MINIMAL SUBMANIFOLDS OF A EUCLIDEAN HYPERSPHERE

By

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§1. **Introduction.** Let M^k be an oriented closed k-dimensional riemannian manifold and $x: M^k \to E^n$ an isometric immersion of M^k into a euclidean n-space E^n. Let ∇ and $\tilde{\nabla}$ denote the covariant differentiations of M^k and E^n, respectively, and let B be the differential of the immersion x. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are two tangent vector fields of M^k, then the Gauss formula gives

$$(1) \qquad \tilde{\nabla}_{B\boldsymbol{u}}(B\boldsymbol{v}) = \nabla_{\boldsymbol{u}}\boldsymbol{v} + \alpha(\boldsymbol{u}, \boldsymbol{v}) ,$$

where $\alpha(\boldsymbol{u}, \boldsymbol{v})$ is a normal vector field on M^k and is called the second fundamental form of M^k in E^n. If $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_k$ is an orthonormal basis in the tangent space $T_p(M^k)$ of M^k at p, then the normal vector

$$(2) \qquad \boldsymbol{H} = (1/n) \sum_{i=1}^{k} \alpha(\boldsymbol{e}_i, \boldsymbol{e}_i) ,$$

is independent of the choice of $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n$, and is called the *mean curvature vector* at p. In the following, let $\boldsymbol{X}$ denote the position vector field of M^k in E^n.

The purpose of this paper is to prove the following theorem:

Theorem 1. *Let M^k be a closed k-dimensional riemannian manifold with an isometric immersion $x: M^k \to E^n$. If the position vector $\boldsymbol{X}$ is parallel to the mean curvature vector $\boldsymbol{H}$ everywhere on M, then M^k is immersed as a minimal submanifold of a hypersphere of E^n.*

§2. **Proof of Theorem 1.** Put

$$(3) \qquad U = \{p \in M^k : \boldsymbol{H} \neq 0 \text{ at } p\} .$$

Then by the fact that there exists no minimal closed submanifold in euclidean space, we know that U is a non-empty open subset of M^k. On the set U, we choose a unit normal vector field $\boldsymbol{e}$ which is parallel to the mean curvature normal $\boldsymbol{H}$. Then, by the assumption that $\boldsymbol{X}$ is parallel to $\boldsymbol{H}$, we can put

$$(4) \qquad \boldsymbol{X} = f\boldsymbol{e} , \quad \text{on } U .$$

Let $\boldsymbol{Y}$ be an arbitrary tangent vector on M^k, we have

$$\tilde{\nabla}_{BY}\boldsymbol{X}=(\boldsymbol{Y}f)\boldsymbol{e}+f\tilde{\nabla}_{BY}\boldsymbol{e}\,. \tag{5}$$

Since, $\boldsymbol{e}$ is a unit vector field and $\boldsymbol{X}$ is the position vector field of M^k in E^n. Hence, we have

$$\langle\tilde{\nabla}_{BY}\boldsymbol{X},\boldsymbol{e}\rangle=0\,,\quad\text{and}\quad\langle\tilde{\nabla}_{BY}\boldsymbol{e},\boldsymbol{e}\rangle=0\,, \tag{6}$$

where $\langle\ ,\ \rangle$ denotes the scalar product of E^n. Therefore, by (5) and (6), we get $\boldsymbol{Y}f=0$ for all tangent vector $\boldsymbol{Y}$ on M^k. Thus, the function f is a constant on M^k. Thus, we get

$$\langle\boldsymbol{X},\boldsymbol{X}\rangle=f^2=\text{constant}\quad\text{on } U\,. \tag{7}$$

On the other hand, by a direct computation of the Laplacian of $\boldsymbol{X}$, we have the following identity:

$$\Delta\boldsymbol{X}=k\boldsymbol{H}\,,\quad\text{on } M^k\,. \tag{8}$$

Hence, by (7) and (8), we get [1]

$$\Delta(\langle\boldsymbol{X},\boldsymbol{X}\rangle)=2k(1+\langle\boldsymbol{X},\boldsymbol{H}\rangle)=0\,,\quad\text{on } U\,. \tag{9}$$

Therefore, by (7), (9) and the assumption $\boldsymbol{X}//\boldsymbol{H}$, we get for every component U^* of U

$$\boldsymbol{X}=c\boldsymbol{H}\,,\quad\text{on } U^*\,,\quad c=-f^2=\text{constant}. \tag{10}$$

Hence, by (7) and (10), we know that $\langle\boldsymbol{H},\boldsymbol{H}\rangle=f^2=\text{constant}$. Therefore by the continuity of $\boldsymbol{H}$ on M^k, we know that

$$U=M^k\,. \tag{11}$$

Consequently, by (3), (8), (10), and (11), we get

$$\Delta\boldsymbol{X}=(k/c)\boldsymbol{X}\,,\quad\text{on } M^k\,. \tag{12}$$

Therefore, by a result due to *Takahashi* [2], we know that M^k is immersed as a minimal submanifold of a hypersphere in E^n. This completes the proof of the theorem.

Remark. Minimal submanifold in a euclidean space is an example said that if the condition of closedness is omitted, then Theorem 1 is not longer true, in general. But in fact, if we assume that M^k is non-minimal in E^n, then we have the same result.

Corollary: *Let M^k be a k-dimensional submanifold of E^n. If we have $\Delta\boldsymbol{X}=f\boldsymbol{X}$ for some functions f on M^k, then M^k is either a minimal submanifold* *of* E^n *or a minimal submanifold of a hypersphere of* E^n.

This corollary follows from (8) and the remark.

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