COEFFICIENTS OF UNIFORMLY NORMAL-BLOCH FUNCTIONS

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1. Introduction.

Let D: $|z|<1$ be the open unit disk. A holomorphic function f in D is *normal* if and only if

$$\frac{|f'(z)|}{1+|f(z)|^2}=O\frac{1}{1-|z|}\ [9]\ .$$

Bagemihl and *Seidel* have pointed out several classes of functions that are normal [1]. Bounded functions, functions having a finite Dirichlet integral, and univalent functions are all normal. Lappan has studied normal functions and found that they do not form a linear space [6], and he found it of interest to consider the linear subspace of uniformly normal functions [7] and [8]. *Pommerenke* has also studied this subspace and has called it the class of Bloch functions [10], also see *Hayman* [5]. A holomorphic function f in D is a *uniformly normal-Bloch* function if and only if $|f'(z)|=O(1/(1-|z|))$.

In this paper we investigate coefficients of uniformly normal-Bloch functions. Also, several classes of normal functions are shown to be uniformly normal. We find that holomorphic functions with finite Dirichlet integrals and Hadamard gap series with bounded coefficients are Bloch functions. Finally, certain fractional integrals of Bloch functions are shown to be Bloch functions. We will use the terms uniformly normal and Bloch interchangeably.

2. Preliminaries.

Hayman has noted that the coefficients of any uniformly normal-Bloch function are bounded [5]. We give an example of a non-normal holomorphic function with bounded coefficients. Theorem 1 is due to *Hayman* but our proof is different.

Theorem 1. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n$ *be unifomly normal, then for each* n, $|a_n|\leqq M$.

Proof. From *Cauchy's* formula we obtain

$$|a_n|=\left|\frac{1}{2\pi in}\int_0^{2\pi}\frac{f'(re^{i\theta})d\theta}{r^{n-1}e^{i(n-1)\theta}}\right|\leqq n^{-1}(1-r)^{-1}r^{-n+1}C\,, \qquad \text{where} \quad |f'(z)|\leqq\frac{C}{1-|z|}\,.$$

The minimum value is achieved for $r=1-(1/n)$ and we obtain $|a_n|\leqq(1-(1/n))^{-n+1}C\leqq eC$, for $n\geqq 2$. We choose $M=\max\{|a_0|, |a_1|, eC\}$ and the theorem is proved. The function $f(z)=\sum_{n=0}^{\infty} z^{2^n}$ is uniformly normal and $\limsup_{n\to\infty}|a_n|=1$.

Pommerenke has shown that if the Bloch function $f(z)=\sum_{n=0}^{\infty}a_n z^n$ has radial limits almost everywhere, then $|a_n|\to 0$ [10]. This is not sufficient for a function to be uniformly normal. For fixed β, $0<\beta<1$, the function $f(z)=(1-z)^{-\beta}=\sum_{n=0}^{\infty}a_n z^n$ is normal, but not uniformly normal. It is known that $a_n=(1/\Gamma(\beta))n^{\beta-1}(1+o(1))$, where Γ is the Gamma function, and hence $|a_n|\to 0$. We conclude this section by exhibiting a non-normal function with bounded coefficients.

Example 1. *There exists a holomorphic function with bounded coefficients which is not normal.*

Proof. The function $g(z)=\log(1-z)=\sum_{n=1}^{\infty}(1/n)z^n$ is uniformly normal and unbounded. By a theorem of *Lappan* [6, Theorem 3, p. 190], there exists a Blaschke product $B(z)=\sum_{n=0}^{\infty}b_n z^n$ such that $f(z)=g(z)B(z)$ is not normal. $B(z)$ is an H^2 function, hence $\sum_{n=0}^{\infty}|b_n|^2<\infty$. We have $f(z)=\sum_{n=0}^{\infty}a_n z^n=\sum_{n=1}^{\infty}(1/n)z^n\sum_{n=0}^{\infty}b_n z^n$, and hence for each n

$$|a_n|=\left|\sum_{k=1}^{n}\frac{1}{k}b_{n-k}\right|\leqq\left\{\sum_{k=1}^{n}\left(\frac{1}{k}\right)^2\right\}^{1/2}\left\{\sum_{k=0}^{n-1}|b_k|^2\right\}^{1/2}\leqq M\,.$$

In the following we will restrict our attention to coefficients of uniformly normal holomophic functions.

3. The Main Results.

We establish sufficient conditions for a holomorphic function to be uniformly normal-Bloch. Then we investigate the necessary conditions.

Theorem 2. *The following are sufficient conditions for* $f(z)=\sum_{n=0}^{\infty}a_n z^n$ *to be uniformly normal.*

(i) *If there exists a fixed integer* J, $J\geqq 0$, *and* $\sum_{k=J}^{n}k^J|a_k|\leqq Cn^J$ *for all* n.

(ii) $\sum_{n=1}^{\infty} n|a_n|^2<\infty$.

