A NEW UNIFORMITY FOR HYPERSPACES

By

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§ 1. Introduction

Let $(X, \mathscr{U})$ be a uniform space and let $2^X(\mathscr{U})=\{A : A\neq\phi, A \text{ is closed}\}$. When $\mathscr{U}$ is the only uniformity being considered, $2^X(\mathscr{U})$ will be denoted by 2^X. For each entourage $U\in\mathscr{U}$, let $H(U)=\{(A, B) : A, B\in 2^X \text{ and } A\subseteq U[B], B\subseteq U[A]\}$. It is well known that $\{H(U) : U\in\mathscr{U}\}$ is a base for a uniformity for 2^X, commonly denoted by $2^{\mathscr{U}}$. The space $(2^X, 2^{\mathscr{U}})$ has been extensively studied in the literature (see [1], [2]).

We propose to study in this paper a new uniformity for 2^X which we shall also denote by $2^{\mathscr{U}}$: for each $U\in\mathscr{U}$, let $G(U)=\{(A, B) : A, B\in 2^X \text{ and } A=B \text{ or } A\times A\cup B\times B\subseteq U\}$. Then $2^{\mathscr{U}}$ is the uniformity for 2^X with $\{G(U) : U\in\mathscr{U}\}$ as base.

In § 2, we obtain for $(X, \mathscr{U})$, characterizations for indiscreteness, discreteness, separation, $\mathscr{U}$ possessing a smallest element, pseudo metrizability in terms of $(2^X, 2^{\mathscr{U}})$. We also show that $\mathscr{U}=\mathscr{V}$ iff $2^{\mathscr{U}}=2^{\mathscr{V}}$.

In § 3, subspace, uniform continuity, completeness and total boundedness are studied.

In § 4, we show that $\mathscr{T}(2^{\mathscr{U}})$ is always O-dimensional and that $\mathscr{T}(\mathscr{U})\neq\mathscr{T}(\mathscr{V})$ implies that $\mathscr{T}(2^{\mathscr{U}})\neq\mathscr{T}(2^{\mathscr{V}})$, but not conversely. We also give another characterization of $\mathscr{U}$ possessing a smallest member in terms of $2^{\mathscr{U}}$.

Set valued maps are considered in § 5. In particular, we give conditions under which the intersection and union of two uniformly continuous set valued maps are uniformly continuous. We show that the map that takes x into its closure is uniformly continuous.

§ 2. Extremal properties, separation, pseudo metrizability

Theorem 2.1. Let $(X, \mathscr{U})$ be a uniform space. Then $(2^X, 2^{\mathscr{U}})$ is a uniform space.

Proof. The theorem follows from the following facts which the reader can easily verify: (i) $\Delta(2^X)$ (the diagonal in $2^X \times 2^X$) is contained in $G(U)$ for all $U \in \mathscr{U}$ (ii) $G(U)=(G(U))^{-1}$ (iii) $G(U) \cap G(V)=G(U \cap V)$ and (iv) $G(U) \circ G(U)=G(U)$.

We will find much use for the following

Lemma 2.2. Let $(X, \mathscr{U})$ be a uniform space and $U \in \mathscr{U}$, $U=U^{-1}$, U closed. If $(x, y) \in U$, then $(c(x), c(x, y)) \in G(U)$, c denoting the closure operator.

Proof. Clearly $c(x) \times c(x)$ and $c(y) \times c(y)$ are subsets of U. Also, $c(x) \times c(y)$ and $c(y) \times c(x)$ are subsets of U. It follows then that $c(x, y) \times c(x, y)$ is a subset of U and hence $(c(x), c(x, y)) \in G(U)$.

Lemma 2.3. Let $(X, \mathscr{U})$ be a uniform space. (i) If $U \subseteq V$ in $\mathscr{U}$, then $G(U) \subseteq G(V)$; (ii) If $G(U) \subseteq G(V)$, $U=U^{-1}$ and U is closed, then $U \subseteq V$.

Proof. Let $(x, y) \in U$. By lemma 2.2, $(c(x), c(x, y)) \in G(U)$ and hence $(c(x), c(x, y)) \in G(V)$. If $c(x)=c(x, y)$, then $(x, y) \in c(x) \times c(x) \subseteq V$. If $c(x) \neq c(x, y)$, then $(x, y) \in c(x, y) \times c(x, y) \subseteq V$.

Theorem 2.4. In a uniform space $(X, \mathscr{U})$, the following are equivalent: (i) $(X, \mathscr{U})$ is indiscrete (ii) $2^X=\{X\}$ (iii) $(2^X, 2^{\mathscr{U}})$ is separated (iv) $(2^X, 2^{\mathscr{U}})$ is indiscrete.

