A STUDY ON THE SPACE OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

By

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1. Introduction and Terminology. Let $\mathfrak{C}$ denote the complex plane, and I be the set of non-negative integers. We write for $n \in I$, $n \geq 1$,

$$\mathfrak{C}^n = \{(z_1, z_2, \cdots, z_n): z_i \in \mathfrak{C}, 1 \leq i \leq n\}, \qquad I^n = \{(p_1, p_2, \cdots, p_n): p_i \in I, 1 \leq i \leq n\}.$$

$\mathfrak{C}^n$ and I^n are respectively Banach and metric spaces under the functions

$$\|(z_1, \cdots, z_n)\| = |z_1| + \cdots + |z_n|; \qquad \|(p_1, \cdots, p_n)\| = p_1 + \cdots + p_n.$$

We are concerned here with the space of all entire functions from $\mathfrak{C}^n$ to $\mathfrak{C}$ under the usual pointwise addition and scalar multiplication. For the sake of simplicity we consider the case when $n=2$, though our results can be easily extended to any finite integer n. Let therefore X be the class of all entire functions f: $\mathfrak{C}^2 \to \mathfrak{C}$, where

$$f(z_1, z_2) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{m,n} z_1^m z_2^n; \qquad a_{m,n} \in \mathfrak{C}, \quad \text{for} \quad m, n \geq 0. \tag{1.1}$$

and assume that X is equipped with the topology T of uniform convergence on compacta in $\mathfrak{C}^2$. It is known that (X, T) is a separable locally convex metrizable complete space, i.e. a separable Fréchet space. Our interest in this paper is two-fold: first, to introduce on X an invariant metric in terms of the determining constants $a_{m,n}$ and then to show that the topology generated by this new metric is equivalent to T and also to characterise continuous linear functionals on X; secondly to give a characterisation of proper absolute bases in X. This paper may be considered as an introduction to the structural study of X.

2. Topology on X. Let for each $f \in X$, define

$$\|f\| = \sup\{|a_{0,0}|; |a_{m,n}|^{1/(m+n)}, \quad m, n \geq 0, m+n \neq 0\}$$

Then $\|f\|$ is a total paranorm or F-norm on X. We now prove

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Theorem 2.1. *$(X, \|f\|)$ is a complete metrizable, non-locally bounded space, i.e. a non-normable Fréchet space, where the invariant metric on X is given by $\|f-g\|$; $f, g \in X$.*

Proof. It is sufficient to show that for each Cauchy sequence $\{f_p\}$ in X, there corresponds a unique $f \in X$, such that $\|f_p - f\| \to 0$ as $p \to \infty$. So, let $\varepsilon > 0$ be given. There exists $Q = Q(\varepsilon)$, such that

$$|a_{0,0}^{(p)} - a_{0,0}^{(q)}|, \ |a_{m,n}^{(p)} - a_{m,n}^{(q)}|^{1/(m+n)} < \varepsilon, \qquad \text{for } p, q \geq Q; m, n \geq 0, m+n \neq 0,$$

where

$$f_p(z_1, z_2) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{m,n}^{(p)} z_1^m z_2^n .$$

Therefore

(2.1) $$a_{m,n}^{(p)} \to a_{m,n}, \quad \text{say, as } p \to \infty, \qquad \text{for } m, n \geq 0 .$$

Now for $m+n \neq 0$, $m, n \geq 0$

$$|a_{m,n}|^{1/(m+n)} \leq |a_{m,n}^{(p)} - a_{m,n}|^{1/(m+n)} + |a_{m,n}^{(p)}|^{1/(m+n)}$$

But for any fixed p, and therefore for $p = Q$, we have

$$|a_{m,n}^{(p)}|^{1/(m+n)} \to 0 \qquad \text{as } \|(m, n)\| \to \infty .$$

Therefore

$$\lim_{\|(m,n)\| \to \infty} |a_{m,n}|^{1/(m+n)} = 0 .$$

Hence the function $f\colon \mathfrak{C}^2 \to \mathfrak{C}$ given by

$$f(z_1, z_2) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{m,n} z_1^m z_2^n$$

is a member of X. Moreover, from (2.1), $\|f_p - f\| \to 0$ as $p \to \infty$.

