness, we use the following notation for the evaluation of a term at $\phi_s^{(p)}=0$, $\forall p \forall s$:

$$(*)|_{\phi_s^{(p)}=0,\ \forall p \forall s} \equiv (*)|_0 .$$

In terms of differential operators A^s and B^{ks}, we can write the infinitesimal transformation as

$$\left.\begin{aligned} \bar{t} &= t + A_s(\phi_s) + \cdots, \\ \bar{x}^k &= x + B^{ks}(\phi_s) + \cdots . \end{aligned}\right\} \tag{2.11}$$

We denote their formal adjoints by $\tilde{A}^s$ and $\tilde{B}^{ks}$. These are defined by the Lagrange identity

$$\int_{t_0}^{t_1} \psi H \phi \, dt = \int_{t_0}^{t_1} \phi \tilde{H} \psi \, dt + [\cdots]_{t_0}^{t_1},$$

where H is a differential operator and $\psi=\psi(t)$ and $\phi=\phi(t)$. The invariance of $J(C)$ under a $G_{r\infty}$ group is defined exactly as in the G_r group case.

Theorem 2. (Noether) If $J(C)$ is an invariant of the group $G_{r\infty}$, then there exist r identities involving the variational derivatives of the form

$$\tilde{B}^{ks}(E_k(L)) - \tilde{A}^s(\dot{x}^k E_k(L)) = 0 . \tag{2.12}$$

The classical proof of these identities is carried out by calculating the total variation ΔJ of J due to simultaneous variations Δt and Δx^k of the independent and dependent variables. This calculation, which is particularly involved for the multiple integral case (see *Gelfand-Fomin* [9]), yields the *fundamental variational formula* for J:

$$\Delta J(\Delta t, \Delta x^k) = \int_{t_0}^{t_1} E_k(L)(\Delta x^k - \dot{x}^k \Delta t) dt + \left[\left(L - \dot{x}^k \frac{\partial L}{\partial \dot{x}^k}\right)\Delta t + \frac{\partial L}{\partial \dot{x}^k}\Delta x^k\right]_{t_0}^{t_1}. \tag{2.13}$$

The invariance of J under $G_{r\infty}$ is the same as

$$\Delta J(A^s(\phi_s), B^{ks}(\phi_s)) = 0 . \tag{2.14}$$

Since the $\phi_s(t)$ are arbitrary, (2.14) holds for $\phi_s(t)$ which vanish along with their derivatives at t_0 and t_1. Hence according to (2.13) and (2.14), we obtain

$$\int_{t_0}^{t_1} E_k(L)(B^{ks}(\phi_s)-\dot{x}^k A^s(\phi_s))dt=0 .$$

Using the adjoint property and again the arbitrariness of the $\phi_s(t)$, we conclude that

$$\int_{t_0}^{t_1} [\tilde{B}^{ks}(E_k(L))-\tilde{A}^s(\dot{x}^k E_k(L))]\phi_s(t)dt=0 .$$

The independence of the $\phi_s(t)$ along with the Fundamental Lemma of the Calculus of Variations (see [9]) yield the Noether Identities (2.12).

We have given this classical proof since we shall refer in Sections 3 and 4 to its latter portion.

3. A New Proof-Single Integral Case

We now give a proof of Theorem 2 which avoids the fundamental variational formula, equation (2.13), and hence the concept of small variations. The direct calculation proceeds from the following evident, but basic, observation concerning the invariance of $J(C)$.

Remark 1. A necessary and sufficient condition for $J(C)$ to be invariant of $G_{r\infty}$ is that

$$L\left(\bar{t}, \bar{x}, \frac{d\bar{x}}{d\bar{t}}\right) d\bar{t}/dt = L(t, x, \dot{x}) , \tag{3.1}$$

for every $\phi_s(t)$, $\phi_s^{(1)}(t), \cdots, \phi_s^{(q)}(t)$. Differentiating (3.1) with respect to $\phi_s^{(p)}$ we obtain

$$L\left(\bar{t}, \bar{x}, \frac{d\bar{x}}{d\bar{t}}\right) \frac{\partial}{\partial \phi_s^{(p)}} \left(\frac{d\bar{t}}{dt}\right) + \frac{\partial}{\partial \phi_s^{(p)}} L\left(\bar{t}, \bar{x}, \frac{d\bar{x}}{d\bar{t}}\right) \frac{d\bar{t}}{dt} = 0 . \tag{3.2}$$