Proof. (i)$\to$(ii) If $(X, \mathscr{U})$ is indiscrete, then $\mathscr{T}(\mathscr{U})$ is indiscrete and $2^X=\{X\}$. (ii)$\to$(iii) A one point space is always separated. (iii)$\to$(i) Let $(2^X, 2^{\mathscr{U}})$ be separated. Then $\Delta(2^X)=\cap\{G(U) : U \in \mathscr{U}\}$; let $(x, y) \in X \times X$. Now $(c(x), c(y)) \in G(U)$ for all $U \in \mathscr{U}$ and hence $c(x)=c(y)$.
It follows then that $\mathscr{T}(\mathscr{U})$ is indiscrete and that $\mathscr{U}$ is indiscrete. (i)$\to$(iv) This follows from the fact that $G(X \times X)=2^X \times 2^X$. (iv)$\to$(i) Let $U \in \mathscr{U}$. Then $G(U) \supseteq 2^X \times 2^X=G(X \times X)$. But $X \times X$ is closed and symmetric and by lemma 2.3, it follows that $U \supseteq X \times X$.

Definition 2.5. Let $i(x)=\{x\}$ for each x in X.

Theorem 2.6. A uniform space $(X, \mathscr{U})$ is discrete iff $\Delta(2^X) \cup i[X] \times i[X] \in 2^{\mathscr{U}}$.

Proof. Suppose that $(X, \mathscr{U})$ is discrete. Then $\Delta(X) \in \mathscr{U}$ and $G(\Delta(X)) \in 2^{\mathscr{U}}$. But $\Delta(2^X) \cup i[X] \times i[X] \supseteq G(\Delta(X))$. Conversely, let $\Delta(2^X) \cup i[X] \times i[X] \in 2^{\mathscr{U}}$. Then $\Delta(2^X) \cup i[X] \times i[X] \supseteq G(U)$ for some $U \in \mathscr{U}$, $U=U^{-1}$. We will show that $U=\Delta(X)$. Let $(x, y) \in U$. Since $\Delta(2^X) \cup i[X] \times i[X] \in 2^{\mathscr{U}}$, it follows that singleton sets are closed. Hence $(\{x\}, \{x, y\}) \in G(U)$ and $(\{x\}, \{x, y\}) \in \Delta(2^X) \cup i[X] \times i[X]$. It follows that $x=y$.

Theorem 2.7. A uniform space $(X, \mathscr{U})$ is separated iff $\cap 2^{\mathscr{U}}=\Delta(2^X) \cup i[X] \times i[X]$.

Proof. Let $(X, \mathscr{U})$ be separated, that is, $\cap \mathscr{U}=\Delta(X)$. Now $(A, B)\in \cap 2^{\mathscr{U}}$ iff $(A, B)\in G(U)$ for all $U\in \mathscr{U}$ iff $A=B$ or $A\times A\cup B\times B\subseteq U$ for all U in $\mathscr{U}$ iff $A=B$ or $A\times A\cup B\times B\subseteq \Delta(X)$ iff $(A, B)\in \Delta(2^X)\cup i[X]\times i[X]$.

Conversely, let $\cap 2^{\mathscr{U}}=\Delta(2^X)\cup i[X]\times i[X]$. Take x, y in X. Then $(\{x\}, \{y\})\in i[X]\times i[X]\subseteq 2^X\times 2^X$ and $\{x\}$ is a closed set. Thus $(X, \mathscr{U})$ is separated.

Theorem 2.8. In a uniform space $(U, \mathscr{U})$, $\mathscr{U}$ has a smallest member iff $2^{\mathscr{U}}$ has a smallest member.

Proof. If U is the smallest in $\mathscr{U}$, then $G(U)$ is the smallest in $2^{\mathscr{U}}$ by lemma 2.3. Conversely, let $G(U)$ be smallest in $2^{\mathscr{U}}$. Let $V\subseteq U$, V closed, $V=V^{-1}$. Then $G(V)\subseteq G(U)\subseteq G(W)$ for all $W\in \mathscr{U}$. Applying lemma 2.3, it follows that $V\subseteq W$ and that V is the smallest element of $\mathscr{U}$.

Theorem 2.9. $(X, \mathscr{U})$ is pseudo metrizable iff $(2^X, 2^{\mathscr{U}})$ is pseudo metrizable.

Proof. If $\{U_n : n\in P\}$ is a base for $\mathscr{U}$, then $\{G(U_n) : n\in P\}$ is a base for $2^{\mathscr{U}}$. Conversely let $\{G(V_n) : n\in P\}$ be a base for $2^{\mathscr{U}}$ and take $U_n\subseteq V_n$, $U_n=U_n^{-1}$, U_n closed for each $n\in P$. Then for each $U\in \mathscr{U}$, $G(U)\supseteq G(V_n)\supseteq G(U_n)$ for some n and applying lemma 2.3, we have $U\supseteq U_n$. It follows then that $\{U_n : n\in P\}$ is a base for $\mathscr{U}$.

