PERTURBATIONS OF THE MONOTONE SHIFT

By

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(Received July 1, 1971)

1. Let H be a Hilbert space with orthonormal basis $\{e_i\}_0^\infty$. If the operator S is defined on H by $Se_i=\alpha_i e_{i+1}$ for $i=0, 1, \cdots$ where $|\alpha_i|\leq|\alpha_{i+1}|\leq M$ for $i=0, 1, 2, \cdots$ then S will be called a monotone shift. For the case $\alpha_i=\alpha_{i+1}$, $i=0, 1, 2, \cdots$ the lattice of invariant subspaces of the shift operator S on $l^2(0, \infty)$ has been characterized by *Beurling* [1] and in [2] *J. Freeman* shows that for a large class of compact operators, the perturbed shift $S+C$ is similar to the unperturbed shift S. A characterization of the operators which are similar to the weighted shift (*i.e.*, not necessarily monotone) using the geometry of invariant subspaces appear in [5].

In general, if one perturbs an arbitrary operator by a compact operator this may markedly change the fine structure of the spectrum. Using the results of *Friedrichs* [3], [4] and *Freeman* [2] we prove that every monotone shift S with $\alpha_0\neq 0$ is similar to $S+P$ where P is a compact operator with strictly lower-triangular matrix.

2. Let $\mathscr{A}$ the algebra of operators on $l^p(0, \infty)(1\leq p\leq\infty)$ with the matrix $A=(a_{nm})_{n,m=0}^\infty$ which satisfies the conditions:

$$\sup_m \sum_{n=0}^\infty |a_{nm}|=a<\infty\ , \quad \sup_n \sum_{m=0}^\infty |a_{nm}|=b<\infty\ ,$$

By a theorem of *M. Riesz* the operators A are bounded on $l^p(0, \infty)(1\leq p\leq\infty)$ and

$$\|A\|_p\leq \max\{\|A\|_1, \|A\|_\infty\}\ ,$$

where $\|A\|_1=a$ and $\|A\|_\infty=b$.

It is easy to see that $\mathscr{A}$ is a Banach algebra with the norm

$$\|A\|=\max\{\|A\|_1, \|A\|_\infty\}\ ,$$

and that the class $\mathscr{L}$ of matrices P such

$$|P|=\sum_{n,m=0}^\infty |p_{nm}|<\infty\ ,$$

is an ideal in $\mathscr{A}$ and $|APB|\leq\|A\||P|\|B\|$ for all A, $B\in\mathscr{A}$, $P\in\mathscr{L}$. By $\mathscr{A}_0$ we denote the subalgebra of the Banach algebra $\mathscr{A}$ consisting of these matrices $A\in\mathscr{A}$ which are lower-triangular, i.e., $a_{mn}=0$ unless $n\geq m$ and $\mathscr{L}_0$ is the subspace of $\mathscr{L}$ consisting of such $P\in\mathscr{L}$ with $p_{mn}=0$ unless $n>m$ (strictly lower-triangular).

Let S_1 the weighted shift defined by $S_1e_i=\dfrac{1}{\bar{\alpha}_i}e_{i+1}$ for $i=0,1,2,\cdots$. The matrices representing S_1 and S_1^* are

$$S_1=\begin{pmatrix} 0 & 0 & 0\cdots \\ 1/\bar{\alpha}_0 & 0 & 0\cdots \\ 0 & 1/\bar{\alpha}_1 & 0\cdots \\ \cdot & \cdot & \cdots \end{pmatrix} \text{ and } S_1^*=\begin{pmatrix} 0 & 1/\alpha_0 & 0 & \cdots \\ 0 & 0 & 1/\alpha_1 & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{pmatrix},$$

and we have that $S_1S^*=E$ and $S^*S_1=I$ where $E=\operatorname{diag}(0,1,1,\cdots)$.

The existence of an invertible operator X such that S is similar to its perturbations with $P\in\mathscr{L}_0$ will be obtained using the formal analogy between the equation $\Delta X=SX-XS=XP$ and the classical differential equations [2], [3], [4].

