UNIFORM APPROXIMATION OF VECTOR-VALUED FUNCTIONS USING A WEIGHT FUNCTION

By

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Abstract: In this paper, uniform linear approximation of functions with values in a real or complex Hilbert space is considered with respect to a generalized weight function. The existence, characterization, unicity and strong unicity of the best approximation and the continuity of the best approximation operator for this type of approximation problem are discussed.

1. Introduction. The problem of linear Chebyshev approximation of continuous functions which assume values in a real or complex Hilbert space was studied by *Zuhovickii and Steckin* [9] and *Lawson* [3] by using gauge functions as measures of approximation. They noted that the results of the classical problem for real or complex functions concerning the existence, the characterization theorem of *Kolmogorov* [2] and the unicity of the best approximation extend nicely to their case of vector-valued functions.

The problem of uniform approximation with respect to a weight function for real valued functions has been considered recently by *Moursund* [5] for the linear case and by *Moursund and Taylor* [6] for the case of rational functions. The purpose of this paper is to consider a generalization of the vector-valued problem of *Zuhovickii and Steckin* [9] and *Lawson* [3] by introducing a suitable weight function. The weight function considered here has a form simpler than the one considered by *Moursund* [5] for the real case. This has been necessitated by the need to avoid the complexities of the vector-valued case and to make the treatment simpler.

2. Preliminaries. Let X be a compact Hausdorff space containing at least $n+1$ points and H be a real or complex Hilbert space with an inner product $\langle , \rangle$, a norm $\| . \|$, and a zero element θ. We denote by $\mathscr{C}(X, H)$ the Banach algebra of functions $f: X \to H$, equipped with the uniform norm $\|f\| = \sup\{\|f(x)\|/ x \in X\}$. Let M be a n-dimensional subspace of $\mathscr{C}(X, H)$, with $\{\Phi_1, \cdots \Phi_n\}$ as a linearly independent set of base functions. We call a map $W: X \times H \to H$ a gener-

alized weight-map with respect to the subspace M, if it satisfies:
(w_1) W is continuous on $X\times H$
(w_2) W is linear in h i.e. $W(x, \alpha h_1+\beta h_2)=\alpha W(x, h_1)+\beta W(x, h_2)$, for all $\alpha, \beta\in \boldsymbol{C}$ and $h_1, h_2\in H$.
(w_2) $p\in M$, $W(x, p(x))=\theta$ on X implies $p(x)=\theta$ on X.

Examples: $W(x, y)=y$; $W(x, y)=w(x)y$, $w(x)$ being a continuous complex function which does not vanish on X, are trivial examples of weight maps. Let $X=[a, b]$, $a>0$, $b>0$. $H=L_2(a, b)$. For the base functions of M, take the linearly independent set $\{\Phi_i(x)=x^i t/i=0,\cdots, n\}$ then $W(x, y)=\sum_{i=0}^{n}\langle\Phi_i(x), y\rangle\, \Phi_i(x)$ gives a nontrivial example of weight map.

For $f\in \mathscr{C}(X, H)$, we define $|f|=\sup_{x\in X}\|W(x, f(x))\|$. It is easily verified that $|\cdot|$ is a semi-norm on $\mathscr{C}$. Let $\inf_{p\in M}|f-p|$ be denoted by $E=E(f)$. We call $p^*\in M$, a best weighted approximation to f in M, provided $|f-p^*|=E(f)$.

Using continuity of $|\cdot|$ and standard compactness arguments one can easily conclude that the set of best weighted approximation to f is nonempty, closed and convex. the case $W(x, y)\equiv y$ corresponds to the one considered by *Zuhovickii* and *Steckin* [9] and $W(x, y)\equiv w(x)y$, gives an approximation problem with ordinary weight for which $w(x)=\dfrac{1}{\min\|f(x)\|}$ ($f(x)\neq\theta$ on X) gives results similar to the ordinary relative error approximation for the real case.