Theorem 2.10. Let $\mathscr{U}$ and $\mathscr{V}$ be uniformities for X for which $\mathscr{T}(\mathscr{U})=\mathscr{T}(\mathscr{V})$. Then $\mathscr{U}\subseteq \mathscr{V}$ iff $2^{\mathscr{U}}\subseteq 2^{\mathscr{V}}$.

Proof. Suppose that $\mathscr{U}\subseteq \mathscr{V}$ and let $G(U)$ be a basic element of $2^{\mathscr{U}}$. But, since $2^X(\mathscr{U})=2^X(\mathscr{V})$, and $U\in \mathscr{V}$ it follows that $G(U)\in 2^{\mathscr{V}}$. Conversely, let $2^{\mathscr{U}}\subseteq 2^{\mathscr{V}}$ and let $U\in \mathscr{U}$. Then $G(U)\in 2^{\mathscr{U}}$ and hence $G(U)\in 2^{\mathscr{V}}$. Thus $G(U)\supseteq G(V)$ for some $V\in \mathscr{V}$, $V=V^{-1}$, V closed. Applying lemma 2 3, it follow that $U\supseteq V$ and hence $U\in \mathscr{V}$.

Corollary 2.11. Let $\mathscr{U}$ and $\mathscr{V}$ be uniformities for X. Then $\mathscr{U}=\mathscr{V}$ iff $2^{\mathscr{U}}=2^{\mathscr{V}}$.

Proof. If $\mathscr{U}=\mathscr{V}$, then $\mathscr{T}(\mathscr{U})=\mathscr{T}(\mathscr{V})$ and $2^{\mathscr{U}}=2^{\mathscr{V}}$ by the above theorem. Conversely, suppose that $2^{\mathscr{U}}=2^{\mathscr{V}}$. But $2^X(\mathscr{U})\times 2^X(\mathscr{U})$ is the largest element in $2^{\mathscr{U}}$ and $2^X(\mathscr{V})\times 2^X(\mathscr{V})$ is the largest element in $2^{\mathscr{V}}$. It follows then that $2^X(\mathscr{U})=2^X(\mathscr{V})$ and hence $\mathscr{T}(\mathscr{U})=\mathscr{T}(\mathscr{V})$. Applying theorem 2.10, we have $\mathscr{U}=\mathscr{V}$.

In regard to theorem 2.10, it seems appropriate to present

Example 2.12. Let X be a set with two or more elements and let $\mathscr{U}$ and

$\mathscr{V}$ be the indiscrete and discrete uniformities respectively. Then $\mathscr{U} \subseteq \mathscr{V}$, but $2^{\mathscr{U}} \not\subseteq 2^{\mathscr{V}}$, for $\{(X, X)\} \in 2^{\mathscr{U}}$, and $\{(X, X)\} \notin 2^{\mathscr{V}}$ since $\Delta(2^X(\mathscr{V})) \not\subseteq \{(X, X)\}$.

§ 3. Subspaces, uniform continuity, completeness, total boundedness

Theorem 3.1. Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be uniform spaces. Then $(2^X, 2^{\mathscr{U}})$ is a subspace of $(2^Y, 2^{\mathscr{V}})$ iff $(X, \mathscr{U})$ is a closed subspace of $(Y, \mathscr{V})$.

Proof. Let $(2^X, 2^{\mathscr{U}})$ be a subspace of $(2^Y, 2^{\mathscr{V}})$. Then $X \in 2^X \subseteq 2^Y$ and hence X is a closed subset of Y. We show next that $\mathscr{U} = X \times X \cap \mathscr{V}$. Let $U \in \mathscr{U}$; then $G(U) \in 2^{\mathscr{U}} = 2^X \times 2^X \cap 2^{\mathscr{V}}$. Thus $G(U) \supseteq 2^X \times 2^X \cap G(V)$ for some $V \in \mathscr{V}$, $V = V^{-1}$, V closed in $Y \times Y$. We will show that $U \supseteq X \times X \cap V$. Let $(x_1, x_2) \in X \times X \cap V$. Since X is closed in Y, it follows that the closure of $\{x\}$ relative to X is the same as the closure of $\{x\}$ relative to Y for every $x \in X$. By lemma 2.2, $(c(x_1), c(x_1, x_2)) \in 2^X \times 2^X \cap G(V)$ and hence $(c(x_1), c(x_1, x_2)) \in G(U)$. It follows that $(x_1, x_2) \in U$. Next, let $V \in \mathscr{V}$. We must show that $X \times X \cap V \in \mathscr{U}$. Now $G(V) \cap 2^X \times 2^X \in 2^{\mathscr{U}}$ and hence $G(V) \cap 2^X \times 2^X \supseteq G(U)$ for some $U \in \mathscr{U}$, $U = U^{-1}$, U closed. Then $X \times X \cap V \supseteq U$; for let $(x_1, x_2) \in U$. By lemma 2.2, $(c(x_1), c(x_1, x_2)) \in G(U)$ and hence $(c(x_1), c(x_1, x_2)) \in G(V)$. It follows that $(x_1, x_2) \in X \times X \cap V$.

