SOME THEOREMS ON FIXED POINTS*

By

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Let X be a metric space. A mapping $T: X \to X$ is called a *contraction mapping* if

$$d(Tx, Ty) \leq kd(x, y) \tag{A}$$

for all $x, y \in X$ and $0<k<1$.

The well-known Banach contraction principle states that a contraction mapping on a complete metric space X has a unique fixed point. A mapping $T: X \to X$ such that

$$d(Tx, Ty) < d(x, y), \qquad (x \neq y) \tag{B}$$

$x, y \in X$, is called a *contractive mapping.* A contractive mapping on a complete metric space need not have a fixed point. For example, let X be the set of real numbers with the usual metric. Let

$$T: X \to X$$

defined by $Tx = x + \frac{\pi}{2} - \text{arc tan } x$. Then T is a contractive mapping but it has no fixed point because arc tan $x < \frac{\pi}{2}$ for every x. However, if the space X is compact, a contractive mapping has a unique fixed point [1].

Recently *Kannan* [3] proved the following theorem:

If T is a mapping of a complete metric space X into itself such that

$$d(Tx, Ty) \leq \alpha \{d(x, Tx) + d(y, Ty)\} \tag{C}$$

for all $x, y \in X$ and $0 < \alpha < \frac{1}{2}$, then T has a unique fixed point.

The aim of this paper is to give extensions of the above theorem by taking $\alpha = \frac{1}{2}$.

Theorem 1. *Let X be a metric space and let $T: X \to X$ be a continuous mapping such that*

$$d(Tx, Ty) < \tfrac{1}{2} \{d(x, Tx) + d(y, Ty)\}$$

for $x \neq y$. If for some $x_0 \in X$, the sequence $\{T^n x_0\}$ has a subsequence $\{T^{n_i} x_0\}$ converging to x, then $T^n x_0$ converges to x and x is a unique fixed point of T.

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Proof. The sequence $d(T^n x_0, T^{n+1} x_0)$ is non-increasing. Since T is continuous we get

$$\lim_{i\to\infty} T^{n_i+1} x_0 = Tx$$

and

$$\lim_{i\to\infty} T^{n_i+2} x_0 = T^2 x$$

Therefore,

$$\begin{aligned} d(x, Tx) &= \lim_{i\to\infty} d(T^{n_i} x_0, T^{n_i+1} x_0) \\ &= \lim_{i\to\infty} d(T^{n_i+1} x_0, T^{n_i+2} x_0) \\ &= \quad d(Tx, T^2 x). \end{aligned}$$

If $x \neq Tx$, then $d(Tx, T^2 x) < d(x, Tx)$. This implies $d(x, Tx) < d(x, Tx)$, impossible, therefore $d(x, Tx) = 0$ i.e. x is a fixed point of T.

Uniqueness follows immediately. Let x and y be two fixed points $x \neq y$. Then $x = Tx$ and $y = Ty$ imply

$$d(Tx, Ty) < \tfrac{1}{2}\{d(x, Tx) + d(y, Ty)\} = 0$$

a contradiction to the fact that $d(Tx, Ty) \geq 0$. Thus the proof.

Theorem 2. *Let T be a continuous map of a metric space X into itself such that*

(1) $d(Tx, Ty) \leq \frac{1}{2}\{d(x, Tx) + d(y, Ty)\}$

(2) *if $x \neq Tx$, then $d(Tx, T^2 x) < d(x, Tx)$,*

(3) *for some $x_0 \in X$, the sequence $\{T^n x_0\}$ has a subsequence $\{T^{n_i} x_0\}$ converging to x. Then the sequence $T^n x_0$ converges to x and x is a unique fixed point*

Proof. By (1) the sequence $d(T^n x_0, T^{n+1} x_0)$ is non-increasing. Since T is continuous we have

$$\begin{aligned} d(x, Tx) &= \lim_{i\to\infty} d(T^{n_i} x_0, T^{n_i+1} x_0) \\ &= \lim_{i\to\infty} d(T^{n_i+1} x_0, T^{n_i+2} x_0) \\ &= \quad d(Tx, T^2 x) \end{aligned}$$

contradiction to (2) unless $x = Tx$. Also, from (1) it follows that for all n, $d(T^{N+n} x_0, x) \leq d(T^N x_0, x)$, whence $T^n x_0 \to x$.

Theorem 3. *Let T be a mapping of a metric space X into itself such that*

(a) *$d(Tx, Ty) \leq d(x,y)$ for all $x, y \in X$,*

(b) *$d(Tx, Ty) < \frac{1}{2}\{d(x, Tx)+d(y, Ty)\}$ for all $x, y \in X$,*

(c) *if for some $x_0 \in X$, the sequence $T^n x_0$ has a subsequence $T^{n_i} x_0$ converging to x then $T^n x_0 \to x$ and x is a unique fixed point.*

Proof. Since T satisfies (a), by a Theorem of *Edelstein* [2], x generates an isometric sequence, i. e. for any integers $m>0, n>0$,

$$d(T^m x, T^n x)=d(T^{m+k} x, T^{n+k} x), \quad k=1,2,3,\cdots.$$

Letting $m=1$, and $n=2$, we have

$$d(Tx, T^2 x)=d(T^{k+1} x, T^{k+2} x); \quad k=1,2,3,\cdots.$$

We conclude that $d(Tx, T^2 x)=0$, since T satisfies (b).

Since T is continuous, we get

$$\lim_{i\to\infty} d(T^{n_i} x, T^{n_i+1} x)=d(x, Tx)$$

and

$$\lim_{i\to\infty} d(T^{n_i+1} x, T^{n_i+2} x)=d(Tx, T^2 x).$$

Thus $$d(x, Tx)=d(Tx, T^2 x)=0.$$

This implies that $x=Tx=\lim_{n\to\infty} T^n x_0$.

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