A function $\eta: X\to H$ with a finite support $S=\{x_1, \cdots, x_n | x_k\in X, \text{ distinct}\}$ i.e. $\eta(x)=\theta$, $x\in X\sim S$ and $\eta(x_k)\neq\theta$, $k=1,\cdots, m$, will be called a unit if $\|\eta(x_k)\|=1$, $k=1,\cdots,m$; and it will be called an extremal unit with respect to the weight W and the subspace M, provided there exists a function $\mu: X\to H$ with the same support S for which $\eta(x_k)=\dfrac{\mu(x_k)}{\|\mu(x_k)\|}$, $k=1,\cdots, m$ and

$$\sum_{k=1}^{m}\langle W(x_k, \mu(x_k)), W(x_k, q(x_k))\rangle=0 \text{ for each } q\in M.$$

3. Characterization of best weighted approximation. We begin with the main characterization theorem. The statement (2) is a generalization of the *Kolmogorov*'s characterization [2] and the statement (5) is a generalization of the characterization due to *Rivlin* and *Shapiro* [8].

Theorem 3.1. Let $f\in \mathscr{C}(X, H)$ and $p\in M$. Then the following five statements are equivalent.

(1) p is a best weighted approximation to f in M

(2) (a) $\max\{\mathrm{Re}\langle W(x,f(x)-p(x)),\ W(x,q(x))\rangle / x\in X\ \ |f-p|=\|W(x,f(x)-p(x))\|\}\geq 0$, for each $q\in M$

or equivalently

(b) $\min\{\mathrm{Re}\langle W(x,f(x)-p(x)),\ W(x,q(x))\rangle / x\in X\ \ |f-p|=\|W(x,f(x)-p(x))\|\}\leq 0$, for each $q\in M$

we shall denote by $X_{f,p}$ the set $\{x\in X/|f-p|=\|W(x,f(x)-p(x))\|\}$

(3) the zero-element $(0,0,\cdots,0)$ of the n-space C^n belongs to the convex-hull of the set of n-tuples:

$$A=\{Z=(\langle W(x,f(x)-p(x)),\ W(x,\Phi_1(x))\rangle, \langle W(x,f(x)-p(x)),\ W(x,\Phi_2(x)))\rangle, \\ \cdots\langle W(x,f(x)-p(x)),\ W(x,\Phi_n(x))\rangle)/x\in X_{f,p}\} .$$

(4) there exist m points $x_1,\cdots,x_m\in X_{f,p}$ and m number $\alpha_k>0$, $k=1,\cdots,m$, $\sum_1^m\alpha_k=1$, (where $m\leq n+1$ for a real Hilbert space H and $m\leq 2n+1$ for a complex Hilbert space H), for which

$$\sum_{k=1}^m\alpha_k\langle W(x_k,f(x_k)-p(x_k)),\ W(x_k,q(x_k))\rangle=0 ,$$

for each $q\in M$.

(5) there exists an extremal unit $\rho: X\to H$ with support $S=\{x_1,\cdots,x_m\}$ of points belonging to $X_{f,p}$ ($m\leq n+1$ or $m\leq 2n+1$) for which

$$\rho(x_k)=\frac{f(x_k)-p(x_k)}{\|f(x_k)-p(x_k)\|} , \quad k=1,\cdots,m .$$

Proof. We establish the implications:

$$(1)\Rightarrow(2)\Rightarrow(3)\Rightarrow(4)\Rightarrow(5)\Rightarrow(1)$$

$(1)\Rightarrow(2)$:

Let p be a best weighted approximation to f. Assume that (2) is false, then there exist a $q\in M$ and $m>0$, such that

$$\max_{x\in X_{f,p}}\{\mathrm{Re}\langle W(x,f(x)-p(x)),\ W(x,q(x))\rangle\}=-2m .$$

By continuity there exists an open subset $G\supset X_{f,p}$ such that,

$$\mathrm{Re}\{\langle W(x,f(x)-p(x)),\ W(x,q(x))\rangle\}<-m, \quad x\in G .$$

For the complement F of G, which is compact,

$$\|W(x,f(x)-p(x))\|<E-\delta,\ x\in F \text{ holds for some } \delta>0 .$$

If we select

$$\varepsilon=\frac{1}{2}\min\left(\frac{m}{|q|^2},\frac{\delta}{2|q|}\right),$$

it is easily verified that for $p_1=p-\varepsilon q\in M$, we get

$$\begin{aligned}\|W(x,f(x)-p_1(x))\|^2=\|W(x,f(x)-p(x)\|^2\\ &+2\varepsilon\mathrm{Re}\langle W(x,f(x)-p(x)),\ W(x,q(x))\rangle\\ &+\varepsilon^2\|W(x,q(x))\|^2\\ &<E^2-\varepsilon m,\quad x\in G\end{aligned}$$

and

$$\begin{aligned}\|W(x,f(x)-p_1(x))\|&\leq\|W(x,f(x)-p(x))\|+\varepsilon\|W(x,q(x))\|\\ &<E-\frac{\delta}{3},\quad x\in F.\end{aligned}$$

Whence

$$|f-p_1|<E,$$

which is a contradiction. 2(b) follows by changing q to $-q$.