Conversely, let $(X, \mathscr{U})$ be a closed subspace of $(Y, \mathscr{V})$. We show firstly that $2^{\mathscr{U}} \subseteq 2^X \times 2^X \cap 2^{\mathscr{V}}$. Let $G(U)$ be basic in $2^{\mathscr{U}}$. But $U = X \times X \cap V$ for some $V \in \mathscr{V}$. Then $G(U) = 2^X \times 2^X \cap G(V)$ as the reader can easily show. Similarly, one shows that $2^{\mathscr{U}} \supseteq 2^X \times 2^X \cap 2^{\mathscr{V}}$.

Definition 3.2. Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be two uniform spaces and suppose $f: X \to Y$, continuity not assumed. Then $\underline{f}: 2^X \to 2^Y$ is defined as follows: $\underline{f}(A) = cf[A]$.

Theorem 3.3. Let $f: (X, \mathscr{U}) \to (Y, \mathscr{V})$ be uniformly continuous. Then $\underline{f}: (2^X, 2^{\mathscr{U}}) \to (2^Y, 2^{\mathscr{V}})$ is uniformly continuous.

Proof. Let $V \in \mathscr{V}$, V closed. There exists a $U \in \mathscr{U}$ such that $f \times f[U] \subseteq V$. It suffices to show that $\underline{f} \times \underline{f}[G(U)] \subseteq G(V)$; let $(A, B) \in G(U)$. We may assume that $A \neq B$. Then $A \times A \cup B \times B \subseteq U$ and hence $f[A] \times f[A] \cup f[B] \times f[B] \subseteq f \times f[U] \subseteq V$. Since V is closed, it follows that $\underline{f}(A) \times \underline{f}(A) \cup \underline{f}(B) \times \underline{f}(B) \subseteq V$ and $\underline{f} \times \underline{f}(A, B) \in G(V)$.

The converse of theorem 3.3 is false as seen in

Example 3.4. Let X be a set with two or more elements and let $\mathscr{U}$ and $\mathscr{V}$ be the indiscrete and discrete uniformity respectively. Let $f: X \to X$ be the identity map. Then $\underline{f}: (2^X, 2^{\mathscr{U}}) \to (2^X, 2^{\mathscr{V}})$ is uniformly continuous, but $f: (X, \mathscr{U}) \to (X, \mathscr{V})$ is not uniformly continuous. Note that in this case, $2^X = \{X\}$ and hence, $\underline{f}$ is a constant.

However, when we add a separation condition, we get

Theorem 3.5. Let $(X, \mathscr{U})$ be a separated uniform space and $(Y, \mathscr{V})$ an arbitrary uniform space. If $\underline{f}: (2^X, 2^{\mathscr{U}}) \to (2^Y, 2^{\mathscr{V}})$ is uniformly continuous, then $f: (X, \mathscr{U}) \to (Y, \mathscr{V})$ is uniformly continuous.

Proof. Let $V \in \mathscr{V}$. Then there exists a $U \in \mathscr{U}$ such that $U = U^{-1}$ and $\underline{f} \times \underline{f}[G(U)] \subseteq G(V)$. It suffices to show that $f \times f[U] \subseteq V$. Let $(x_1, x_2) \in U$. Applying lemma 2.2, we get $(\{x_1\}, \{x_1, x_2\}) \in G(U)$ and hence $(\underline{f}(\{x_1\}), \underline{f}(\{x_1, x_2\})) \in G(V)$. If $\underline{f}(\{x_1\}) = \underline{f}(\{x_1, x_2\})$, then $f(x_2) \in cf(x_1)$ and $f(x_2) \in V[f(x_1)]$. Hence $(f(x_1), f(x_2)) \in V$. If $\underline{f}(\{x_1\}) \neq \underline{f}(\{x_1, x_2\})$, then $(f(x_1), f(x_2)) \in \underline{f}(\{x_1, x_2\}) \times \underline{f}(\{x_1, x_2\}) \subseteq V$.

Theorem 3.6. $(2^X, 2^{\mathscr{U}})$ is complete.