Evaluating at $\phi_s^{(p)}=0$, $\forall p \forall s$, and noting from (2.10) that

$$\left(\frac{\partial \bar{t}}{\partial \phi_s^{(p)}}\right)_0 = a_p^s , \quad \left(\frac{\partial \bar{x}^k}{\partial \phi_s^{(p)}}\right)_0 = b_p^{ks} , \quad \left(\frac{d\bar{t}}{dt}\right)_0 = 1 ,$$

equation (3.2) becomes

$$L(t, x, \dot{x}) \left(\frac{\partial}{\partial \phi_s^{(p)}} \frac{d\bar{t}}{dt}\right)_0 + \frac{\partial L}{\partial t} a_p^s + \frac{\partial L}{\partial x^k} b_p^{ks} + \frac{\partial L}{\partial \dot{x}^k} \frac{\partial}{\partial \phi_s^{(p)}} \left(\frac{d\bar{x}^k}{d\bar{t}}\right)_0 = 0 . \tag{3.3}$$

The second factor in the first term of the last expression is

$$\frac{\partial}{\partial\phi_s^{(p)}}\left(\frac{d\bar{t}}{dt}\right)=\frac{\partial}{\partial\phi_s^{(p)}}\left(\frac{\partial\bar{t}}{\partial t}+\frac{\partial\bar{t}}{\partial x^k}\dot{x}^k+\frac{\partial t}{\partial\phi_s^{(p)}}\dot{\phi}_s^{(p)}\right).$$

Evaluated at $\phi_s^{(p)}=0$, $\forall s\forall p$, this expression becomes

$$\frac{\partial}{\partial\phi_s^{(p)}}\left(\frac{d\bar{t}}{dt}\right)_0=\frac{\partial a_p^s}{\partial t}+\dot{x}^k\frac{\partial a_p^s}{\partial x^t}=\frac{da_p^s}{dt}\,. \tag{3.4}$$

To compute the last term in (3.3) we note that

$$\frac{d\bar{x}^k}{dt}=\frac{d\bar{x}^k}{d\bar{t}}\frac{d\bar{t}}{dt}\,.$$

Consequently,

$$\frac{\partial}{\partial\phi_s^{(p)}}\left(\frac{d\bar{x}^k}{dt}\right)=\frac{d\bar{x}^k}{d\bar{t}}\frac{\partial}{\partial\phi_s^{(p)}}\left(\frac{d\bar{t}}{dt}\right)+\frac{\partial}{\partial\phi_s^{(p)}}\left(\frac{d\bar{x}^k}{d\bar{t}}\right)\frac{d\bar{t}}{dt}\,. \tag{3.5}$$

Since

$$\frac{d\bar{x}^k}{dt}=\frac{\partial\bar{x}^k}{\partial t}+\frac{\partial\bar{x}^k}{\partial x^l}\dot{x}^l+\frac{\partial\bar{x}^k}{\partial\phi_s^{(p)}}\dot{\phi}_s^{(p)}\,,$$

and

$$\frac{d\bar{t}}{dt}=\frac{\partial\bar{t}}{\partial t}+\frac{\partial\bar{t}}{\partial x^l}\dot{x}^l+\frac{\partial\bar{t}}{\partial\phi_s^{(p)}}\dot{\phi}_s^{(p)}\,,$$

equation (3.5) can be written at $\phi_s^{(p)}=0$ as

$$\frac{\partial b_p^{ks}}{\partial t}+\frac{\partial b_p^{ks}}{\partial x^l}\dot{x}^l=\dot{x}^k\left(\frac{\partial a_p^s}{\partial t}+\dot{x}^l\frac{\partial a_p^s}{\partial x^l}\right)+\frac{\partial}{\partial\phi_s^{(p)}}\left(\frac{\partial\bar{x}^k}{\partial\bar{t}}\right)_0,$$

or

$$\frac{\partial}{\partial\phi_s^{(p)}}\left(\frac{\partial\bar{x}^k}{d\bar{t}}\right)_0=\frac{db_p^{ks}}{dt}-\dot{x}^k\frac{da_p^s}{dt}\,. \tag{3.6}$$

The substitution of (3.6) and (3.4) into (3.3) enables us to state the following theorem.