(2)⇒(3):

The set $X_{f,p}$ is compact and the map $x\to z$ is continuous. Hence, A is compact and the convex-hull A_c is compact convex. Now, by (2) (a)

$$\max_{x\in X_{f,p}}\{\mathrm{Re}\sum_{i=1}^{n}\bar{c}_i\langle W(x,f(x)-p(x)),\ W(x,\Phi_i(x))\rangle\}\geq 0,$$

for arbitrary c_i. This implies that if, for some c_i and a_0,

$$\mathrm{Re}\{\sum_{i=1}^{n}\bar{c}_i\langle W(x,f(x)-p(x)),\ W(x,\Phi_i(x))\rangle\}\leq a_0.$$

holds for all $x\in X_{f,p}$, then $a_0\geq 0$. Thus the origin $(0,\cdots,0)$ in $\boldsymbol{C}^n$ belongs to each closed half-space which contains the set A and, hence, it also belongs to the convex hull A_c of A.

(3)⇒(4):

This follows by the characterization of the convex-hull given in *Caratheodory*'s theorem (cf. *Cheney* [1], pp. 17).

(4)⇒(5):

We have

$$\sum_{k=1}^{n}\alpha_k\langle W(x_k,f(x_k)-p(x_k)),\ W(x_k,q(x_k))\rangle=0,$$

for arbitrary $q\in M$. Define

$$\rho(x_k)=\frac{f(x_k)-p(x_k)}{\|f(x_k)-p(x_k)\|},$$

$$\mu(x_k)=\alpha_k(f(x_k)-p(x_k))\ ,\quad k=1,\cdots,m\ ,$$

and

$$\rho(x)=\mu(x)=\theta\ ,\quad x\neq x_k\ ,\quad k=1,\cdots,m\ ,$$

then

$$\sum_{k=1}^{m}\langle W(x_k\,,\mu(x_k)),\ W(x_k\,,q(x_k))\rangle=0\ ,\quad \text{for each}\quad q\in M\ ,$$

hence ρ is an extremal unit.

(5)$\Rightarrow$(1)

Let ρ be an extremal unit satisfying (5), with

$$\mu(x_k)=\alpha_k(f(x_k)-p(x_k))\ ,\quad \alpha_k>0$$

and

$$\rho(x_k)=\frac{\mu(x_k)}{\|\mu(x_k)\|}\ ,\quad k=1,\cdots,m\ .$$

Now for an arbitrary $r\in M$, we have

$$\sum_{k=1}^{m}\langle W(x_k),\mu(x_k)),\ W(x_k\,,p(x_k)-r(x_k))\rangle=0\ ,$$

which gives,

$$\begin{aligned}|f-r||f-p|\sum_{k=1}^{m}\alpha_k&=|f-r|\sum_{k=1}^{m}\|W(x_k\,,\mu(x_k))\|\\ &\geq\sum_{k=1}^{m}|\langle W(x_k\,,\mu(x_k)),\ W(x_k\,,f(x_k)-r(x_k))\rangle|\\ &\geq\left|\sum_{k=1}^{m}\langle W(x_k\,,\mu(x_k)),\ W(x_k\,,f(x_k)-r(x_k))\rangle\right|\\ &=\left|\sum_{k=1}^{m}\langle W(x_k\,,\mu(x_k)),\ W(x_k\,,f(x_k)-p(x_k))\rangle\right|\\ &=|f-p|^2\sum_{k=1}^{m}\alpha_k\ .\ (*)\end{aligned}$$

whence $|f-r|\geq|f-p|$, and p is a best weighted approximation to f. This completes the proof of the theorem.

The following generalizations of the results of *de la Vallée-Poussin* and *Zuhovickii* are immediate consequences.

Theorem 3.2. (*de la Vallée-Poussin*)

Let $\rho: X\to H$ be an extremal unit with support $S=\{x_1\,,\cdots,x_m\}$ with respect to the subspace M and the weight W. If there exists a $p\in M$ and $\delta\geq0$, such that

$$\rho(x_k)=\frac{f(x_k)-p(x_k)}{\|f(x_k)-p(x_k)\|}\ ,$$

and $\| W(x_k, f(x_k)-p(x_k))\| > \delta$, $k=1,\cdots,m$; then $E(f)\geq\delta$. In fact, using the estimate (*), one obtains $|f-r|\geq\delta$ for each $r\in M$.