Proof. Let $S: D \to 2^X$ be a cauchy net and let $x \in X$. We will show that either S is eventually constant, or that S converges to $c(x)$. Suppose S does not converge to $c(x)$. Then $c(x)$ is not a cluster point of S and hence there exists a $U \in \mathscr{U}$ such that S is eventually in $\mathscr{C}G(U)[c(x)]$. There exists a $d_1 \in D$ such that $S(m) \in \mathscr{C}G(U)[c(x)]$ for all $m \geqq d_1$. But there exists a $d_2 \geqq d_1$ such that $(S(m), S(n)) \in G(U)$ for all $m, n \geqq d_2$. Now $S(d_2) \notin G(U)[c(x)]$ and since $c(x) \times c(x) \subseteq U$, it follows that $S(d_2) \times S(d_2) \not\subseteq U$. Since $(S(m), S(d_2)) \in G(U)$ for all $m \geqq d_2$, it is clear that $S(m) = S(d_2)$ for all $m \geqq d_2$ and S is eventually a constant.

Theorem 3.7. $(2^X, 2^{\mathscr{U}})$ is totally bounded iff for each $U \in \mathscr{U}$, $\{B : B \in 2^X$ and $B \times B \not\subseteq U\}$ is finite.

Proof. Suppose that $(2^X, 2^{\mathscr{U}})$ is totally bounded and let $U \in \mathscr{U}$. There exist then $E_1, \cdots, E_n$ in 2^X such that $2^X = G(U)[E_1, \cdots, E_n]$. Suppose $B \in 2^X$ and $B \times B \not\subseteq U$; $(E_i, B) \in G(U)$ for some i and it follows then that $B = E_i$. Thus $\{B : B \in 2^X$ and $B \times B \not\subseteq U\}$ is finite.

Conversely, suppose that for each $U \in \mathscr{U}$, $\{B : B \in 2^X$ and $B \times B \not\subseteq U\}$ is finite. Let $U \in \mathscr{U}$; let $B_i \times B_i \not\subseteq U$ for $i = 1, \cdots, n$ and $E \times E \subseteq U$ for all $E \in 2^X$, $E \neq B_i$. Take $x \in X$ arbitrary. Then $2^X = G(U)[c(x), B_1, \cdots, B_n]$.

§4. The topology of the hyperspace

Definition 4.1. Let $(X, \mathscr{U})$ be a uniform space and suppose that $U \in \mathscr{U}$. Then $\mathscr{A}(U) = \{A : A \in 2^X$ and $A \times A \subseteq U)$, $\mathscr{B}(U) = \{B : B \in 2^X$ and $B \times B \not\subseteq U\}$,

$\mathscr{A}=\cap\{\mathscr{A}(U): U\in\mathscr{U}\}$, $\mathscr{B}=\cup\{\mathscr{B}(U): U\in\mathscr{U}\}$.

Theorem 4.2. Let $(X, \mathscr{U})$ be a uniform space. Then (i) $2^X=\mathscr{A}\cup\mathscr{B}=\mathscr{A}(U)\cup\mathscr{B}(U)$ (ii) $B\in\mathscr{B}$ implies that $\{B\}$ is open and closed in $(2^X, 2^{\mathscr{U}})$ (iii) $A\in\mathscr{A}$ implies that $\mathscr{A}=2^c(\{A\})$ (2^c is the closure operator in $(2^X, 2^{\mathscr{U}})$ (iv) $\mathscr{A}(U)$ is open and closed for each $U\in\mathscr{U}$.

Proof. (i) is clear (ii) Let $B\in\mathscr{B}$. Then $B\times B\not\subseteq U$ for some $U\in\mathscr{U}$ and hence $\{B\}=G(U)[B]$ (iii) If $A\in\mathscr{A}$, then $2^c(\{A\})=\cap\{G(U)[A]: U\in\mathscr{U}\}=\mathscr{A}$ (iv) Let $C\notin\mathscr{U}(U)$. Then $C\times C\not\subseteq U$ and hence $G(U)[C]=\{C\}\subseteq\mathscr{C}\mathscr{A}(U)$. Thus $\mathscr{A}(U)$ is closed. Let now $A\in\mathscr{A}(U)$. Then $G(U)[A]\subseteq\mathscr{A}(U)$ and hence $\mathscr{A}(U)$ is open.

Theorem 4.3. $(2^X, 2^{\mathscr{U}})$ is Lindelöf iff $\mathscr{B}(U)$ is countable for each $U\in\mathscr{U}$.