Theorem 3. A necessary condition for J to be an invariant of the group $G_{r\infty}$ is that the following $r+q$ identities hold true:

$$L\frac{da_p^s}{dt}+\frac{\partial L}{\partial t}a_p^s+\frac{\partial L}{\partial x^k}b_p^{ks}+\frac{\partial L}{\partial\dot{x}^k}\left(\frac{db_p^{ks}}{dt}-\dot{x}^k\frac{da_p^s}{dt}\right)=0\,. \tag{3.7}$$

These *invariance identities* (3.7) can be calculated directly from the Lagrangian L and the quantities which define the group $G_{r\infty}$. It should be noted that these identities contain only first derivatives of the functions $x^k(t)$ which determine the curve C on the manifold R^n. In addition, these identities are more fundamental than Noether's identities (2.12) in that (3.7) implies (2.12), but not conversely.

To prove this last remark we introduce the variational derivative $E_k(L)$ into (3.7) via the following device. We observe that

$$\frac{\partial L}{\partial t}a_p^s=\frac{dL}{dt}a_p^s-\frac{\partial L}{\partial x^k}\dot{x}^k a_p^s-\frac{\partial L}{\partial \dot{x}^k}\ddot{x}^k a_p^s\ ,$$

and

$$\frac{\partial L}{\partial \dot{x}^k}\frac{db_p^{ks}}{dt}=\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}b_p^{ks}\right)-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)b_p^{ks}\ ,$$

and

$$-\frac{\partial L}{\partial \dot{x}^k}\dot{x}^k\frac{da_p^s}{dt}-\frac{\partial L}{\partial \dot{x}^k}\ddot{x}^k a_p^s=-\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}^k}\dot{x}^k a_p^s\right]+\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\dot{x}^k a_p^s\ .$$

By successively substituting these last three identities into (3.7) we obtain after simplification the identities

$$E_k(L)(b_p^{ks}-\dot{x}^k a_p^s)=\frac{d}{dt}\left[\left(L-\dot{x}^k\frac{\partial L}{\partial \dot{x}^k}\right)a_p^s+\frac{\partial L}{\partial \dot{x}^k}b_p^{ks}\right]. \tag{3.8}$$

In passing, we note that along the extremal curve we have $E_k(L)=0$, whence the expressions in the bracket on the right-hand-side of (3.8) are constant. We therefore state the following corollary concerning *conservation laws.*

Corollary. If J is an invariant of the group $G_{r\infty}$, then there exist $r+q$ first integrals of the Euler-Lagrange equations of the form

$$\left(L-\dot{x}^k\frac{\partial L}{\partial \dot{x}^k}\right)a_p^s+\frac{\partial L}{\partial \dot{x}^k}b_p^{ks}=\text{constant}\ . \tag{3.9}$$

Continuing, we multiply the $p=0$ equation in (3.8) by $\phi_s(t)$, the $p=1$ equation by $\phi_s^{(1)}(t)$, and so on until finally we multiply the $p=q$ equation by $\phi_s^{(q)}(t)$. Adding the resulting $q+1$ equations, we get

$$E_k(L)[B^{ks}(\phi_s)-\dot{x}^k A^s(\phi_s)]=\frac{d}{dt}\left[\left(L-\dot{x}^k\frac{\partial L}{\partial \dot{x}^k}\right)A^s(\phi_s)+\frac{\partial L}{\partial \dot{x}^k}B^{ks}(\phi_s)\right]. \tag{3.10}$$

Integrating (3.10) from t_0 to t_1 and then proceeding as in tne proof in Section 2, we obtain the identities (2.12).