To complete the proof, we do here something more than what we are required to do. Indeed, we show that T is equivalent to the topology T^* generated by $\|f\|$ on X. Let $f_p \to f$ in T^*. Let now $\varepsilon > 0$, $\eta > 0$; $R_1, R_2 > 0$ be taken arbitrarily, such that

$$\eta R_1, \eta R_2 < 1 ; \qquad \eta + \eta^2 R_1 R_2 / \{(1 - \eta R_1)(1 - \eta R_2)\} < \varepsilon .$$

Now for $p \geq Q = Q(\eta)$

$$|a_{0,0}^{(p)} - a_{0,0}| < \eta , \qquad |a_{m,n}^{(p)} - a_{m,n}| < \eta^{m+n} ; \qquad m, n \geq 0 , \qquad m+n \neq 0 .$$

Tnen for $|z_1| \leq R_1$, $|z_2| \leq R_2$

$$|f_p(z_1, z_2)-f(z_1, z_2)|<\eta+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(\eta R_1)^m(\eta R_2)^n$$
$$=\eta+\frac{\eta R_1}{1-\eta R_1}\,\frac{\eta R_2}{1-\eta R_2}<\varepsilon\,, \qquad \text{for all} \quad p\geq Q\,.$$

Hence $T\subset T^*$. On the other hand suppose now $f_p\to f$ in T. Then in particular $a_{0,0}^{(p)}\to a_{0,0}$. Choose $\varepsilon>0$. We may find R_1, R_2, such that $R_1, R_2\geq 1/\varepsilon$. There exists $Q=Q(\varepsilon, R_1, R_2)$ such that

$$|a_{0,0}^{(p)}-a_{0,0}|<\varepsilon\,, \qquad p\geq Q$$
$$|f_p(z_1, z_2)-f(z_1, z_2)|\leq 1\,, \qquad p\geq Q,\ |z_1|\leq R_1,\ |z_2|\leq R_2\,.$$

Using *Cauchy*'s inequality for two variables, one finds

$$|a_{m,n}^{(p)}-a_{m,n}|\,R_1^mR_2^n\leq M(R_1, R_2; f_p-f)\leq 1\,, \qquad p\geq Q\,,$$

where

$$M(R_1, R_2, g)=\max_{|z_1|\leq R_1, |z_2|\leq R_2}|g(z_1, z_2)|\,, \qquad g\in X\,.$$

Therefore, for $m+n\neq 0$

$$|a_{m,n}^{(p)}-a_{m,n}|^{1/(m+n)}\leq\frac{1}{R_1^{m/(m+n)}R_2^{n/(m+n)}}\leq\frac{1}{R}\leq\varepsilon\,, \qquad p\geq Q\,,$$

where $R=\max(R_1, R_2)\geq\varepsilon^{-1}$. Thus $T^*\subset T\Rightarrow T=T^*$.

Returning to the last assertion of the theorem, consider G to be an arbitrary neighbourhood of $0\in X$. Then for $\varepsilon>0$ and $n\in I$, $n\geq 1$, we have

$$\{f: f\in X, P_n(f)<\varepsilon\}\subset G\,,$$
$$p_n(f)=\max_{|z_1|\leq n, |z_2|\leq n}|f(z_1, z_2)|\,.$$

Define $f_p\in X$, such that

$$f_p(z_1, z_2)=\frac{\varepsilon}{2}\left(\frac{z_1}{n}\right)^p\left(\frac{z_2}{n}\right)^p, \qquad p\in I\,,$$

and set $\varepsilon_p=2^{-2p}$, $p\in I$. Since $\max\limits_{|z_1|, |z_2|\leq 2n}|\varepsilon_p f_p(z_1, z_2)|=\varepsilon/2>\varepsilon/4$, one concludes that $\varepsilon_p f_p\notin\{f: f\in X, p_{2n}(f)<\varepsilon/4\}$ and so $\varepsilon_p f_p\not\to 0$ as $p\to\infty$. Therefore no neighbourhood G of $0\in X$ is bounded with respect to T and therefore with respect T^*. The proof of the result is now complete.

Cotinuous Linear Functionals on X. We now proceed to characterize continuous linear functionals on X in a most simple and effective manner. In the discussion that follows, we will make use of

Lemma 2.1. Consider the sequence $\{a_{m,n} : m, n \geq 0\}$ from I^2 into $\mathfrak{C}$, satisfying the condition

$$\lim_{\|m,n\| \to \infty} |a_{m,n}|^{1/(m+n)} = 0 .$$

Then

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{m,n} a_{m,n}$$

converges if and only if $\{|c_{0,0}|; |c_{m,n}|^{1/(m+n)}, m, n \geq 0, m+n \neq 0\}$ is bounded.