If we define an operator γ on $\mathscr{L}_0$ by

$$\gamma(P)=\sum_{k=0}^{\infty}S_1^{*k}PS^k\,, \tag{1}$$

then $\gamma(P)$ is a strictly lower-triangular matrix and

$$\|\gamma(P)\|\leq|P|\,. \tag{2}$$

Indeed, since

$$S_1^{*k}PS^k=\begin{pmatrix} \frac{1}{\alpha_0}\cdots\frac{1}{\alpha_{k-1}}\alpha_0\cdots\alpha_{k-1}p_{kk} & \frac{1}{\alpha_0}\cdots\frac{1}{\alpha_{k-1}}\alpha_1\cdots\alpha_k p_{k,k+1}\cdots \\ \frac{1}{\alpha_1}\cdots\frac{1}{\alpha_k}\alpha_0\cdots\alpha_{k-1}p_{k+1,k} & \frac{1}{\alpha_1}\cdots\frac{1}{\alpha_k}\alpha_1\cdots\alpha_k p_{k+1,k+1}\cdots \\ \cdots\cdots & \cdots\cdots \\ \frac{1}{\alpha_i}\cdots\frac{1}{\alpha_{i+k-1}}\alpha_0\cdots\alpha_{k-1}p_{k+i,k} & \frac{1}{\alpha_i}\cdots\frac{1}{\alpha_{i+k-1}}\alpha_1\cdots\alpha_k p_{k+i,k+1} \\ \cdots\cdots & \cdots\cdots \end{pmatrix},$$

and if we consider

$$\sum_{k=0}^{\infty}\frac{1}{\alpha_i}\cdots\frac{1}{\alpha_{i+k-1}}\alpha_j\cdots\alpha_{j+k-1}p_{k+i,k+j}\,,$$

we have

$$\sup_j\sum_{i=0}^{\infty}\Big|\sum_{k=0}^{\infty}\frac{1}{\alpha_i}\cdots\frac{1}{\alpha_{i+k-1}}\alpha_j\cdots\alpha_{j+k-1}p_{k+i,k+j}\Big|\leq\sup_j\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\left|\frac{\alpha_j}{\alpha_i}\right|\left|\frac{\alpha_{j+1}}{\alpha_{i+1}}\right|\cdots\left|\frac{\alpha_{j+k-1}}{\alpha_{i+k-1}}\right|$$

$$\times |p_{k+i,k+j}| \leq \sup_j \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} |p_{k+i,k+j}| \leq |P| ,$$

and similarly

$$\sup_i \sum_{j=0}^{\infty} | \sum_{k=0}^{\infty} \frac{1}{\alpha_i} \cdots \frac{1}{\alpha_{i+k-1}} \alpha_j \cdots \alpha_{j+k-1} p_{k+i,k+j} | \leq |P| .$$

Then, if we denote by

$$[\gamma(P)]_{ij} = \sum_{k=0}^{\infty} \frac{1}{\alpha_i} \cdots \frac{1}{\alpha_{i+k-1}} \alpha_j \cdots \alpha_{j+k-1} p_{k+i,k+j} ,$$

we obtain that $\|\gamma(P)\|_1 \leq |P|$ and $\|\gamma(P)\|_\infty \leq |P|$. Therefore $\|\gamma(P)\| \leq |P|$.

Now if

$$\gamma_n(P) = \sum_{k=0}^{n} S_1^{*k} P S^k ,$$

is the sequence of partial sums of series (1) we have

$$\gamma(P) - \gamma_{n-1}(P) = \gamma(S_1^{*n} P S^n) ,$$

and hence

$$\|\gamma(P) - \gamma_{n-1}(P)\| = \|\gamma(S_1^{*n} P S^n)\| \leq |S_1^{*n} P S^n| \leq \sum_{i=n}^{\infty} \sum_{j=n}^{\infty} |p_{ij}| \to 0 \quad \text{for} \quad n \to \infty$$

Let the operator

$$\gamma_P : Q \to \gamma(Q) P .$$

If $Q, P \in \mathscr{L}_0$, we have that

$$|\gamma(Q) P| \leq \|\gamma(Q)\| \, |P| \leq |Q| \, |P| .$$

For the iterates of γ_P we give

Lemma 2.1. *If $P, Q \in \mathscr{L}_0$ then $\gamma_P^n(Q) \in \mathscr{L}_0$ and*

$$\|\gamma_P^n(Q)\| \leq \frac{|P|^n}{n!} |Q| .$$

Proof. It is clear that $\gamma_P(Q) \in \mathscr{L}_0$ and that

$$[\gamma_P(Q)]_{ij} = \sum_{j < m_1 < i} \sum_{s_1=0}^{\infty} \frac{1}{\alpha_{s_1}} \cdots \frac{1}{\alpha_{s_1+i-1}} \alpha_{s_1} \cdots \alpha_{s_1+j-1} q_{i+s_1+1, m_1+s_1} p_{m_1, j}$$