(iii) *If there exists an* $\alpha>1$ *and a subsequence* $\{n_j\}$ *for which* $n_{j+1}>\alpha n_j$, *and if for each* n_j, $|a_{n_j}|\leqq M$, *then* $f(z)=\sum_{j=0}^{\infty} a_{n_j}z^{n_j}$ *is uniformly normal.*

Proof for (i). *Titchmarsh* has shown [11, p. 225] that if $g(z)=\sum_{n=0}^{\infty} c_n z^n$ is holomorphic in D and if for each n, $\sum_{k=1}^{n} |c_k|\leqq Mn$, then $|g(z)|\leqq M/(1-|z|)$. For $J=1$, we set $g(z)=f'(z)$ and the proof is immediate. For $J>1$ it is easy to verify that $|f^{[J]}(z)|\leqq C/(1-|z|)^J$ and the result follows by successive integration.

Proof for (ii). Let $\sum_{n=1}^{\infty} n|a_n|^2=M$. From the *Cauchy-Schwarz* inequality we obtain

$$\sum_{k=1}^{n} k|a_k|\leqq\left\{\sum_{k=1}^{n} k\right\}^{1/2}\left\{\sum_{k=1}^{n} k|a_k|^2\right\}^{1/2}\leqq M^{1/2}n\,,$$

and the proof follows from part (i), with $J=1$.

Proof for (iii). Let n be a positive integer and let k_j $(j=0, 1, \cdots, J)$ be the k_j which are less than or equal to n. Then we have $\sum_{k_j\leqq n} k_j|a_{k_j}|=\sum_{j=0}^{J} k_j|a_{k_j}|$ $\leqq M(\alpha^{-J}+\alpha^{-J+1}+\cdots+1)k_J\leqq M(\alpha/\alpha-1)n$, and the proof of the theorem is complete.

Remark. If $\int_0^{2\pi}\int_0^1 |f'(re^{i\theta})|^2 r dr d\theta=\pi\sum_{n=1}^{\infty} n|a_n|^2<\infty$, then f is said to have a finite Dirichlet integral. We remark that holomorphic functions with finite Dirichlet integrals are uniformly normal. The functions of (iii) are called Hadamard gap series. We remark that Hadamard gap series with bounded coefficients are Bloch functions.

We now investigate necessary conditions for the coefficients of uniformly normal holomorphic functions. The first result shows when the sufficient condition of theorem 2 is a necessary condition.

Theorem 3. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n$ *be uniformly normal. If for each* n, $\alpha\leqq\arg(a_n)$ $\leqq\alpha+\pi/2$, *then* $\sum_{k=1}^{n} k|a_k|\leqq Cn$, *for all* n.

Proof. We may assume that $\alpha=0$. Since f is uniformly normal we have $\left|\sum_{k=1}^{\infty} ka_k z^{k-1}\right|=|f'(z)|\leqq C/(1-|z|)$, and $\left|\sum_{k=1}^{\infty} k\bar{a}_k z^{k-1}\right|=|\overline{f'(\bar{z})}|\leqq C/(1-|z|)$, and we obtain $\left|\sum_{k=1}^{\infty} k\operatorname{Re} a_k z^{k-1}\right|\leqq C/(1-|z|)$, where $0\leqq \operatorname{Re} a_k$. A similar result holds for $\operatorname{Im} a_k$ and we obtain $\left|\sum_{k=1}^{\infty} k|a_k|z^{k-1}\right| \leqq 2C/(1-|z|)$. *Hardy* and *Littlewood* have shown [4, Theorem 96, p. 155], [11, p. 226] that if $g(z)=\sum_{n=0}^{\infty} c_n z^n$ and $|g(z)| \leqq K/(1-|z|)$, where $0 \leqq c_n$, then for each n, $\sum_{k=0}^{n} c_k \leqq Kn$. We set $f'(z)=g(z)$ and the proof is immediate.

Corollary. *If $F(z)=\sum_{n=0}^{\infty} |a_n| z^n$ is uniformly normal, then $f(z)=\sum_{n=0}^{\infty} a_n z^n$ is uniformly normal.*

The following example shows that the restrictions on the coefficients in theorem 3 were necessary.

Example 2. *Let $1<p<3/2$. The holomorphic function $f(z)=\sum_{n=1}^{\infty} n^{p-2} e^{i(2p-2)n} z^n$ is uniformly normal and $\sum_{k=1}^{\infty} k|a_k| = \sum_{k=1}^{n} k^{p-1} \approx Cn^p$.*

Proof. *Hardy* [3] has shown that if $F(z)=\sum_{n=1}^{\infty} n^{-\beta} e^{i\alpha n} z^n$, where $0<\alpha<1$, then F is unbounded if $1-\beta-\frac{1}{2}\alpha>0$, and $|F(z)|=O(1/((1-|z|)^{1-\beta-(1/2)\alpha}))$. For $F(z) = \sum_{n=1}^{\infty} n^{p-1} e^{i(2p-2)n} z^n$ we obtain $|F(z)|=O(1/(1-|z|))$. Setting $F(z)=zf'(z)$ it follows that f is uniformly normal and the proof is complete.

