INVARIANT SUBSPACES OF POLYNOMIALLY COMPACT OPERATOR IN BANACH SPACES

By

NAI-HUNG HSU

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For discussion the existence of invariant subspaces, in 1954, *N. Aronszajn* and *K. T. Smith* [1] has proved the theorem: Let B be a Banach space and T a compact operator in Banach spaces, then there exists a proper invariant subspaces of T. But for general bounded operator, even in Hilbert apace, it is not yet known that whether there always exists a proper invariant subspace. Recently in 1966, *A. R. Bernstein and A. Robinson* [2] has proved the theorem: If A is a linear bounded operator on a Hilbert space H of dimension greater than 1 and if p is a non-zero polynomial such that $p(A)$ is compact, then there exists a non-trivial subspace of H invariant under A. The proof was based on the framework of Non-standard analysis. And at the same time *P. R. Halmos* [3] has proved the same theorem that was expressed in ths standard framework of classical analysis.

Now, in this present paper, I want to show that the result [2] can be extended to the case of general Banach spaces, that is, if A is a linear bounded operator in a Banach space B of infinite dimension and if p is a non-zero polynomial such that $p(A)$ is compact, then there exists a non-trivial subspace of B invariant under A.

Let A be a *linear bounded operator* in a Banach space B, $A(B)\subset B$. *A closed linear subspace* $L\subset B$ is said to be a proper invariant subspace under A, if $(0)\neq L\neq B$, then $A(L)\subset L$.

A compact operator (*completely continuous operator*) A in B means that if, for any bounded subset E of B, $A(E)$ is relatively compact in B. An equivalent condition is that for any bounded sequence $\{x_n\}$ in B, there is a subsequence $\{x_{n_k}\}$ such that the sequence $\{A(x_{n_k})\}$ converges in B.

Theorem. *If A is a linear bounded operator in a Banach space B of infinite dimension and if p is a non-zero polynomial such that $p(A)$ is compact, then there exists a non-trivial subspace of B invariant under A.*

Proof. Consider an arbitrary $f\neq 0$ in B. The closed subspace $\{A^n f\}_0^\infty$ generated by f and its successive images, $Af, A^2f, \cdots$ is clearly an invariant subspace of A.

Hence we can limit ourselves to the case that

$$\{A^n f\}_0^\infty = B \tag{1}$$

This formula implies the following properties:

(2) *B is separable.*

(3) *All the elements $A^n f$ are $\neq 0$ and linearly independent.*

Suppose that we have the relation

$$a_1 A^{n_1} f + a_2 A^{n_2} f + \cdots + a_k A^{n_k} f = 0 \quad \text{where} \quad a_i \neq 0$$
$$i = 1, 2, \cdots, k, \quad \text{and} \quad 0 \leqslant n_1 < n_2 < \cdots < n_k$$

Then we have $A^{n_k} f = \left(-\dfrac{1}{a_k}\right)(a_1 A^{n_1} f + a_2 A^{r_2} f + \cdots + a_{k-1} A^{n_{k-1}} f)$

and hence all the $A^n f's$ would lie in the subspace generated by those with indices $n < n_k$, Which is in contradiction to (1) and the infinite dimension of B.

Since in every separable Banach space we can define an equivalent strictly convex norm, i. e. such that if $x \neq y$ and $\|x\| = \|y\| \neq 0$, then $\|x+y\| < \|x\| + \|y\|$ (see *J. A. Clarkson* [4]). We shall suppose that the norm in a separable Banach space B is strictly convex.

Now we consider an arbitrary finite dimensional subspace $L \subset B$. For every $x \in B$ we can consider the minimal distance $d(x, L)$ from x to L. Since L is of finite dimension, the shortest distance is certainly attained and in view of the strict convexity of the norm it is immediately proved that there exists a unique point $Px \in L$ which realizes this minimal distance, i. e.

$$\|x - Px\| = d(x, L) = \min_{y \in L} \|x - y\|.$$

Px represents an operator in B, in general non-linear, we shall call P the *metric projection on L*, or brief, the projection on L.

By the definition of projection P we have the following properties:

(a-1) *P is idempotent*: $P = P^2$

(a-2) *P is homogenuous*: $P(ax) = aPx$ *for every* $a \in k$ *(field)*

(a-3) *P is quasi-additive*: $P(y+x) = y + Px$ *for every* $y \in L$

(a-4) *P is bounded*: $\|Px - x\| \leqslant \|x\|, \qquad \|Px\| \leqslant 2\|x\|$.

(a-5) $|\|x - Px\| - \|y - Px\|| \leqslant \|x - y\|$.

(a-6) *If $L' \subset L$ and P' is the projection on L' then*
$$\|x - Px\| \leqslant \|x - P'x\|$$

Consider now a sequence of closed subspace $\{L_k\}$, where $L_k \subset B$.

Definition. *If $\underline{\lim}\ L_k$ = set of all $x \in B$ such that for some $x_k \in L_k$, $x_k \longrightarrow x$, then we called $\underline{\lim}\ L_k$ is the limit inferior of the sequences $\{L_k\}$.*

By the above definition, we have the following properties:

(b-1) *$\underline{lim}$ L_k is a closed subspace.*

(b-2) *If every L_k is finite dimensional then $x \in \underline{lim}\ L_k$ if and only if $p_k x \longrightarrow x$, where P_k is the projection on L_k.*

Now we prove the main theorem, with f satisfying (1). We construct the k-dimensional subspace.