4. The Multiple Integral Case

Let $t=(t^1, t^2, \cdots, t^\nu)$ denote a point in some given domain $D\subseteq R^\nu$, and let $x=(x^1, x^2, \cdots, x^n)$ denote a set of n variables. In the configuration space $R^{\nu+n}$, a ν dimensional hypersurface C_ν can be represented parametrically in the form

$$x^k=x^k(t)\ ,$$

where the index k, like all other small Latin indicies, $i, j, l, \cdots$ in the sequel, has range $k=1, 2, \cdots, n$. Small Greek indicies will have range $1, 2, \cdots, \nu$. We assume that the functions $x^k(t)$ are of class C^2 and we denote their partial derivatives by

$$\dot{x}^k_\alpha \equiv \frac{\partial x^k}{\partial t^\alpha} .$$

We further assume that $\operatorname{rank}(\dot{x}^k_\alpha)=\nu$. Given a function $L: R^{\nu+n+\nu n} \to R^1$ which is of class C^2 in all of its arguments, we form the ν-fold variational integral

$$J(C_\nu)=\int_D L(t, x, \dot{x})dt , \tag{4.1}$$

where $dt=dt^1 \cdots dt^\nu$, and $\dot{x}=(\dot{x}^k_\alpha)$.

In order not to unduly complicate the notation, we will for multiple integral problems discuss transformation groups which depend on r arbitrary, indpendent functions $\phi_s(t)$, $t \in D$, and *only* their first derivative $\phi_{s,\alpha}(t)=\partial\phi_s/\partial t^\alpha$. For convenience, we denote

$$\phi_{s,0}(t) \equiv \phi_s(t) ,$$

so that we can use the notation $\phi_{s,\zeta}(t)$, $\zeta=0, 1, \cdots, \nu$. We assume, then, that the invariance transformation is of the form

$$\bar{t}^\alpha=\bar{t}^\alpha(t, x, \phi_{s,\zeta}) , \qquad \bar{x}^k=\bar{x}^k(t, x, \phi_{s,\zeta}) , \tag{4.2}$$

where the identity is obtained when $\phi_{s,\zeta}=0$ for all s and ζ. The infinitesimal form of (4.2) becomes

$$\left.\begin{aligned} \bar{t}^\alpha&=t^\alpha+a^{\alpha s\zeta}\phi_{s,\zeta}(t)+\cdots , \\ \bar{x}^k&=x^k+b^{ks\zeta}\phi_{s,\zeta}(t)+\cdots , \end{aligned}\right\} \tag{4.3}$$

where the a's and b's, which are coefficients in the Taylor series, are functions of t^α and x^k and the 'dots' denote terms involving products of the $\phi_{s,\zeta}$. We denote the above transformation by $G_{r\infty}$. Further, by Δ we denote the functional determinant

$$\Delta=\det\left(\frac{\partial\bar{t}^\alpha}{\partial t^\beta}\right) . \tag{4.4}$$

It is evident that

$$\Delta|_0=1 , \tag{4.5}$$

where we have again used the notation

$$(*)|_0=(*)|\phi_{s,\zeta}=0,\ \forall s \forall \zeta .$$

We are able to characterize the invariance of the variational integral (4.1) with respect to (4.3) by the following observation.

Remark 2. A necessary and sufficient condition for $J(C_\nu)$ to be an invariant of the group $G_{r\infty}$ is that

$$L\left(\bar{t}^\alpha, \bar{x}^k, \frac{\partial \bar{x}^k}{\partial \bar{t}^\alpha}\right)\Delta = L(t^\alpha, x^k, \dot{x}^k_\alpha)\,, \tag{4.6}$$

for all $\phi_{s,\zeta}(t)$; $s=1, \cdots, r$; $\zeta=0, 1, \cdots, \nu$. We proceed as in the single integral case, that is by differentiating (4.6) with respect to $\phi_{s,\zeta}$ and setting $\phi_{s,\zeta}=0$ for all s and ζ. We now, however, have the added difficulty of differentiating the functional determinant Δ. Since the right-hand-side of (4.6) is independent of $\phi_{s,\zeta}$, we obtain using (4.5)

$$L\left(\frac{\partial \Delta}{\partial \phi_{s,\zeta}}\right)_0 + \frac{\partial}{\partial \phi_{s,\zeta}} L\left(\bar{t}, \bar{x}^k, \frac{\partial \bar{x}^k}{\partial \bar{t}^\alpha}\right)\Bigg|_0 = 0\,.$$