Proof. Assume first of all that there is a positive constant M, such that $|c_{0,0}| \leq M$, $|c_{m,n}| \leq M^{m+n}$; $m, n \geq 0$, $m+n \neq 0$. From the hypothesis on $\{a_{m,n}\}$, we find an integer N, such that

$$|a_{m,n}| \leq \left[\frac{1}{2M}\right]^{m+n}, \qquad \text{for} \quad \|(m, n)\| \geq N ,$$
$$\Rightarrow |a_{m,n} c_{m,n}| \leq 2^{-m-n} , \qquad \|(m, n)\| \geq N .$$

Therefore $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |c_{m,n} a_{m,n}|$ converges and the sufficiency of the lemma follows.

To prove the necessity, assume that the series in question is convergent but

$$\{|c_{0,0}|, |c_{m,n}|^{1/(m+n)} , \qquad m, n \geq 0, m+n \neq 0\}$$

is unbounded. Then in general there exist increasing sequences $\{m_k\}$, $\{n_l\} \subset I$, such that

$$|c_{m_k, n_l}| \geq (k+l)^{m_k + n_l}$$

Define

$$a_{m,n} = 0 \text{ if } m \neq m_k , \quad n \neq n_l$$
$$a_{m,n} = (k+l)^{-m-n} \quad \text{if} \quad m = m_k, n = n_l .$$

Then

$$\lim_{\|(m,n)\| \to \infty} |a_{m,n}|^{1/(m+n)} = 0 ;$$

moreover

$$|a_{m_k, n_l} c_{m_k, n_l}| \geq 1 , \qquad k, l \geq 1 .$$

Therefore $|a_{m,n} c_{m,n}| \not\to 0$ as $\|(m, n)\| \to \infty$. Hence $\sum \sum a_{m,n} c_{m,n}$ does not converge and this contradiction completes the proof of the lemma.

One may now completes the proof of

Theorem 2.2. *Let us consider X equipped with either T or T^*. Then every continuous linear functional ϕ on X is of the form*

$$\phi(f)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{m,n}c_{m,n}\,, \qquad f(z_1, z_2)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{m,n}z_1^m z_2^n\,, \tag{2.2}$$

where

$$\{|c_{0,0}|;\ |c_{m,n}|^{1/(m+n)},\quad m, n\geq 0, m+n\neq 0\}\quad \textit{is bounded.} \tag{2.3}$$

Moreover, if any double sequence $\{c_{m,n} : m, n\geq 0\}$ satisfies (2.2), *then the mapping $\phi : X\rightarrow \mathfrak{C}$ whose value at any $f\in X$ is given by* (2.1), *represents a continuous linear functional on X.*

Proof. First, assume that $\phi\in X'$, the topological dual of X and that

$$\phi(\delta_{m,n})=c_{m,n}\,, \qquad m, n\geq 0\,,$$

where $\delta_{m,n}(z_1, z_2)=z_1^m z_2^n$. Let

$$f(z_1, z_2)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{m,n}z_1^m z_2^n\,; \qquad f_{m,n}(z_1, z_2)=\sum_{l=0}^{n}\sum_{k=0}^{m} a_{k,l}z_1^k z_2^l\,.$$

Then $f_{m,n}\rightarrow f$ as $\|(m, n)\|\rightarrow\infty$ uniformly on compacta in $\mathfrak{C}^2$ and therefore

$$\lim_{\|(m,n)\|\rightarrow\infty}\phi(f_{m,n})=\phi(f)\,;$$

moreover

$$\phi(f_{m,n})=\sum_{l=0}^{n}\sum_{k=0}^{m} a_{k,l}c_{k,l}\,; \qquad m, n\geq 0\,.$$

Hence

$$\phi(f)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{m,n}c_{m,n}\,.$$

Since

$$\lim_{\|(m,n)\|\rightarrow\infty} |a_{m,n}|^{1/(m+n)}=0\,,$$

we find that $\{c_{m,n} : m, n\geq 0\}$ satisfies (2.3).