Theorem 3.3 (*Zuhovickii*) If p is a best weighted approximation to $f\in C(X, H)$ on X, then there exists a certain finite subset of $X_{f,p}$, which consists of $m(m\leq 2n+1$ or $m\leq n+1)$ points, such that p is also best on this set.

4. **Uniqueness.** The conditions on the base functions used by *Zuhovickii* and *Steckin* [9] in the cases when the Hilbert Space is finite and infinite dimensional, though they are necessary as well as sufficient for the unicity of the best approximation for the respective cases, do not seem to lead to the strong unicity and the continuity of the best approximation operator as in the classical case of the Haar condition.

Here, we introduce two alternative conditions: the first is sufficient and the second necessary in the case of the finite dimensional Hilbert space, for the unicity of the best approximation. The first condition also yields the strong unicity and the continuity of best approximation operator in the case of a real Hilbert space.

Theorem 4.1. Let the subspace M and the weight W satisfy the following condition:
(T) For each $f\notin M$ and each best weighted approximation p to f in M. The equation $\langle W(x, f(x)-p(x)), W(x, q(x))\rangle=0$ can have at most $n-1$ distinct solutions in X, unless $W(x, q(x))\equiv\theta$ i.e. $q(x)\equiv\theta$ on X. Then each $f\in C(X, H)$ has a unique best weighted approximation in M.

Proof. We need the following Lemma.

Lemma 1. If M and W satisfy the condition (T), then the set $X_{f,p}$ contains at least $n+1$ points for each $f\in \mathscr{C}(X, H)$.

Suppose $X_{f,p}$ contains points $\leq n$. By theorem 3.1 (4) there exist distinct points $x_1,\cdots, x_m\in X_{f,p}$, $m\leq n$ and the numbers $\alpha_k>0$, $k=1, 2,\cdots, m$; such that

$$\sum_{k=1}^{m}\alpha_k\langle W(x_k, f(x_k)-p(x_k)), W(x_k, q(x_k))\rangle=0\ , \quad q\in M\,.$$

Define evaluation functionals $L_k\in M^*$, $k=1,\cdots, m$; by $L_k(q)=\langle W(x_k, f(x_k)-p(x_k)), W(x_k, q(x_k))\rangle$, $q\in M$. Then $\sum_1^m \alpha_k L_k=0$. Let $x_{m+1},\cdots x_n$ be distinct points of X different from $x_1,\cdots x_m$, if $m<n$. Then we have

$$\sum_{k=1}^{m}\alpha_k L_k=0\ , \quad \text{by choosing} \quad \alpha_{m+1}=\cdots=\alpha_n=0\ ,$$

if $m>n$.
Thus L_k, $k=1,\cdots,n$ do not span M^*. By Hahn Banach theorem, select a functional $\mathscr{L}\in M^{**}$, such that $\mathscr{L}(L_k)=0$, $k=1,\cdots,n$. By reflexivity of finite dimensional spaces, there exists a polynomial $q_1\in M$, such that $L_k(q_1)=\langle W(x_k, f(x_k)-p(x_k)), W(x_k, q_1(x_k))\rangle=0$, $k=1,\cdots,n$. This contradicts condition (T) and establishes the Lemma. Proof of the theorem can now be completed by routine arguments. Assume that p and P are two best approximations to f in M, then $q=\dfrac{p+P}{2}$ is also a best approximation to f and for the points $x\in X_{f,q}$, one gets

$$2E(f)=\| W(x,f(x)-p(x)+f(x)-P(x))\|\le\| W(x,f(x)-p(x)\| \\ +\| W(x,f(x)-P(x))\|\le 2E(f)\ .$$

Hence the equality occurs throughout this string of inequalities, which implies $x\in X_{f,p}$ and $x\in X_{f,P}$. By the strict convexity of the Hilbert space norm and hypothesis (w_3) for the weight W, $p(x)=P(x)$ for $x\in X_{f,q}$; whence $p(x)\equiv P(x)$ by Lemma 1 and the condition (T). This completes the proof of the theorem.