Noting that

$$\left(\frac{\partial \bar{t}^\alpha}{\partial \phi_{s,\zeta}}\right)_0 = a^{\alpha s\zeta} \quad \text{and} \quad \left(\frac{\partial \bar{x}^k}{\partial \phi_{s,\zeta}}\right)_0 = b^{ks\zeta}\,,$$

the last equation becomes

$$L\left(\frac{\partial \Delta}{\partial \phi_{t,\zeta}}\right)_0 + \frac{\partial L}{\partial t^\alpha} a^{\alpha s\zeta} + \frac{\partial L}{\partial x^k} b^{ks\zeta} + \frac{\partial L}{\partial \dot{x}^k_\alpha}\left(\frac{\partial}{\partial \phi_{s,\zeta}} \frac{\partial \bar{x}^k}{\partial \bar{t}^\alpha}\right)_0 = 0\,. \tag{4.7}$$

In order to compute the first term, we denote the cofactors of the matrix $(d\bar{t}^\alpha/dt^\beta)$ by T^α_β. Then,

$$\begin{aligned}\frac{\partial \Delta}{\partial \phi_{s,\zeta}} &= \frac{\partial}{\partial \phi_{s,\zeta}}\left(\frac{d\bar{t}^\alpha}{dt^\beta}\right) T^\beta_\alpha \\ &= \frac{\partial}{\partial \phi_{s,\zeta}}\left(\frac{\partial \bar{t}^\alpha}{\partial t^\beta} + \frac{\partial \bar{t}^\alpha}{\partial x^k}\dot{x}^k_\beta + \frac{\partial \bar{t}^\alpha}{\partial \phi_{s,\zeta}}\frac{\partial \phi_{s,\zeta}}{\partial t^\beta}\right) T^\beta_\alpha\,.\end{aligned}$$

Carrying the differentiation, evaluating at $\phi_{s,\zeta}=0$, and using

$$(T^\beta_\alpha)_0 = \delta^\alpha_\beta\,,$$

we obtain

$$\left(\frac{\partial \Delta}{\partial \phi_{s,\zeta}}\right)_0 = \left(\frac{\partial a^{\alpha s\zeta}}{\partial t^\beta} + \frac{\partial a^{\alpha s\zeta}}{\partial x^k}\dot{x}^k_\beta\right)\delta^\beta_\alpha = \frac{d a^{\alpha s\zeta}}{dt^\alpha}\,. \tag{4.8}$$

It remains to compute the last term in (4.7). To do this, we note that in terms of the barred coordinate system the hypersurface C_ν is given by

$$\bar{x}^k = \bar{x}^k(\bar{t}^\alpha)\,.$$

According to the chain rule, we then have

$$\frac{d\bar{x}^k}{dt^\beta}=\frac{\partial \bar{x}^k}{\partial \bar{t}^\alpha}\frac{d\bar{t}^\alpha}{dt^\beta}\,.$$

Expanding the terms, we get

$$\frac{\partial \bar{x}^k}{\partial t^\beta}+\frac{\partial \bar{x}^k}{\partial x^h}\dot{x}^h_\beta+\frac{\partial \bar{x}^k}{\partial \phi_{s,\zeta}}\frac{d\phi_{s,\zeta}}{dt^\beta}=\frac{\partial \bar{x}^k}{\partial \bar{t}^\alpha}\left(\frac{\partial \bar{t}^\alpha}{\partial t^\beta}+\frac{\partial \bar{t}^\alpha}{\partial x^h}\dot{x}^h_\beta+\frac{\partial \bar{t}^\alpha}{\partial \phi_{s,\zeta}}\frac{d\phi_{s,\zeta}}{dt^\beta}\right).$$