Theorem 4. *Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be a Bloch function. Then for each fixed integer $P \geqq 0$ we have $\sum_{k=p}^{n} k^P |a_k| \leqq Cn^{P+1/2}$, for all n.*

Proof. *Hardy* has shown [3], [2, Lemma 1, p. 45] that if $g(z)=\sum_{n=0}^{\infty} c_n z^n$ is holomorphic in D and $|g(z)|=O(1/((1-|z|)^{\alpha}))$, $\alpha>0$, then for $-\infty<\gamma<\alpha+\frac{1}{2}$,

$$\sum_{n=1}^{\infty} n^{-\gamma} |c_n| |z|^n = O\left(\frac{1}{(1-|z|)^{\alpha-\gamma+1/2}}\right).$$

We set $f'(z)=g(z)$, $\alpha=1$, $\gamma=-P+1$ and obtain $\sum_{n=1}^{\infty} n^P |a_n| |z|^{n-1} = O(1/(1-|z|)^{P+1/2})$. Estimating $1/(1-|z|)^{P+1/2}$ we obtain

$$\sum_{n=1}^{\infty} n^P |a_n| |z|^n \leqq C \sum_{n=0}^{\infty} n^{P-1/2} |z|^n .$$

Titchmarsh has shown [11, p. 224] that the above inequality implies that $\sum_{k=1}^{n} k^P |a_k| \leqq Cn^{P+1/2}$, and the theorem is proved.

We now consider fractional integrals of uniformly normal-Bloch functions.

Let $\beta>0$ and $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be holomorphic in D, the fractional integral of f of order β is $f_{[\beta]}(z)=\sum_{n=0}^{\infty}(n!/\Gamma(n+1+\beta))a_n z^n$.

Theorem 5. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n$ *be uniformly normal. Then,*

i) *If* $\beta \geqq \frac{1}{2}$, *then* $f_{[\beta]}$ *is uniformly normal, and*

ii) *If* $\beta>0$, *and if for each* n, $\alpha \leqq \arg(a_n) \leqq \alpha+\pi/2$, *then* $f_{[\beta]}$ *is uniformly normal.*

Proof. $f'_{[\beta]}(z)=\sum_{n=0}^{\infty}(n!/\Gamma(n+1+\beta))na_n z^{n-1}$.

For $n \geqq 1$ an elementary calculation shows that $n!/\Gamma(n+1+\beta) \leqq Mn^{-1/2}$. From *Hardy's* result [3], $|f'(z)|=O(1/(1-|z|))$ implies that $\sum_{n=1}^{\infty} n^{-1/2}n|a_n||z|^{n-1}=O(1/(1-|z|))$. It follows that $|f'_{[\beta]}(z)|=O(1/(1-|z|))$ and the first part is proved. The proof of the second part follows from theorem 3.

Fractional derivatives are defined similarly to fractional integrals. We remark that no fractional derivative of $f(z)=\sum_{n=0}^{\infty} z^{2^n}$ is uniformly normal.

REFERENCES

[1] F. Bagemihl and W. Seidel: *Sequential and Continuous Limits of Meromorphic Functions*, Ann. Sci. Fenn. Ser. AI, no. 280 (1960), 17 pp.

[2] P. L. Duren, B. W. Romberg and A. L. Shields: *Linear Functionals on H^p Spaces with* $0<p<1$, J. Reine Angew. Math. 238 (1969), 32-60.

[3] G. H. Hardy: *A Theorem Concerning Taylor's Series*, Quart. J. Math. 44 (1913), 147-160.

[4] G. H. Hardy: *Divergent Series*, Oxford, London, 1967.

[5] W. K. Hayman: *Uniformly Normal Families*, Lectures on functions of a Complex Variable (Univ. of Michigan Press, 1955).

[6] P. Lappan: *Non-Normal Sums and Products of Unbounded Normal Functions*, Mich. Math. J. 8 (1961), 187-192.

[7] P. Lappan: *Some Sequential Properties of Normal and Non-Normal Functions with Applications to Automorphic Functions*, Comment. Math. Univ. St. Pauli, 12 (1964), 41-57.

[8] P. Lappan: *Fatou Points of Harmonic Normal Functions and Uniformly Normal Functions* Math. Z. 102 (1967), 110-114.

[9] O. Lehto and K. I. Virtanen: *Boudary Behavior and Normal Meromorphic Functions*, Acta Math. 97 (1957), 47-65.

[10] Ch. Pommerenke: *On Bloch Functions*, J. London Math. Soc. (2), 2 (1970), 689-695.

[11] E. C. Titchmarsh, *The Theory of Functions*, Oxford, London, 1964.

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