To prove the other part of the theorem, let ϕ be as mentioned in the hypothesis. In view of lemma 2.1, ϕ is well defined on X and is a linear functional on X. To prove continuity of ϕ, let $f_p\rightarrow 0$ in (X, T) as $p\rightarrow\infty$, where $f_p\in X$, $p\geq 1$, and

$$f_p(z_1, z_2)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{m,n}^{(p)}z_1^m z_2^n, \qquad p\in I, p\geq 1\,.$$

Suppose

$$M=\sup \{|c_{0,0}|;\ |c_{m,n}|^{1/(m+n)},\quad m, n\geq 0, m+n\neq 0\}\ .$$

Let $\varepsilon>0$ be chosen so that $\varepsilon M<1$. Then there exists $Q=Q(\varepsilon)$, such that

$$|a_{0,0}^{(p)}|,\ |a_{m,n}^{(p)}|^{1/(m+n)}<\varepsilon,\quad p\geq Q;\qquad m, n\geq 0, m+n\neq 0\ .$$

Now

$$\begin{aligned}|\phi(f_p)|&<\varepsilon M+\sum_{\substack{n=0\\ m+n\neq 0}}^{\infty}\sum_{m=0}^{\infty}|a_{m,n}^{(p)},\ c_{m,n}|\ ;\qquad \text{for}\quad p\geq Q\\ &<\varepsilon M+\sum_{\substack{n=0\\ m+n\neq 0}}^{\infty}\sum_{m=0}^{\infty}(\varepsilon M)^{m+n}\ ;\qquad \text{for}\quad p\geq Q\\ &\to 0\qquad \text{as}\quad \varepsilon\to 0\ .\end{aligned}$$

Therefore ϕ is continuous and the proof of the result is complete.

3. Bases in X. The sequence $\{z_1^m z_2^n : m, n\in I;\ z_1, z_2\in \mathfrak{C}$ and are fixed$\}$ plays a significant role in determining uniquely the representation of $f(z_1, z_2)$ where $f\in X$; one is apparantly tempted therefore to think of $\{\delta_{m,n} : m, n\geq 0\}$, $\delta_{m,n}(z_1,z_2)=z_1^m z_2^n$ as a Hamel base for X. However, this is not true, since the function $f\in X$, $f(z_1, z_2)=e^{z_1+z_2}$ cannot be represented as a finite linear combination of $\{\delta_{m,n} : m, n\geq 0\}$. In fact, $\{\delta_{m,n}\}$ is a base in the sense of the definition that follows hereafter.

Definition 3.1. A sequence $\{\alpha_{m,n} : m, n\geq 0\}\subset(X, T)$ is said to be a *base* for X, if for each $f\in X$, there exists a unique sequence $\{a_{m,n} : m, n\geq 0\}\subset\mathfrak{C}$, such that

$$f=\lim_{||(m,n)||\to\infty}\sum_{j=0}^{n}\sum_{k=0}^{m}a_{j,k}\alpha_{j,k}\ (\text{in } T)\ ,$$

or

$$f=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_{m,n}\alpha_{m,n}\ ,$$

where the convergence of this double series is with respect to the topology of of uniform convergence on compacta in $\mathfrak{C}^2$. The members $a_{m,n}$ are called the base functions.

In view of this definition we find that $\{\delta_{m,n}\}$ is a base for X and moreover, for this base, the base functions satisfy the following condition:

(3.1) $$\lim_{||(m,n)||\to\infty}|a_{m,n}|^{1/(m+n)}=0\ .$$

However, for all bases in X, the corresponding coefficients do not necessarily satisfy (3.1). For instance, consider $\{\alpha_{m,n}\}$, where $\alpha_{m,n}(z_1, z_2)=z_1^m z_2^n/((m+n)/2)!$ Then by properly scaling the coefficients in the representation (1.1) of any $f \in X$, we find that $\{\alpha_{m,n}\}$ is a base. Now

$$e^{z_1 z_2}=\sum_{m-n=0}^{\infty} \alpha_{m,n}(z_1, z_2)$$

and so $a_{m,n}=1$, for all $m=n\geq 0$, $a_{m,n}=0$ for $m\neq n$. Thus

$$\lim_{\|(m,n)\|\to\infty} |a_{m,n}|^{1/(m+n)}=1\ ,$$

and our assertion follows.

Definition 3.2. A sequence $\{\alpha_{m,n} : m, n\geq 0\}\subset(X, T)$ will be called an *absolute base* for X, if each $f \in X$ can be uniquely expressed as $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{m,n}\alpha_{m,n}$, where this double series is absolutely convergent on compacta in $\mathfrak{C}^2$.