Upon differentiating with respect to $\phi_{s,\zeta}$ we obtain

$$\begin{aligned}\frac{\partial^2 \bar{x}^k}{\partial t^\beta \partial \phi_{s,\zeta}}&+\frac{\partial^2 \bar{x}^k}{\partial x^h \partial \phi_{s,\zeta}}\dot{x}^h_\beta+\frac{\partial}{\partial \phi_{s,\zeta}}\left(\frac{\partial \bar{x}^k}{\partial \phi_{s,\zeta}}\frac{d\phi_{s,\zeta}}{dt^\beta}\right)\\ &=\frac{\partial \bar{x}^k}{\partial \bar{t}^\alpha}\left(\frac{\partial^2 \bar{t}^\alpha}{\partial t^\beta \partial \phi_{s,\zeta}}+\frac{\partial^2 \bar{t}^\alpha}{\partial x^h \partial \phi_{s,\zeta}}\dot{x}^h_\beta+\frac{\partial}{\partial \phi_{s,\zeta}}\left(\frac{\partial \bar{t}^\alpha}{\partial \phi_{s,\zeta}}\frac{d\phi_{s,\zeta}}{d\bar{t}^\beta}\right)\right)\\ &\quad+\frac{\partial}{\partial \phi_{s,\zeta}}\left(\frac{\partial \bar{x}^k}{\partial \bar{t}^\alpha}\right)\left(\frac{\partial \bar{t}^\alpha}{\partial t^\beta}+\frac{\partial \bar{t}^\alpha}{\partial x^h}\dot{x}^h_\beta+\frac{\partial \bar{t}^\alpha}{\partial \phi_{s,\zeta}}\frac{d\phi_{s,\zeta}}{dt^\beta}\right).\end{aligned}$$

The evaluation of this expression at $\phi_{s,\zeta}=0\ \forall s,\zeta$ yields

$$\frac{\partial b^{ks\zeta}}{\partial t^\beta}+\dot{x}^h_\beta\frac{\partial b^{ks\zeta}}{\partial x^h}=\dot{x}^k_\alpha\left(\frac{\partial a^{\alpha s\zeta}}{\partial t^\beta}+\dot{x}^h_\beta\frac{\partial a^{\alpha s\zeta}}{\partial x^h}\right)+\left(\frac{\partial}{\partial \phi_{s,\zeta}}\frac{\partial \bar{x}^k}{\partial \bar{t}^\alpha}\right)_0\delta^\alpha_\beta\,.$$

or

$$\left(\frac{\partial}{\partial \phi_{s,\zeta}}\frac{\partial \bar{x}^k}{\partial \bar{t}^\beta}\right)_0=\frac{db^{ks\zeta}}{dt^\beta}-\dot{x}^k_\alpha\frac{da^{\alpha s\zeta}}{dt^\beta}\,. \tag{4.9}$$

Finally, upon substituting (4.9) and (4.8) into (4.7), we obtain the ν-dimensional analog of Theorem 3.

Theorem 4. A necessary condition for $J(C_\nu)$ to be an invariant of the group $G_{r\infty}$ is that the following identities hold true:

$$L\frac{da^{\alpha s\zeta}}{dt^\alpha}+\frac{\partial L}{\partial t^\alpha}a^{\alpha s\zeta}+\frac{\partial L}{\partial x^k}b^{ks\zeta}+\frac{\partial L}{\partial \dot{x}^k_\alpha}\left(\frac{db^{ks\zeta}}{dt^\alpha}-\dot{x}^k_\beta\frac{da^{\beta s\zeta}}{dt^\alpha}\right)=0\,. \tag{4.10}$$