and for its iterates we have

$$[\gamma_P^k(Q)]_{ij} = \sum_{j < m_1 \leq \cdots \leq m_k < i} \sum_{0 \leq s_1 \leq \cdots \leq s_k < \infty} \frac{1}{\alpha_i} \cdots \frac{1}{\alpha_{i+s_1-1}} \alpha_{m_1} \cdots \alpha_{m_1+s_1-1}$$

$$\cdot\frac{1}{\alpha_{i+s_1}}\cdots\frac{1}{\alpha_{i+s_2-1}}\alpha_{m_2+s_1}\cdots\alpha_{m_2+s_2-1}\frac{1}{\alpha_{i+s_2}}\cdots\frac{1}{\alpha_{i+s_3-1}}\alpha_{m_2+s_2}\cdots$$
$$\cdots\alpha_{m_{k-1}+s_{k-1}-1}\frac{1}{\alpha_{i+s_{k-1}}}\cdots\frac{1}{\alpha_{i+s_k-1}}\alpha_{m_k+s_{k-1}}\cdots\alpha_{m_k+s_k-1}$$
$$\times q_{i+s_k,m_k+s_k}p_{m_k+s_{k-1},m_{k-1}+s_{k-1}}\cdots p_{m_2+s_1,m_1+s_1}p_{m_1,j}\,.$$

Indeed

$$[\gamma_P^{k+1}(Q)]_{ij}=\sum_{m=j+1}^{i-1}\sum_{s=0}^{\infty}\frac{1}{\alpha_i}\cdots\frac{1}{\alpha_{i+s-1}}\alpha_m\cdots\alpha_{m+s-1}[\gamma_P^k(Q)]_{i+s,m+s}p_{mj}$$
$$=\sum_{m=j+1}^{i-1}\sum_{s=0}^{\infty}\sum_{m+s<m_1\cdots<m_k<i+s}\sum_{0\le s_1\le\cdots<s_k\le\infty}\frac{1}{\alpha_i}\cdots\frac{1}{\alpha_{i+s-1}}\alpha_m\cdots\alpha_{m+s-1}$$
$$\times\frac{1}{\alpha_{i+s}}\cdots\frac{1}{\alpha_{i+s+s_1-1}}\times\cdots$$
$$\times\alpha_{m_k+s_{k-1}}\cdots\alpha_{m_k+s_k-1}q_{i+s+s_k,m_k+s_k}\cdots p_{m_2+s_1,m_1+s_1,}p_{m_1,m+s}p_{mj}\,.$$

and if denotes $m=\bar{m}_1$, $m_1-s=\bar{m}_2$, $\cdots$ $m_k-s=\bar{m}_{k+1}$, $s=\bar{s}_1$, $s_1+s=\bar{s}_2$, $\cdots$, $s_k+s=\bar{s}_{k+1}$, we obtain

$$[\gamma_P^{k+1}(Q)]_{ij}=\sum_{j<\bar{m}_1<\cdots\bar{m}_{k+1}<i}\sum_{0<\bar{s}_1<\cdots<\bar{s}_{k+1}<\infty}\frac{1}{\alpha_i}\cdots\frac{1}{\alpha_{i+\bar{s}_1-1}}\alpha_{\bar{m}_1}\cdots\alpha_{\bar{m}_1+\bar{s}_1-1}\frac{1}{\alpha_{i+\bar{s}_1}}\cdots$$
$$\cdots\alpha_{\bar{m}_k+\bar{s}_k-1}\frac{1}{\alpha_{i+\bar{s}_k}}\cdots\frac{1}{\alpha_{i+\bar{s}_{k+1}-1}}\alpha_{\bar{m}_{k+1}+\bar{s}_k}\cdots\alpha_{\bar{m}_{k+1}+\bar{s}_{k+1}-1}$$
$$\cdot q_{i+\bar{s}_{k+1},\bar{m}_{k+1}+\bar{s}_{k+1}}p_{\bar{m}_{k+1}+\bar{s}_k,\bar{m}_k+\bar{s}_k}\cdots p_{\bar{m}_2+\bar{s}_1,\bar{m}_1+\bar{s}_1}p_{\bar{m}_1,j}\,.$$