Proof. Suppose that $(2^X, 2^{\mathscr{U}})$ is a Lindelöf space. Let $U\in\mathscr{U}$. Now $2^X=\mathscr{A}(U)\cup\mathscr{B}(U)=\mathscr{A}(U)\cup\cup\{\{B\}: B\in\mathscr{B}(U)\}$. By theorem 4.2, $\mathscr{A}(U)$ is open and $\{B\}$ is open for all $B\in\mathscr{B}(U)$. It follows then that $\mathscr{B}(U)$ is countable. Conversely, suppose that $\mathscr{B}(U)$ is countable for each $U\in\mathscr{U}$ and let $2^X=\cup\{\mathscr{O}_\alpha:\alpha\in\Delta\}$, each $\mathscr{O}_\alpha$ being an open set. Let $x\in X$. Then $c(x)\in 2^X$ and hence $c(x)\in\mathscr{O}_\alpha$ for some $\alpha\in\Delta$. There exists then a $U\in\mathscr{U}$ such that $G(U)[c(x)]\subseteq\mathscr{O}_\alpha$. But $\mathscr{A}(U)=G(U)[c(x)]$ and hence $\mathscr{C}\mathscr{O}_\alpha\subseteq\mathscr{C}\mathscr{A}(U)=\mathscr{B}(U)$, a countable set. It follows then that $(2^X, 2^{\mathscr{U}})$ is Lindelöf.

Theorem 4.4. Let $(X, \mathscr{U})$ be a pseudo metrizable uniform space. Then $(2^X, 2^{\mathscr{U}})$ is a second axiom space iff $\mathscr{B}$ is countable.

Proof. If $(2^X, 2^{\mathscr{U}})$ is a second axiom space, then $\mathscr{B}$ is countable since $\{B\}$ is open for each $B\in\mathscr{B}$ by (ii) of theorem 4.2. Conversely, suppose that $\mathscr{B}$ is countable and that $\{U_n: n\in P\}$ is a countable base for $\mathscr{U}$. Then $\{\{B\}: B\in\mathscr{B}\}\cup\{\mathscr{A}(U_n): n\in P\}$ is a countable open base for $\mathscr{T}(2^{\mathscr{U}})$ as the reader can easily check.

Theorem 4.5. Let $\mathscr{U}$ and $\mathscr{V}$ be uniformities for X and suppose that $\mathscr{T}(\mathscr{U})\neq\mathscr{T}(\mathscr{V})$. Then $\mathscr{T}(2^{\mathscr{U}})\neq\mathscr{T}(2^{\mathscr{V}})$.

Proof. $\mathscr{T}(\mathscr{U})\neq\mathscr{T}(\mathscr{V})$ implies that $2^X(\mathscr{U})\neq 2^X(\mathscr{V})$. But $2^X(\mathscr{U})$ is the largest element of $\mathscr{T}(2^{\mathscr{U}})$ and $2^X(\mathscr{V})$ is the largest element in $\mathscr{T}(2^{\mathscr{V}})$. Thus $\mathscr{T}(2^{\mathscr{U}})\neq\mathscr{T}(2^{\mathscr{V}})$.

The converse of theorem 4.4 is false as shown by

Example 4.6. Let X consist of the positive integers and let $V_n=\{(a, b): a=b$ or $a\geqq n, b\geqq n\}$. If $\mathscr{V}$ is the uniformity for X with $\{V_n: n\in P\}$ as base and if

$\mathscr{U}$ is the discrete uniformity for X, then $\mathscr{T}(\mathscr{U})=\mathscr{T}(\mathscr{V})$. But $\mathscr{T}(2^{\mathscr{U}})\neq\mathscr{T}(2^{\mathscr{V}})$. To see this, observe that $\Delta(2^X)\cup i[X]\times i[X]\in 2^{\mathscr{U}}$ since $\mathscr{U}$ is discrete (see theorem 2.6) and $(\Delta(2^X)\cup i[X]\times i[X])[\{1\}]=i[X]$. Hence $i[X]$ is a neighborhood $(\mathscr{T}(2^{\mathscr{V}}))$ of $\{1\}$. But $G(V_n)[\{1\}]\subseteq i[X]$ holds for no integer n since $\{m : m\geqq n\}\in G(V_n)[\{1\}]$ and $\{m : m\geqq n\}\notin i[X]$.

Theorem 4.7. $(2^X, \mathscr{T}(2^{\mathscr{U}}))$ is O-dimensional.

Proof. $\{\{B\} : B\in\mathscr{B}\}\cup\{\mathscr{A}(U) : U\in\mathscr{U}\}$ is a base for $\mathscr{T}(2^{\mathscr{U}})$ which consists of sets which are both open and closed (see definition 4.1 and theorem 4.2). The details are left to the reader.

Lemma 4.8. Let $(X, \mathscr{U})$ be a uniform space. Then $\cap\mathscr{U}=\cup\{A\times A : A\in\mathscr{A}\}$.