Definition 3.3. A sequence $\{\alpha_{m,n} : m, n\geq 0\}\subset(X, T)$ is called a *proper base* for X if

(i) $\{\alpha_{m,n}\}$ is an absolute base for X, and

(ii) for any sequence $\{a_{m,n} : m, n\geq 0\}\subset\mathfrak{C}$, the series $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{m,n}\alpha_{m,n}$ converges absolutely on compacta in $\mathfrak{C}^2$ if and only if

$$\lim_{\|(m,n)\|\to\infty} |a_{m,n}|^{1/(m+n)}=0\ .$$

Let now for $f \in X$ and $R_1, R_2>0$

$$\|f; R_1, R_2\|=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |a_{m,n}|\, R_1^m R_2^n\ .$$

From *Cauchy*'s inequality we have

$$M(R_1, R_2; f)\leq\|f; R_1, R_2\|\leq M(2R_1, 2R_2; f)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} 2^{-m-n}=4M(2R_1, 2R_2; f)\ .$$

Therefore the topology T_1 generated by the family $\{\|f; R_1, R_2\| : R_1, R_2>0\}$ is equivalent to T.

4. Characterisation of Proper Bases. Our discussion in this direction will require a number of intermediary results. We start with

Lemma 4.1. Let $\{\alpha_{m,n} : m, n\geq 0\}$ be a sequence of entire functions defined on $\mathfrak{C}^2$, such that $\sum\sum |\alpha_{m,n}|$ converges on compacta in $\mathfrak{C}^2$ to a function bounded on compacta in $\mathfrak{C}^2$. Then given $\lambda>1$; and $R_1, R_2>0$, there exists an integer $N>0$, such that $\|(m, n)\|\geq N$ implies

$$\max_{|z_1|=R_1,|z_2|=R_2} |\alpha_{m,n}(z_1, z_2)|^{1/(m+n)} < \lambda .$$

Proof. Suppose the conclusion of the lemma is false. Then we may find sequences $\{m_k\}$, $\{n_l\}$, such that

$$\max_{|z_1|=R_1,|z_2|=R_2} |\alpha_{m_k,n_l}(z_1, z_2)|^{1/(m_k+n_l)} > \lambda$$
$$\Longrightarrow \max_{|z_1|=R_1,|z_2|=R_2} |\alpha_{m_k,n_l}(z_1, z_2)| > \lambda^{m_k+n_l} \to \infty ,$$

as $\|(m_k, n_l)\| \to \infty$, contrary to the fact that the given series converges absolutely on compacta in $\mathfrak{C}^2$ to a function bounded on compacta in $\mathfrak{C}^2$

Now we have

Theorem 4.1. *Let $\{\alpha_{m,n} : m, n \geq 0\} \subset X$. Suppose m,n be an arbitrary sequence contained in $\mathfrak{C}$, such that*

$$\lim_{\|(m,n)\|\to\infty} |c_{m,n}|^{1/(m+n)} = 0 . \tag{4.1}$$

Then $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |c_{m,n}\alpha_{m,n}|$ converges to a function bounded on compacta in $\mathfrak{C}^2$ if and only if for each $R_1, R_2 > 0$,

$$\lim_{\|(m,n)\|\to\infty} [M_{m,n}(R_1, R_2)]^{1/(m+n)} < +\infty , \tag{4.2}$$

where

$$M_{m,n}(R_1, R_2) = M(R_1, R_2; \alpha_{m,n}) ; \qquad m, n \geq 0 .$$

Proof. (Necessity). Suppose the double series be convergent absolutely with respect to T, and that (4.2) is not true. Hence for some $R_1, R_2 > 0$, there exist sequences $\{m_k\}$, $\{n_l\} \subset I$, such that

$$M_{m_k,n_l}(R_1, R_2) > (k+l)^{m_k n_l} .$$

Define for $\lambda > 1$

$$c_{m,n} = \begin{cases} \lambda^{m_k+n_l}/M_{m_k,n_l}(R_1, R_2) , & \text{for } \quad m = m_k, n = n_l \\ 0 , & m \neq m_k, n = n_l \end{cases}$$

Then (4.1) holds. But

$$\max_{|z_1|=R_1,|z_2|=R_2} |c_{m_k,n_l}\alpha_{m_k,n_l}(z_1, z_2)|^{1/(m_k+n_l)} = \lambda ,$$

and this contradicts lemma 4.1.