Remark. We wish to note here that the condition (w_3) of the weight W is used with full effect in the preceding theorem and also in the subsequent discussion. However, for purposes of section 3 only, it can be clearly dispensed with. The condition (T) also implies the following condition (T′) the equation $W(x,q(x))=\theta$, $q\in M$, can have no more than $n-1$ distinct solutions in X, unless $q(x)\equiv\theta$ on X.

(T′) does not seem to be sufficient for the unicity of best weighted approximation, however, in the next theorem we prove that it is necessary in case H is finite dimensional.

Theorem 4.2. Let H be finite dimensional then if for each $f\in\mathscr{C}(X,H)$, there exists a unique best weighted approximation in M. Then M and W satisfy the condition (T′).

Proof. We assume the contrary, that (T′) is not satisfied. Then there exists a $Q\in M$, $Q\neq\theta$ and distinct points $x_1, x_2\cdots, x_n\in X$, such that $W(x_j, Q(x_j))=0$, $j=1,\cdots,n$. We may assume that $|Q|<1$. Now select $h\in H$ such that $\|h\|=1$ then

$$\mathrm{Det}\,|\langle W(x_j, h),\ W(x_j, \Phi_i(x_j))\rangle|_1^n=0\ .$$

Select a non-zero vector $(\beta_1,\cdots\beta_n)$ orthogonal to the rows of the corresponding matrix. Let $J=\{j/\beta_j\neq 0,\ W(x_j, h)\neq\theta\}$, $X_J=\{x_j/j\in J\}$ and $\boldsymbol{C}_J=\{\mathrm{sgn}\,\beta_j/j\in J\}$. We may assume that h has been so selected that the set J is non-empty. Define the maps $\eta: X_J\to\boldsymbol{C}$, $\mu:\boldsymbol{C}_J\to H$ by $\eta(x_j)=\mathrm{sgn}\,\beta_j$ and

$$\mu(\operatorname{sgn}\beta_j)=\frac{(\operatorname{sgn}\beta_j)h}{\|W(x_j, h)\|}.$$

By Tietze's extension theorem, they have continuous extensions $\bar{\eta}: X\to \boldsymbol{C}$ and $\bar{\mu}: \boldsymbol{C}\to H$ satisfying $\|\bar{\eta}\|\le 1$ and $\|W(x, \mu(x))\|\le 1$, $\alpha\in \boldsymbol{C}$, $x\in X$. The composed map $\bar{f}: X\to H$, $\bar{f}-\mu\circ\bar{\eta}\in \mathscr{C}(X, H)$ satisfies

$$\text{(a)}\qquad \bar{f}(x_j)=\frac{(\operatorname{sgn}\beta_j)h}{\|W(x_j, h)\|}, \qquad j\in J.$$

$$\text{(b)}\qquad |\bar{f}|\le 1.$$

At this stage, we need

Lemma 2. For each $\bar{f}\in \mathscr{C}(X, H)$ satisfying (a) and (b), we have $E(\bar{f})=1$. By (b), $E(\bar{f})\le 1$. If we assume that there exists a $q\in M$, $|\bar{f}-q|<1$, then

$$\begin{aligned}\|W(x_j, \bar{f}(x_j)-q(x_j))\|^2&=\|W(x_j, \bar{f}(x_j))\|^2\\ &-2\operatorname{Re}\langle W(x_j, \bar{f}(x_j)), W(x_j, q(x_j))\rangle+\|W(x_j, q(x_j))\|^2<1.\end{aligned}$$

Using (a), this gives for $j\in J$, $\operatorname{Re}\langle W(x_j, \bar{f}(x_j)), W(x_j, q(x_j))\rangle>0$ i.e.

$$\operatorname{Re}\{\beta_j\langle W(x_j, h), W(x_j, q(x_j))\rangle\}>0.$$

whence

$$\operatorname{Re}\{\sum_{i=1}^{n}\beta_j\langle W(x_j, h), Wx_j, q(x_j))\rangle\}>0.$$

This contradicts the choice of the vector $(\beta_1,\cdots,\beta_n)$ and establishes the lemma.

For the proof of the main theorem, now take

$$F(x)=\bar{f}(x)(1-\|W(x, Q(x))\|).$$

Then $F(x)$ satisfies (a) and (b) of Lemma 2, hence $E(F)=1$. Finally, for $0\le\lambda\le 1$

$$\|W(x, F(x)-\lambda Q(x))\|\le 1-\|W(x, Q(x))\|+\lambda\|W(x, Q(x))\|\le 1.$$

This proves the theorem.

