CONDITIONS IMPLYING CONTINUITY OF FUNCTIONS IN TOPOLOGICAL SPACES

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(Received March 22, 1971)

Introduction. *Halfer* [1] uses the notion of neighbourhoods to introduce the definition of removable discontinuity of functions in topologicol spaces. He shows that a function having at worst a removable discontinuity is continuous under certain conditions. *Pervin and Levine* [2] define removable discontinuity of functions with the help of convergent sequences. They have noted that if the domain space is first axiom, then the two definitions are equivalent. They have also proved certain results implying continuity of functions having at worst a removable discontinuity. In this paper, we attempt to extend the notion of removable discontinuity to ordinary discontinuity of functions in topological spaces. In section A, along with certain other results, we prove some theorems implying continuity of functions having at worst ordinary discontinuity. In section B, we prove two theorems on the continuity of functions under certain general conditions.

All spaces under our consideration are at least Hausdorff spaces. By $f: X \to Y$ we mean a single-valued mapping of X into Y. A function $f: X \to Y$ will be said to be connected if and only if the image of every connected set of X is connected in Y [2] and f will be said to have closed point inverse if for each $y \in Y$, $f^{-1}(y)$ is a closed subset of X [1]. The terms mapping and function are synonymous. The rest of the terminology is standard. We use the notation $Cl\,A$ for closure of A.

Section A

Definition A [1]. A function $f: X \to Y$ has at worst a removable discontinuity at $p \in X$ if there exists a point $y \in Y$ such that for each neighbourhood V of Y, there is a neighbourhood U of p such that $f(U-[p]) \subset V$.

Definition 1. A function $f: X \to Y$ has at worst an ordinary discontinuity at $p \in X$, if there exists a set S, say, consisting of the finite number of points $y_1, y_2, y_3, \cdots, y_n$ of Y such that for every open neighbourhood V_α of y_α, $\alpha=1, 2, \cdots, n$ and every open neighbourhood V of p $f(V) \cap V_\alpha \neq \phi$, $\alpha=1, 2, \cdots, n$ there exists an open neighbourhood U of p such that

$$f(U-[p])\subset \bigcup_{\alpha=1,2,\cdots,n} [V_\alpha] .$$

Remark 1. If f has a removable discontinuity at $p\in X$, then it has an ordinary discontinuity there but not conversely.

Theorem 1. *Let f be a connected mapping of a locally connected space X into Y. Then f is continuous at p if and only if f has at worst an ordinary discontinuity there.*

Proof. Let f have at worst an ordinary discontinuity at p and let S consist of the points $y_1, y_2, y_3, \cdots, y_n$. If possible, let f be not continuous at p. Then there exists at least one y_i which is different from $f(p)$. Since Y is Hausdorff, there exist disjoint open neighbourhoods $V, V_1, V_2, \cdots, V_n$ of $f(p), y_1, y_2, \cdots, y_n$ respectively. And since f has an ordinary discontinuity at p, there exists an open neighbourhood U of p such that

$$f(U)\subset V\cup V_1\cup V_2\cup \cdots \cup V_n .$$

Since X is locally connected, there exists a connected open set C such that $p\in C\subset U$. Therefore $f(C)\subset V\cup V_1\cup V_2\cup \cdots \cup V_n$ which is a contradiction, since $f(C)$ is connected. Hence f is continuous at p. The necessary part is clear. This proves the theorem.

Theorem 2. *Let X be regular and f be a closed function with closed point inverses and let f have at worst an ordinary discontinuity at $p\in X$. Then f is continuous at p.*

Proof. If p is isolated in X, the result is immediately true. Let, therefore, p be non-isolated. If possible, let f be not continuous at p and S consist of the points $y_1, y_2, \cdots, y_n$ where at least one y_i is different from $f(p)$. We may, however, clearly suppose that $f(p)\neq y_i$ for $i=1, 2, \cdots, n$. Since point inverses are closed, $f^{-1}(y_i)$ are closed sets in X. Again since X is regular and $p\notin f^{-1}(y_i)$, there exist open neighbourhoods U_i of p such that $f^{-1}(y_i)\cap Cl U_i=\phi$. Let $\bigcap_{i=1,2,\cdots,n} U_i=U$ Therefore $p\in U$ and since

$$Cl U=Cl\{\bigcap_{i=1,2,\cdots,n} U_i\}\subset \bigcap_{i=1,2,\cdots,n} Cl U_i ,$$

