A NOTE ON MALCEV AND QUASI-LIE ALGEBRAS

By

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(Received August 17, 1972)

1. Throughout this paper the algebras considered are non-associative (i.e., not necessarily associative) and $J(x, y, z)$ will denote $(xy)z+(yz)x+(zx)y$ under usual multiplication operation, which expression, for simplicity, may be written as

$$xy\cdot z+yz\cdot x+zx\cdot y \ .$$

A. A. Sagle [1] studied *Malcev* algebras. In this note, firstly we give a characterization of *Maclev* algebras in terms of *Jacobi-Teichmüller* identity (Theorem 2.2), and then use the same to give alternative simpler proofs of some of the results proved by *Sagle* (Propositions 4.1 and 5.1). *Kass* and *Witthoft* [3] found the irreducible homogeneous polynomial identities of degree less that or equal to four in anticommutative algebras over a field of characteristic different from two. We use his fifth polynomial

$$J(x, y, z)w-J(w, x, y)z+J(z, w, x)y-J(y, z, w)x \ ,$$

([3], Theorem 2) to define a concept of quasi-lie algebra, and to show that Malcev algebras and extended lie algebras (see *Sagle* [2]) are not comparable with quasi-lie algebras. Sagle introduced the concepts of the 'Lie subsets' and 'Nucleus' and showed that, in Malcev algebra, they form a subalgebra and an ideal respectively. In this note, we construct some examples to show that the lie subset and the nucleus may not be so in quasi-lie algebras.

2. We first state the following lemma of *Sagle* [1].

Lemma 2.1. An algebra A of characteristic not two is a Malcev algebra if and only if A satisfies $xy=-yx$ and

$$xy\cdot zw=x(wy\cdot z)+w(yz\cdot x)+y(zx\cdot w)+z(xw\cdot y) \text{ for all } x, y, z, w \text{ in } A.$$

It is well known that the Teichmüller identity

$$(wx, y, z)-(w, xy, z)+(w, x, yz)-w(x, y, z)-(w, x, y)z=0 \ ,$$

holds in all non-associative algebras (see Kleinfield [4]) where (a, b, c) is the associator $(ab)c-a(bc)$.

Now we give a characterization of Malcev algebras:

Theorem 2.2. An anticommutative algebra of characteristic not two, is Malcev if and only if it satisfies the identity

$$J(wx, y, z)-J(w, xy, z)+J(w, x, yz)-wJ(x, y, z)-J(w, x, y)z=0\,. \qquad \ldots.(1)$$

(We shall call this identity as Jacobi-Teichüller)

Proof. Let A satisfy the Jacobi-Teichmüller identity. Put $w=x$ in (1). Then the above expression reduces to $-J(x, xy, z)-xJ(x, y, z)=0$, since *the other two terms* reduce to zero.

Hence, we have by interchanging y and z, $J(x, y, xz)=J(x, y, z)x$ for all x, y, z in A, which defines Malcev algebra (Sagle [1]).

Conversely let A be a Malcev algebra, then by lemma 2.1, we have $yx\cdot zw=y(wx\cdot z)+w(xz\cdot y)+x(zy\cdot w)+z(yw\cdot x)$. Adding to it the Teichmüller identity $(wx, y, z)-(w, xy, z)+(w, x, yz)-w(x, y, z)-(w, x, y)z=0$ which holds in every non-associative algebra and adjusting the corresponding terms, we obtain

$$J(wx, y, z)-J(w, xy, z)+J(w, x, yz)-wJ(x, y, z)-J(w, x, y)z=0\,.$$

Hence the proof is complete.

3. We shall call a non-associative algebra a quasi-lie algebra if it is anticommutative and satisfies

$$J(x, y, z)w-J(w, x, y)z+J(z, w, x)y-J(y, z, w)x=0\,.$$

We see immediately from the definition that any anti-commutative algebra of dimension not exceeding three is a quasi-lie algebra.

Consider the following examples:

Ex. 1. The algebra A having basis $\{e_1, e_2, e_3\}$ with the multiplication table:

	e_1	e_2	e_3
e_1	0	e_1	e_2
e_2	$-e_1$	0	$-e_3$
e_3	$-e_2$	e_3	0

is a quasi-lie algebra of dimension three but it can be easily checked that it is neither Malcev nor extended lie, i.e. it does not satisfy the identity $J(x, y, xy)=0$, [2].

Ex. 2. The algebra A having basis $\{e_1, e_2, e_3, e_4\}$ with the multiplication:

	e_1	e_2	e_3	e_4
e_1	0	e_3	e_4	0
e_2	$-e_3$	0	0	e_1
e_3	$-e_4$	0	0	e_2
e_4	0	$-e_1$	$-e_2$	0

is a quasi-lie algebra of dimension four, but not a Malcev algebra since $J(e_1, e_2, e_1e_3) = e_2 \neq J(e_1, e_2, e_3)e_1 = 0$. It is not even an extended lie algebra since $J(e_1, e_2, e_1e_2) = e_1 \neq 0$.