We next prove a strong unicity theorem for this approximation problem.

Theorem 4.3. Let H be a real Hilbert space. Suppose that M and W satisfy the conditon (T) as in theorem 4.1 and let p be a best weighted approximation to f in M. Then there exists a constant $\lambda=\lambda(f)>0$, such that for any $q\in M$,

$$|f-q|\ge|f-p|+\lambda(f)|p-q|.$$

Proof. If $|f-p|=0$, then $\lambda=1$ satisfies the required inequality. Next, assume that $|f-p|>0$. By the characterization theorem 2.1 there exist distinct points $x_k\in X_{f,p}$ and numbers $\alpha_k>0$, $k=1,\cdots,m$, $m\le n+1$, such that

$$\frac{1}{|f-p|}\sum_{k=1}^{m}\alpha_k\langle W(x_k, f(x_k)-p(x_k)),\ W(x_k, p(x_k))\rangle=0\ ,$$

for each $q\in M$. Using the same idea as in the proof of Lemma 1, we conclude that $m\geq n+1$. Hence $m=n+1$.

Next, since $\alpha_k>0$, by the condition T) we infer that for at least one index k,

$$\frac{\langle W(x_k, f(x_k)-p(x_k)),\ W(x_k, q(x_k))\rangle}{|f-p|}>0\ ,$$

consequently,

$$\max_{1\leq k\leq n+1}\frac{\langle W(x_k, f(x_k)-p(x_k)),\ W(x_k, q(x_k))\rangle}{|f-p|}\ ,$$

is a positive continuous function of q.

Hence,

$$\lambda(f)=\min_{q\in M,\,|q|=1}\frac{\max\langle W(x_k, f(x_k)-p(x_k)),\ W(x_k, q(x_k))\rangle}{|f-p|}>0\ ,$$

by a continuity and compactness argument.

Now let $q\in M$. If $|p-q|=0$, the inequality to be proved is trivial. Otherwise, let $r=\dfrac{p-q}{|p-q|}$, then $|r|=1$ and we have for some index k

$$\begin{aligned}|f-q||f-p|&\geq\langle W(x_k, f(x_k)-p(x_k)),\ W(x_k, f(x_k)-q(x_k))\rangle\\&=\|W(x_k, f(x_k)-p(x_k))\|^2\\&\quad+\langle W(x_k, f(x_k)-p(x_k)),\ W(x_k, p(x_k)-q(x_k))\rangle\\&\geq|f-p|^2+\lambda(f)|f-p||p-q|\ .\end{aligned}$$

Hence $|f-q|\geq|f-p|+\lambda(f)|p-q|$, and the proof is complete.

5. Continuity of the best approximation operator. We assume that M and W satisfy the condition (T). For each $f\in\mathscr{C}(X, H)$, let us denote by $\mathscr{T}_f$ the best weighted approximation to f in M. As in the classical case, we can show that $\mathscr{T}$ is a continuous operator. In fact, we show then $\mathscr{T}$ satisfies a Lipschitz condition.

Theorem 5.1. $\mathscr{T}$ is a continuous mapping on $\mathscr{C}(X, H)$ to M. For each f, there corresponds a number $r(f)$ such that

$$|\mathscr{T}_f-\mathscr{T}_g|\leq r(f)|f-g|\ ,\quad\text{for all}\quad g\in\mathscr{C}(X, H)\ .$$

Proof. By the strong unicity theorem 4.3, one gets a constant $\lambda(f)$ such that

$$|f-q|\geq|f-\mathscr{T}_f|+\lambda(f)|\mathscr{T}_f-q|\ .\quad\text{Taking}\quad q=\mathscr{T}_g\ ,$$

we have

$$\begin{aligned}\lambda(f)|\mathscr{T}_f-\mathscr{T}_g| &\le |f-\mathscr{T}_g|-|f-\mathscr{T}_f| \\ &\le |f-g|+|g-\mathscr{T}_g|-|f-\mathscr{T}_f| \\ &\le |f-g|+|g-\mathscr{T}_f|-|f-\mathscr{T}_f| \\ &\le |f-g|+|g-f|+|f-\mathscr{T}_f|-|f-\mathscr{T}_f| \\ &=2|f-g| \,.\end{aligned}$$

Thus we get $r(f)=2\lambda^{-1}(f)$ as the Lipschitz constant.

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