we have

$$Cl U\cap \{\bigcup_{i=1,2,\cdots,n} f^{-1}(y_i)\}=\phi .$$

Since f is closed, $f(Cl U)$ is closed and since $y_i\notin f(Cl U)$, there exist open neighbourhoods V_i of y_i such that $V_i\cap f(Cl U)=\phi$. Therefore $\{\bigcup_{i=1,2,\cdots,n} V_i\}\cap \{f(Cl U)\}=\phi$. Since

f has an ordinary discontinuity at p, there exists an open neighbourhood W of p such that

$$f(W-[p])\subset \bigcup_{i=1,2,\cdots,n} V_i .$$

As p is non-isolated, $(U\cap W)-p\neq\phi$. Again

$$\phi\neq f(W-[p])\cap f(ClU)\subset\{\bigcup_{i=1,2,\cdots,n} V_i\}\cap f(ClU)=\phi ,$$

which is a contradiction. Hence f is continuous at p.

Definition B [2]. Let f be a mapping of X into Y. For every point p in X let the set of limit points of f at p denoted by $L(f; p)$ be the set of all points p^* in Y for which there exists a sequence p_n of points in X such that

$$\lim p_n=p \qquad \text{and} \qquad \lim f(p_n)=p^* .$$

Pervin and Levine [2] show that $L(f; p)$ is a closed and connected set provided that f, X and Y satisfy certain conditions.

Definition 2. Let f be a mapping of X into Y. Then for every point $p\in X$ the set $D(f; p)$ consists of points y_α, $\alpha\in I$ where I is an index set and $y_\alpha\in Y$, if and only if for every open neighbourhood V_α of y_α and for every open neighbourhood U of p, $f(U)\cap V_\alpha\neq\phi$.

Remark 2. It is clear that $D(f; p)$ always contains $f(p)$.

Theorem 3. *Let f be a mapping of the first axiom space X into the sequentially compact space Y. Then f is continuous at p if and only if $D(f; p)$ consists only of $f(p)$.*

Proof. Supsose that $D(f; p)$ consists only of $f(p)$ and if possible, let f be not continuous at p. Since X is first axiom, there exists a monotone descending sequence of open neighbourhoods $\{U_i\}$ forming a base at p. Again since f is not continuous at p, there exists an open neighbourhood V of $f(p)$ such that for every U_i there exists at least one point $p_i\in U_i$ satisfying $f(p_i)\not\subset V$. It then follows that the sequence $\{f(p_i)\}$ has a subsequence $\{f(p_{\alpha i})\}$ which converges to p^*, say where $p^*\notin V$. Again $\{p_n\}\to p$. Therefore for arbitrary open neighbourhoods V_1 and U of p^* and p respectively, we have $f(p_{\alpha i})\in V_1$ for $\alpha_i\geqslant N_1$ and $p_i\in U$ for $i\geqslant N_2$. Hence $f(p_{\alpha i})\in f(U)\cap V_1$ for $i\geqslant N=\max.(N_1, N_2)$. Therefore $f(U)\cap V_1\neq\phi$ implying that $p^*\in D(f; p)$ which is contradictory. Consequantly f is continuous at p. The necessary part is clear. This proves the theorem.

Theorem 4. $L(f; p) \subset D(f; p)$.

Proof. Let $p^* \in L(f; p)$. Then there exists a sequence of points $\{p_n\}$ in X such that $\lim p_n = p$ and $\lim f(p_n) = p^*$. If V be any open neighbourhood of p^*, $f(p_n) \in V$ for $n \geqslant N_1$ and if U be any open neighbourhood of p, $p_n \in U$ for $n \geqslant N_2$. If $N = \max.(N_1, N_2)$, then $f(p_n) \in f(U) \cap V$ for $n \geqslant N$. Consequently $p^* \in D(f; p)$.

Theorem 5. *$D(f; p)$ is a closed subset of Y for a fixed point $p \in X$.*

Proof. Let p^* be a limit point of $D(f; p)$. Then every open set O^* of Y containing p^* contains at least one point q^* (say) of $D(f; p)$ and different from p^*. Since $q^* \in D(f: p)$ for every open neighbourhood U of p, we have $f(U) \cap O^* \neq \phi$. Hence $p^* \in D(f; p)$ and consequently $D(f; p)$ is closed.

Theorem 6. *If f be a connected mapping of the locally connected first axiom space X into the first axiom compact space Y, then $D(f; p)$ is a connected subset of Y.*

The proof can be constructed in a way similar to that of theorem 3.7 [2].