Proof. $\cup\{A\times A : A\in\mathscr{A}\}\subseteq\cap\mathscr{U}$ by definition of $\mathscr{A}$ (see definition 4.1). Conversely, let $(x, y)\in\cap\mathscr{U}$. Since $\cap\mathscr{U}$ is a closed symmetric set containing $\Delta(X)$, it follows that $c(\{x, y\})\times c(\{x, y\})\subseteq\cap\mathscr{U}$. Then $c(\{x, y\})\in\mathscr{A}$ and $(x, y)\in c(\{x, y\})\times c(\{x, y\})$.

Theorem 4.9. $\cap\mathscr{U}\in\mathscr{U}$ iff $\Delta(2^X)\cup\mathscr{A}\times\mathscr{A}\in 2^{\mathscr{U}}$ (see theorem 2.8).

Proof. Let $\Delta(2^X)\cup\mathscr{A}\times\mathscr{A}\in 2^{\mathscr{U}}$. Then there exists a closed symmetric V in $\mathscr{U}$ such that $\Delta(2^X)\cup\mathscr{A}\times\mathscr{A}\supseteq G(V)$. It suffices to show that $\cup\{A\times A : A\in\mathscr{A}\}\supseteq V$ (see lemma 4.8). Let $(x, y)\in V$; then $(c(x), c(x, y))\in G(V)$ by lemma 2.2. If $c(x)=c(x, y)$, then $(x, y)\in c(x)\times c(x)\subseteq\cup\{A\times A : A\in\mathscr{A}\}$ since $c(x)\in\mathscr{A}$. If $c(x)\neq c(x, y)$, then $(c(x), c(x, y))\in\mathscr{A}\times\mathscr{A}$ and $c(x, y)\in\mathscr{A}$. Thus $(x, y)\in c(x, y)\times c(x, y)\subseteq\cup\{A\times A : A\in\mathscr{A}\}$. Conversely, let $\cap\mathscr{U}\in\mathscr{U}$. By lemma 4.8, $\cup\{A\times A : A\in\mathscr{A}\}\in\mathscr{U}$. Let $U=\cup\{A\times A : A\in\mathscr{A}\}$. Then $\Delta(2^X)\cup\mathscr{A}\times\mathscr{A}\supseteq G(U)$. To see this, let $(C, D)\in G(U)$; if $C=D$, then $(C, D)\in\Delta(2^X)$ and if $C\neq D$, then $C\times C\cup D\times D\subseteq U=\cup\{A\times A : A\in\mathscr{A}\}=\cap\mathscr{U}$. Hence C and D are in $\mathscr{A}$ and $(C, D)\in\mathscr{A}\times\mathscr{A}\subseteq\Delta(2^X)\cup\mathscr{A}\times\mathscr{A}$.

Thus $\mathscr{U}$ has a smallest element iff $\Delta(2^X)\cup\mathscr{A}\times\mathscr{A}\in 2^{\mathscr{U}}$.

§ 5. Set valued uniformly continuous maps

Let c be the function from X into 2^X which takes each point x into its closure.

Theorem 5.1. Let $(X, \mathscr{U})$ be a uniform space. Then (i) $c : (X, \mathscr{U})\to(2^X, 2^{\mathscr{U}})$ is uniformly continuous and (ii) $(c[X], c[X]\times c[X]\cap 2^{\mathscr{U}})$ is indiscrete.

Proof. It suffices to show (ii); but (ii) follows from the fact that $c[X]\times$

$c[X] \subseteq G(U)$ for all $U \in \mathscr{U}$.

Corollary 5.2. If $(X, \mathscr{U})$ is a separated uniform space and i is the function from X into 2^X which takes x into $\{x\}$, then (i) $i : (X, \mathscr{U}) \rightarrow (2^X, 2^{\mathscr{U}})$ is uniformly continuous and (ii) $(i[X], i[X] \times i[X] \cap 2^{\mathscr{U}})$ is indiscrete.

Lemma 5.3. Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be uniform spaces and suppose that $\varphi_2 : (X, \mathscr{U}) \rightarrow (2^Y, 2^{\mathscr{V}})$ is uniformly continuous and that $\varphi_1 : X \rightarrow 2^Y$, continuity not assumed. If $\varphi_1(x) \subseteq \varphi_2(x)$ for each x and if $\varphi_1(x_1) = \varphi_1(x_2)$ whenever $\varphi_2(x_1) = \varphi_2(x_2)$, then $\varphi_1 : (X, \mathscr{U}) \rightarrow (2^Y, 2^{\mathscr{V}})$ is uniformly continuous.

Proof. The lemma follows from the fact that $(\varphi_2 \times \varphi_2)^{-1} G(V) \subseteq (\varphi_1 \times \varphi_1)^{-1} G(V)$.