But

$$\left|\frac{1}{\alpha_i}\cdot\alpha_{m_1}\right|\le 1\,,\quad\left|\frac{1}{\alpha_{i+s_1-1}}\cdot\alpha_{m_1+s_1-1}\right|\le 1\,,\ \cdots,$$
$$\cdots,\left|\frac{1}{\alpha_{i+s_k-1}}\cdot\alpha_{m_k+s_k-1}\right|\le 1\quad\text{since}\quad i\ge m_1,\ \cdots,\ i+s_k-1\ge m_k+s_k-1$$

and therefore

$$|\gamma_P^n(Q)|\le\sum_{0\le j\le m_1\le\cdots\le m_n\le i<\infty}\sum_{0\le s_1\le\cdots\le s_n\le\infty}$$
$$\times|q_{i+s_n,m_n+s_n}p_{m_n+s_{n-1},m_{n-1}+s_{n-1}}\cdots p_{m_2+s_1,m_1+s_1}p_{m_1,j}|$$
$$\le\sum_{0\le j\le m_1\le\cdots\le m_n<\infty}\sum_{0\le s_1\le\cdots\le s_{n-1}<\infty}$$
$$\times\Big[\sum_{i=m_n}^{\infty}\sum_{s_n=s_{n-1}}|q_{i+s_n,m_n+s_n}|\Big]|p_{m_n+s_{n-1},\,m_{n-1}+s_n}\cdots p_{m_2+s_1,m_1+s_1}p_{m_1,j}|$$
$$\le|Q|\sum_{0\le j\le m_1\le\cdots\le m_n<\infty}\sum_{0\le s_1\le\cdots s_{n-1}<\infty}|p_{m_n+s_{n-1},m_{n-1}+s_{n-1}}\cdots p_{m_2+s_1,m_1+s_1}p_{m_1,j}|\,.$$

Since P is strictly lower-triangular the nonzero terms of the above sum are products of the form

(3) $$|p_{u_n v_n} \cdots p_{u_2 v_2} p_{u_1 v_1}| ,$$

with $u_n > v_n \geq \cdots \geq u_2 > v_2 \geq u_1 > v_1$ which contains any given entry p_{mn} of P at most once and each product (3) occurs at most once in the above sum, since the indices $j, m_1, \cdots, m_n$ and $s_1, \cdots, s_{n-1}$ are determined recursively by $u_1, \cdots, u_n$ and $v_1, \cdots, v_n$; $j = v_1$, $m_1 = u_1$, $m_1 + s_1 = v_2$, $m_2 + s_1 = u_2$, etc. The lemma follows from the fact that each product (3) occurs exactly $n!$ times when the product $|P|^n = (\sum |p_{ij}|)^n$ is expanded.

In a similar way as in [2] we define an indefinite integral $\Gamma(P)$ of integrable operators P which will be used to find an invertible operator X such that $S = X(S+P)X^{-1}$.

Let

$$\Gamma(P) = \sum_{k=0}^{\infty} S_1^{*k+1} P S^k ,$$

where $P \in \mathscr{L}_0$.

Proposition 2.1. a) *Γ maps $\mathscr{L}_0$ in $\mathscr{A}_0$ and for $P \in \mathscr{L}_0$, $\|\Gamma(P)\| \leq \frac{|P|}{|\alpha_0|}$.*

b) *$\Delta\Gamma(P) = P$.*

c) *the equation*

$$X = I + \Gamma(XP) ,$$

is uniquely solvable for $X \in \mathscr{A}_0$ and the solution is given by the Peano series

$$X = I + \Gamma(P) + \Gamma[\Gamma(P)P] + \Gamma[\Gamma[(\Gamma(P)P]P] + \cdots ,$$

which converges absolutely in $\mathscr{A}_0$.

d) *$\Delta X = XP$.*

Proof. a) It is easy to see that

$$\Gamma(P) = S_1^* \gamma(P) ,$$

and since for $P \in \mathscr{L}_0$, $\gamma(P)$ is a strictly lower-triangular matrix and $\|\gamma(P)\| \leq |P|$ we obtain that

$$\|\Gamma(P)\| \leq \frac{|P|}{|\alpha_0|} ,$$

where $\frac{1}{|\alpha_0|} = \|S_1^*\|$.