Theorem 7. *Suppose that f is a connected mapping of the locally connected first axiom space X into the compact first axiom space Y, then f is continuous at p if and only if $D(f; p)$ is finite or denumerable.*

Proof is similar to that of theorem 3.8 of [2].

Section B

Theorem 8. *Let f be an open mapping of a first axiom space X onto a sequentially compact space Y having the property that if U_1 and U_2 be any two open sets in X, then $U_1 \cap U_2 = \phi$ implies $f(U_1) \cap f(U_2) = \phi$. Then f is continuous.*

Proof. If possible, let f be not continuous at a point $p \in X$. Since X is first axiom, there axists a monotone descending sequence of open sets $\{U_i\}$ forming a base at p. Discontinuity of the function f at p implies the existence of an open set V containing $f(p)$ such that for every open set U_i there exists at least one point $p_i \in U_i$ satisfying $f(p_i) \notin V$. The sequence $\{p_i\}$ corresponding to the sequence of open sets $\{U_i\}$ converges to p. Since Y is sequentially compact, the sequence of points $\{f(p_i)\}$ has a convergent subsequence $\{f(p_{\alpha i})\}$, say. Let $\{f(p_{\alpha i})\}$ converge to the point $f(\beta)$ which lies outside V. Since $p \neq \beta$ and X is Hausdorff, there exist two open sets U_1 and U_2 such that $p \in U_1$, $\beta \in U_2$ and $U_1 \cap U_2 = \phi$. As $\{p_i\}$ converges to p, $p_i \in U_1$ for $i \geqslant N_1$. Since $f(U_2)$ is an open set containing $f(\beta)$ and $\{f(p_{\alpha i})\}$ converges to $f(\beta)$, $f(p_{\alpha i}) \in f(U_2)$ for $\alpha_i \geqslant N_2$. Let $\max.(N_1, N_2) = N$. Therefore $f(p_{\alpha i})\in f(U_1)$, $f(U_2)$ for $i\geqslant N$. This implies that $f(U_1)\cap f(U_2)\neq\phi$ which leads to a contradiction. Hence f is continuous at p.

Remark 3. If the last of the given condition of the above theorem be not satisfied, the function may fail to be continuous as shown by

Example 1. Let $X=[0, 3]$ and $Y=[0, 2]$ and let f: $X\to Y$ be defined as

$$\begin{aligned} f(x)&=x \qquad \text{for} \quad 0\leqslant x<1\,, \\ &=3-x \quad \text{for} \quad 1\leqslant x\leqslant 3\,. \end{aligned}$$

Let $U_1=(0, 1/2)$ and $U_2=(5/2, 3)$. Then $U_1\cap U_2=\phi$ does not imply $f(U_1)\cap f(U_2)=\phi$ and f is not continuous. It is clear that the other conditions of the theorem are satisfied.

Theorem 9. *Let f be a connected mapping of a first axiom space X into a sequentially compact space Y and having the property that under f the inverse of every open set in Y is connected in X. Then f is continuous.*

Proof. If possible, let f; $X\to Y$ be discontinuous at a point $p\in X$. As in the previous theorem, there exists a monotone descending sequence of open set $\{U_i\}$ forming a base at p and there exists an open set V containing $f(p)$ such that there exists at least one point p_i belonging to U_i with the property that $f(p_i)$ does not belong to V. It may be noted that the sequence $\{p_i\}$ converges to p. Now Y being sequentially compact, the sequence $\{f(p_i)\}$ has a convergent subsequence $\{f(p_{\alpha i})\}$, say. Let the subsequence $\{f(p_{\alpha i})\}$ converge to q^* where $q^*\notin V$. Y being Hausdorff and q^* being different from $f(p)$, these can be separated strongly. Without any loss of generality we may assume that the open sets V_1 and V separate them strongly. $f^{-1}(V)$ and $f^{-1}(V_1)$ are connected subsets of X where $p\in f^{-1}(V)$ and $f^{-1}(V_1)$ contains all points of $\{p_{\alpha i}\}$ except at most finite number. The subsequence $\{p_{\alpha i}\}$ converges to the point p and hence p is a limit point of the subset $f^{-1}(V_1)$. Thus $f^{-1}(V_1)\cup\{p\}=A$ is connected. Putting $f^{-1}(V)=B$, we have A and B are connected subsets of X and since each of them contains the point p, $A\cup B$ is also connected. But $f(A\cup B)=f(A)\cup f(B)\subset V\cup V_1$ which is a contradiction since f is connected. Consequently is continuous at p.

The author is thankful to Professor *B. K. Lahiri* for his helpful suggestions in the preparation of the paper.

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