(Sufficiency). Let (4.2) be satisfied. Then for each $R_1, R_2 > 0$, there exists a constant $M(R_1, R_2)$, such that

$$M_{m,n}(R_1, R_2) \le [M(R_1, R_2)]^{m+n}, \qquad m, n \ge 0 .$$

Now for $|z_1| \le R_1$, $|z_2| \le R_2$

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |c_{m,n} \alpha_{m,n}(z_1, z_2)| \le \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |c_{m,n}| [M(R_1, R_2)]^{m+n} < \infty , \tag{4.3}$$

since $|c_{m,n}| < [M(R_1, R_2)/2]^{m+n}$ for $\|(m, n)\| \ge N$ and the second series in (4.3) can be broken into two parts, one with $\|(m, n)\| < N$ and the other with $\|(m, n)\| \ge N$, the latter being obviously convergent on account of preceding arguments and the proof of the result is complete.

Theorem 4.2. *Let* $\{\alpha_{m,n} : m, n \ge 0\} \subset X$ *and* $\{c_{m,n} : m, n \ge 0\}$ *be an arbitrary sequence in* $\mathfrak{C}$, *such that*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{m,n} \alpha_{m,n}$$

converges absolutely on compacta in $\mathfrak{C}^2$ *to a function bounded on compacta in* $\mathfrak{C}^2$. *Then*

$$\lim_{\|(m,n)\| \to \infty} |c_{m,n}|^{1/(m+n)} = 0 \tag{4.4}$$

if and only if

$$\lim_{R_1, R_2 \to \infty} \{ \varliminf_{\|(m,n)\| \to \infty} [M_{m,n}(R_1, R_2)]^{1/(m+n)} \} = +\infty . \tag{4.5}$$

Proof. (Necessity). Let (4.4) hold good and suppose (4.5) is not true. Then for each $R_1, R_2 > 0$

$$\varliminf_{\|(m,n)\| \to \infty} [M_{m,n}(R_1, R_2)]^{1/(m+n)} < M < +\infty ,$$

since $M_{m,n}(R_1, R_2)$ is monotonocally increasing in $R_1, R_2 > 0$ for each fixed pair (m, n). There exist sequences $\{m_k\}$, $\{n_l\} \subset I$, such that

$$M_{m_k, n_l}(R_1, R_2) < M^{m_k + n_l} .$$

If $R_1, R_2 > 0$ are given, then this inequality yields

$$|\alpha_{m_k, n_l}(z_1, z_2)| < M^{m_k + n_l}, \qquad \text{for} \quad |z_1| \le R_1, |z_2| \le R_2 .$$

Define

$$C_{m,n} = \begin{cases} (2M)^{-1/(m_k + n_l)}, & m = m_k, n = n_l \\ 0, & m \ne m_k, n \ne n_l . \end{cases}$$

Hence the series $\sum\sum c_{m,n} \alpha_{m,n}$ converges absolutely on compacta in $\mathfrak{C}^2$. But

$$\overline{\lim_{\|(m,n)\|\to\infty}} |c_{m,n}|^{1/(m+n)} = \frac{1}{2M} \neq 0 \,,$$

and this contradicts (4.4).

(Sufficiency). Suppose (4.5) is true and in turn (4.4) is false, i.e.

$$\text{(4.6)} \qquad \overline{\lim_{\|(m,n)\|\to\infty}} |c_{m,n}|^{1/(m+n)} > \mu > 0 \,.$$

Now for $R_1, R_2 > 0$, one finds from (4.5)

$$\underline{\lim_{\|(m,n)\|\to\infty}} [M_{m,n}(R_1, R_2)]^{1/(m+n)} > \frac{2}{\mu} \,,$$

$$\Rightarrow M_{m,n}(R_1, R_2) > \left(\frac{2}{\mu}\right)^{m+n} , \qquad \text{for all large } \|(m, n)\| \,.$$

Also for $m = m_k$, $n = n_l$, $k, l \in I$, we find from (4.6) that

$$|c_{m_k, n_l}| > \mu^{m_k + n_l} \,, \qquad \text{where} \quad \|(m_k, n_l)\| \to \infty \quad \text{with} \quad \|(k, l)\| \,.$$

Therefore

$$|c_{m_k, n_l} M_{m_k, n_l}(R_1, R_2)|^{1/(m_k + n_l)} > 2 \,,$$

and this contradicts lemma 4.1. The proof of the result is now complete.

Combining Theorem 4.1 and 4.2, we have the following main result of this section.

Theorem 4.3. *Let* $\{\alpha_{m,n} : m, n \geq 0\}$ *be an absolute basis in* (T, X). *Then* $\{\alpha_{m,n}\}$ *is proper if and only if* (4.2) *and* (4.5) *hold.*

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