Corollary 5.4. Let $\varphi : (X, \mathscr{U}) \rightarrow (2^Y, 2^{\mathscr{V}})$ be uniformly continuous and suppose that $\alpha : (X, \mathscr{U}) \rightarrow (2^Y, 2^{\mathscr{V}})$ is a constant map. If $\varphi(x) \cap \alpha(x) \neq \phi$ for each x, then $\varphi \cap \alpha : (X, \mathscr{U}) \rightarrow (2^Y, 2^{\mathscr{V}})$ is uniformly continuous.

Proof. Let $\varphi_2 = \varphi$ and $\varphi_1 = \varphi \cap \alpha$. Apply lemma 5.3.

Corollary 5.5. Let $\varphi_2 : (X, \mathscr{U}) \rightarrow (2^Y, 2^{\mathscr{V}})$ be uniformly continuous and one to one and suppose that $\varphi_1 : X \rightarrow 2^Y$ (continuity not assumed). If $\varphi_1(x) \subseteq \varphi_2(x)$ for each x, then $\varphi_1 : (X, \mathscr{U}) \rightarrow (2^Y, 2^{\mathscr{V}})$ is uniformly continuous.

Proof. $\varphi_1(x_1) = \varphi_2(x_2)$ implies that $x_1 = x_2$ which implies that $\varphi_1(x_1) = \varphi_1(x_2)$. Apply lemma 5.3.

Corollary 5.6. Let $\varphi : (X, \mathscr{U}) \rightarrow (2^Y, 2^{\mathscr{V}})$ be uniformly continuous and one to one and suppose that $\alpha : X \rightarrow 2^Y$, continuity not assumed. If $\varphi(x) \cap \alpha(x) \neq \phi$ for all x in X, then $\varphi \cap \alpha : (X, \mathscr{U}) \rightarrow (2^Y, 2^{\mathscr{V}})$ is uniformly continuous.

Proof. Let $\varphi_2 = \varphi$ and $\varphi_1 = \varphi \cap \alpha$ in corollary 5.5.

Theorem 5.7. Let $\varphi_i : (X, \mathscr{U}) \rightarrow (2^Y, 2^{\mathscr{V}})$ be uniformly continuous for $i = 1, 2$ and suppose that $\varphi_1(x) \cap \varphi_2(x) \neq \phi$ for all x in X. If $\varphi_1(x) \cup \varphi_2(x) = \varphi_1(y) \cup \varphi_2(y)$ whenever $\varphi_1(x) = \varphi_1(y)$ ar $\varphi_2(x) = \varphi_2(y)$, then $\varphi_1 \cup \varphi_2 : (X, \mathscr{U}) \rightarrow (2^Y, 2^{\mathscr{V}})$ is uniformly continuous.

Proof. Let V and W be symmetric entourages and suppose that $V \circ W \subseteq V$. It suffices to show that $((\varphi_1 \cup \varphi_2) \times (\varphi_1 \cup \varphi_2))^{-1} G(V) \supseteq (\varphi_1 \times \varphi_1)^{-1} G(W) \cap (\varphi_2 \times \varphi_2)^{-1} G(W)$. Let $(x, y) \in (\varphi_1 \times \varphi_1)^{-1} G(W) \cap (\varphi_2 \times \varphi_2) G(W)$. If $\varphi_1(x) = \varphi_1(y)$ or $\varphi_2(x) = \varphi_2(y)$, then $\varphi_1(x) \cup \varphi_2(x) = \varphi_1(y) \cup \varphi_2(y)$ and $(x, y) \in ((\varphi_1 \cup \varphi_2) \times (\varphi_1 \cup \varphi_2))^{-1} G(V)$. Assume then that $\varphi_1(x) \neq \varphi_1(y)$ and $\varphi_2(x) \neq \varphi_2(y)$. Then $\varphi_i(x) \times \varphi_i(x) \cup \varphi_i(y) \times \varphi_i(y) \subseteq W$ for $i = 1, 2$. By symmetry, it suffices to show that $(\varphi_1(x) \cup \varphi_2(x)) \times (\varphi_1(x) \cup \varphi_2(x)) \subseteq V$. Let $y' \in \varphi_1(x)$, $y'' \in \varphi_2(x)$ and $y''' \in \varphi_1(x) \cap \varphi_2(x)$. Then $(y', y''') \in W$ and $(y'', y''') \in W$ and hence

$(y', y'') \in V$.

Corollary 5.8. Let $\varphi_i : (X, \mathscr{U}) \to (2^Y, 2^{\mathscr{V}})$ be uniformly continuous for $i=1, 2$ and suppose that $\varphi_1(x) \cap \varphi_2(x) \neq \phi$ for all x. If φ_1 and φ_2 are each one to one, then $\varphi_1 \cup \varphi_2 : (X, \mathscr{U}) \to (2^Y, 2^{\mathscr{V}})$ is uniformly continuous.

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