Equations (4.10) represent a set of identities which the Lagrangian L and its derivatives must satisfy under the invariance hypothesis. To derive Noether's Identities from (4.10), we observe that

$$\frac{\partial L}{\partial t^\alpha}=\frac{dL}{dt^\alpha}-\frac{\partial L}{\partial x^h}\dot{x}^h_\alpha-\frac{\partial L}{\partial \dot{x}^h_\beta}\ddot{x}^h_{\alpha\beta}\,,$$

$$\frac{\partial L}{\partial \dot{x}^k_\alpha}\frac{db^{ks\zeta}}{dt^\alpha}=\frac{d}{dt^\alpha}\left(\frac{\partial L}{\partial \dot{x}^k_\alpha}b^{ks\zeta}\right)-\frac{d}{dt^\alpha}\left(\frac{\partial L}{\partial \dot{x}^k_\alpha}\right)b^{ks\zeta}\,,$$

and

$$\frac{\partial L}{\partial \dot{x}^k_\alpha}\dot{x}^k_\beta \frac{da^{\beta s\zeta}}{dt^\alpha}+\frac{\partial L}{\partial \dot{x}^k_\alpha}\ddot{x}^k_{\alpha\beta}\, a^{\beta s\zeta}=\frac{d}{dt^\alpha}\left(\frac{\partial L}{\partial \dot{x}^k_\alpha}\dot{x}^k_\beta\, a^{\beta s\zeta}\right)-\frac{d}{dt^\alpha}\left(\frac{\partial L}{\partial \dot{x}^k_\alpha}\right)\dot{x}^k_\beta a^{\beta s\zeta}\,.$$

By successively substituting these three expressions into (4.10), we find after simplifying that

$$-E_k(L)(b^{ks\zeta}+\dot{x}^k_\alpha a^{\alpha s\zeta})=\frac{d}{dt^\alpha}\left(\left(L\delta^\alpha_\beta-\dot{x}^k_\beta\frac{\partial L}{\partial \dot{x}^k_\alpha}\right)a^{\beta s\zeta}+\frac{\partial L}{\partial \dot{x}^k_\alpha}b^{ks\zeta}\right),$$

where the $E_k(L)$ are the variational derivatives defined by

$$E_k(L)=\frac{\partial L}{\partial x^k}-\frac{d}{dt^\alpha}\left(\frac{\partial L}{\partial \dot{x}^k_\alpha}\right). \tag{4.12}$$

If the Euler-Lagrange equations hold for the system, i. e. C_ν is an extremal surface, then $E_k(L)=0$ and

$$\frac{d}{dt^\alpha}\left(H^\alpha_\beta a^{\beta s\zeta}-\frac{\partial L}{\partial \dot{x}^k_\alpha}b^{ks\zeta}\right)=0\,, \tag{4.13}$$

where H^α_β is the *Hamiltonian tensor* defined by

$$H^\alpha_\beta=-L\delta^\alpha_\beta+\dot{x}^k_\beta\frac{\partial L}{\partial \dot{x}^k_\alpha}\,.$$

The vanishing divergences given by (4.13) are interpreted in the usual way as *conservation laws* for the system. By multiplying (4.11) by $\phi_{s,\zeta}$, integrating over D, and using the arbitrariness of the functions $\phi_{s,\zeta}$, we obtain

$$\int_D E_k(L)(b^{ks\zeta}\phi_{s,\zeta}-\dot{x}^k_\alpha a^{\alpha s\zeta}\phi_{s,\zeta})dt=0\,. \tag{4.14}$$

The boundary integral vanishes since the ϕ_s can be chosen such that they along with their derivatives vanish on Bd D. Integration by parts in (4.14) to remove the derivatives from the ϕ_s yields

$$\int_D\left\{\left(b^{ks0}-b^{ks\beta}\frac{\partial}{\partial t^\beta}\right)E_k(L)-\left(a^{\alpha s0}-a^{\alpha s\beta}\frac{\partial}{\partial t^\beta}\right)(\dot{x}^k_\alpha E_k(L))\right\}\phi_s(t)\,dt=0\,.$$

Using the independence and arbitrariness of the $\phi_s(t)$, we conclude, as in the proof in Section 2, that

$$\left(b^{ks0}-b^{ks\beta}\frac{\partial}{\partial t^\beta}\right)E_k(L)-\left(a^{\alpha s0}-a^{\alpha s\beta}\frac{\partial}{\partial t^\beta}\right)(\dot{x}^k_\alpha E_k(L))=0\,. \tag{4.15}$$

These are the Noether identities in the ν-dimensional case.

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