Ex. 3. The non-associative algebra with the basis $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ having the multiplication table:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	$2e_2$	$2e_3$	$2e_4$	$-2e_5$	$-2e_6$	$-2e_7$
e_2	$-2e_2$	0	$2e_7$	$-2e_6$	e_1	0	0
e_3	$-2e_3$	$-2e_7$	0	$2e_5$	0	e_1	0
e_4	$-2e_4$	$2e_6$	$-2e_5$	0	0	0	e_1
e_5	$2e_5$	$-e_1$	0	0	0	$-2e_4$	$2e_3$
e_6	$2e_6$	0	$-e_1$	0	$2e_4$	0	$-2e_2$
e_7	$2e_7$	0	0	$-e_1$	$-2e_3$	$2e_2$	0

is a Malcev algebra, but not a quasi-lie algebra, because

$$J(e_4, e_5, e_6)e_7 - J(e_7, e_4, e_5)e_6 + J(e_6, e_7, e_4)e_5 - J(e_5, e_6, e_7)e_4 = 36e_4 \neq 0 .$$

The above examples exhibit that neither the Malcev nor the extended lie algebras are generalization of quasi-lie algebra and conversely.

4. *Sagle* [1] defines that a subset B of non-associative algebra A is a lie subset of A if $J(B, B, B) = 0$. B is a maximal lie subset of A provided B is a maximal subset of A such that $J(B, B, B) = 0$.

We now give a simpler proof of a *Sagle's* theorem 4.1 [1] and show that it is not true in the case of quasi-lie algebras.

Proposition 4.1. Every maximal lie subset B of a Malcev algebra A of characteristic not two is a subalgebra of A.

Proof. Let $w, x, y, z \in B$. Since B is a lie subset of the Malcev algebra A, both $J(x, y, z)$ and $J(w, x, y)$ are equal to zero. Now using Theorem 2.2 above

$$J(wx, y, z) - J(w, xy, z) + J(w, x, yz) = 0 \qquad \ldots .(a)$$

for all w, x, y, z of B. Consider $J(x, y, z)w - J(w, x, y)z + J(z, w, x)y - J(y, z, w)x$

in B which is identically zero since B is a lie subset, but one can see that in a Malcev algebra of characteristic different from 2, this identity reduces to

$$J(wx, y, z)+J(w, x, yz)=0 \qquad \ldots.(b)$$

by *Sagle* [1], Prop. 2.23.

From (a) and (b), it follows that $J(w, xy, z)=0$ for all $w, x, y, z \in B$. Thus B is a subalgebra.

We have seen that the algebra with basis $\{e_1, e_2, e_3\}$ considered in Ex. 1 is a quasi lie algebra and can easily be seen that it has a subspace B generated by $\{e_1, e_3\}$ as a maximal lie subset. This B is not a subalgebra as in the Malcev case of A since $e_1e_3=e_2 \notin B$.

The following example further shows that a maximal lie subset of a Malcev algebra need not be an ideal.

Ex. 4. The algebra A with basis $\{e_1, e_2, e_3, e_4\}$ having the multiplication table:

	e_1	e_2	e_3	e_4
e_1	0	$-e_2$	$-e_3$	e_4
e_2	e_2	0	$2e_4$	0
e_3	e_3	$-2e_4$	0	0
e_4	$-e_4$	0	0	0

is a Malcev algebra. The subspace generated by $\{e_1, e_2, e_4\}$ is a maximal lie subset, but not an ideal.

5. The nucleus N of a Malcev algebra A is defined as

$$N=\{x \in A \mid J(x, y, z)=0 \qquad y, z \in A\}.$$

This implies that N is the maximal subset of A such that $J(N, A, A)=0$. It can further be noted by definition of quasi-lie algebra that if N is the nucleus of the quasi-lie algebra then N satisfies $NJ(A, A, A)=0$ and is also a subalgebra.

It may be remarked that a simpler proof of Sagle's lemma 5.13 can be obtained by using our theorem 2.2 and the Jacobi-Teichmüller identity which we have introduced.

Proposition 5.1. The nucleus of a Malcev algebra A is an ideal of A.

Proof. Let $w \in N$ and $x, y, z \in A$ where A is a Malcev algebra, N be its nucleus. From the Jacobi-Teichmuller identity it follows that

$$J(wx, y, z)-wJ(x, y, z)=0\,. \qquad \ldots.(c)$$

Again in the same identity, assuming that $x \in N$ and $x, y, z \in A$, we obtain

$$J(wx, y, z)-J(w, xy, z)=0 .$$

Because of the anticommutativity, the above expression implies that

$$-J(xw, y, z)-J(xy, z, w)=0 .$$

Using the result (c), this gives us

$$-xJ(w, y, z)-xJ(y, z, w)=0 ,$$

$$\text{i.e.} \quad 2xJ(w, y, z)=0 .$$

Since the characteristic is different from 2, we have $xJ(w, y, z)=0$ whenever $x \in N$. Therefore, if $w \in N$, and $x, y, z \in A$, we have $J(wx, y, z)=0$ implying that N is an ideal of A.

Acknowledgements

The author wishes to express his gratitude to Prof. *M. A. Kazim* and *Dr. Surjeet Singh* for their help in the preparation of this paper. The author is also thankful to *Dr. Orihara* for his helpful comments.

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