b) Since $SS_1^* = E$ where $E = \text{diag}(0, 1, 1, 1, \cdots)$ we have

$$\begin{aligned} \Delta\Gamma(P) &= S\Gamma(P) - \Gamma(P)S = SS_1^*\gamma(P) - S_1^*\gamma(P)S = SS_1^*\gamma(P) - [\gamma(P) - P] \\ &= \gamma(P) - \gamma(P) + P = P . \end{aligned}$$

c) If we define the operator $\Gamma_P: \mathscr{L}_0 \to \mathscr{L}_0$ by $\Gamma_P(Q)=\Gamma(Q)P$, then $\Gamma_P^n(Q)= S_1^{*}\gamma_P^n(Q)$ since $\Gamma(Q)=S_1^{*}\gamma(Q)=\gamma(S_1^{*}Q)$ and by Lemma 2.1 and a) we have

$$\|\Gamma(\Gamma_P^n(Q))\| \leq \frac{|\Gamma_P^n(Q)|}{|\alpha_0|} \leq \frac{|P|^n|Q|}{|\alpha_0|^{n+1}n!} .$$

Now assuming the existence of a solution $X \in \mathscr{A}_0$ we have by successive substitutions

$$X=I+\Gamma(P)+\Gamma(\Gamma(P)P)+ \cdots +\Gamma(\Gamma_P^n(Q))+\Gamma(\Gamma_P^{n+1}(XP)) ,$$

and the above reason for $Q=XP$ gives the uniqueness of the solution. The absolute convergence of the Peano series follows taking $Q=P$ in the same inequality. It is clear that the series satisfies the integral equation and applying Δ to both sides it follows $\Delta X=XP$.

In a similar way as in [2] we define the product integral

$$\hat{\Gamma}(I+P)=\prod_{k=0}^{\infty}(I+S_1^{*k+1}PS^k) ,$$

for $P\in\mathscr{L}_0$ (where $\prod_{k=0}^{n} A_k=A_nA_{n-1}\cdots A_1A_0$ for $A_k\in\mathscr{A}$). For the same reason as in [2] we conclude that for $P\in\mathscr{L}_0$ the infinite products $\prod_{k=0}^{\infty}(I+S_1^{*k+1}PS^k)$ converges in $\mathscr{A}_0$ and that the integral equation $X=I+\Gamma(XP)$ is solved by $\hat{\Gamma}(I+P)$ (the product $\prod_{k=0}^{\infty}A_k$ is said to be convergent if there exists an n_0 such that A_k is nonsingular ($A_k^{-1}\in\mathscr{A}$) for $k\geq n_0$ and $\prod_{k=n_0}^{n}A_k$ converges to a nonsingular element of $\mathscr{A}$ as $n\to\infty$).

The main result of this note is the following.

Theorem. *If $P=(p_{ij})$ is a strictly lower-triangular matrix with $|P|=\sum_{i,j=0}^{\infty}|p_{ij}| <\infty$ and $p_{i+1,i}\neq-\alpha_i$, then $S+P$ and S represents similar operators on $l^p(0,\infty)$ $(1\leq p\leq\infty)$ and the product integral $\hat{\Gamma}(I+P)$ represents an operator which implements the similarity.*

Proof. Since the similarity of $S+P$ with S is equivalent to the solvability of $\Delta X=XP$ by an invertible operator X and we proved that for $P\in\mathscr{L}_0$ the matrix equation $\Delta X=XP$ is solved by $X=\hat{\Gamma}(I+P)\in\mathscr{A}_0$. For to prove that $\hat{\Gamma}(I+P)$ is an invertible operator on $l^p(0,\infty)$ we remarks that since the infinite product $\prod_{k=0}^{\infty}(I+S_1^{*k+1}PS^k)$ is convergent, then it can be factored

$$\hat{\Gamma}(I+P)=\prod_{k=n_0+1}^{\infty}(I+S_1^{*k+1}PS^k)(\prod_{k=0}^{n_0}(I+S_1^{*k+1}PS^k) ,$$

where $\prod_{k=n_0+1}^{\infty}(I+S_1^{*k+1}PS^k)$ is invertible and his inverse is in $\mathscr{A}_0$. Since by the

Riesz theorem these matrices all represents bounded operators on $l^p(0, \infty)$ it remains to show that the operator $\prod_{k=0}^{n_0}(I+S_1^{*k+1}PS^k)$ is invertible on $l^p(0, \infty)$. This assertion follows from the fact that -1 is not an eigenvalue of $S_1^{*k+1}PS^k$ and these operators are completely continuous for which the Fredholm alternative implies that -1 is a regular point.

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