(4) $$L^{(k)} = \{A^n f\}_0^{k-1}$$

We denote by $P^{(k)}$, the metric projection on $L^{(k)}$, by (1) it is clearly $\underline{lim}$ $L^{(k)} = B$. And by (a-2) we have

(5) $$P^{(k)}x \longrightarrow x \quad \text{for all } x \in B.$$

We can use the classical result that it may be represented by a triangular matrix which gives that there exists an increasing sequence of subspaces.

(6) $$0 = L^{(k,0)} \subset L^{(k,1)} \subset L^{(k,2)} \subset \cdots \quad \subset L^{(k,k)} = L^{(k)}$$

and $P^{(k,i)}$ denotes the projection on $L^{(k,i)}$, where $i \leqslant k$.

The following Lemma 1 and Corollary 1 are due [1].

Lemma 1. *Let $\{k_m\}$ and $\{i_m\}$ be sequences of integers such that $k_m \nearrow \infty$ and $0 \leqslant i_m \leqslant k_m$. Further, let $x_m \in L^{(k_m, i_m)}$. If $Ax_m \longrightarrow y$ then $y \in \underline{lim}\ L^{(k_m, i_m)}$.*

Corollary 1. *For any sequence $\{k_m\}$ and $\{i_m\}$ satisfying the condition of the lemma 1, then $\underline{lim}$ $L^{(k_m, i_m)}$ is an invariant subspace of A.*

Lemma 2. *Let $\{k_m\}$ and $\{i_m\}$ be sequences of integers such that $k_m \nearrow \infty$ and $0 \leqslant i_m \leqslant k_m$. if the lim of every subsequence of $L^{(k_m, i_m)}$ is equal to zero and $p(z)$ is a non-zero polynomial, i.e. $p(z) \neq 0$, such that $p(A)$ is compact operator in B, then for any bounded sequence $\{x_m\}$, $x_m \in L^{(k_m, i_m)}$, we have $p(A) \longrightarrow 0$.*

Proof By compact operator $p(A)$, the bounded sequence $\{x_m\}$ is transformed into a relatively compact sequence $\{p(A)x_m\}$. Therefore it is enough to prove that if any subsequence $\{p(A)x_{m_j}\}$ converges to some y, then $y=0$. By hypothesis and (5), we have

$$\|p(A)x_{m_j} - P^{(k_{m_j})}p(A)x_{m_j}\| \leqslant \|y - P^{(k_{m_j})}y\| + \|p(A)x_{m_j} - y\| \longrightarrow 0.$$

and $\|y - P^{(k_{m_j})}p(A)x_{m_j}\| \leqslant \|y - p(A)x_{m_j}\| + \|p)A)x_{m_j} - P^{(k_{m_j})}p(A)x_{m_j}\| \longrightarrow 0,$

where $p^{(k_{m_t})}$ is the proiection on $L^{(k_{m_j})}$.

By definition of inferior limit, we get $y \in \underline{lim}\ L^{(k_{m_j}, i(k_{m_j}))}$ and

$\underline{lim}\ L^{(k_{m_j}, i(k_{m_j}))} \subset \underline{lim}\ L^{(k_m, i(k_m))}$, hence $y=0$.

We choose now an arbitrary a with

(7) $\quad 0<a<1, \qquad \|p(A)f\|>a\|p(A)\|\cdot\|f\|$

Since $f\in L^{(k)}$, we have by (6) and (a-6)

$$\|f\|=\|f-P^{(k,0)}f\|\geqslant\|f-P^{(k,1)}f\|\geqslant\cdots\geqslant\|f-P^{(k,k)}f\|=0$$

There exists therefore for each $k=1,2,\cdots$, a unique indice $i(k)$, $0\leqslant i(k)<k$, such that

(8) $\quad \|f-P^{(k,i(k))}f\|\geqslant a\|f\|>\|f-P^{(k,i(k)+1)}f\|$

Let z_k, $k=1,2,\cdots\cdots$, be an element of $L^{(k,i(k)+1)}$ such that

(9) $\quad \|z_k\|=1, \qquad P^{(k,i(k))}z_k=0.$

Such an element can be obtained from an arbitrary element

$u\in L^{(k,i(k)+1)}-L^{(k,i(k))}$, by putting $z_k=\|u-P^{(k,i(k))}u\|^{-1}(u-P^{(k,i(k))}u)$

by (a-2) and (a-3), then (9) is proved.

Since the dimensions of $L^{(k,i(k)+1)}$ and $L^{(k,i(k))}$ differ by 1. Hence every element $y\in L^{(k,i(k)+1)}$ is representable in a unique way in the form $y=x+bz_k$ with $x=P^{(k,i(k))}y$ correspondingly, we shall put

(10) $\quad P^{(k,i(k)+1)}f=x_k+b_k z_k$

$\quad P^{(k,i(k)+1)}Af=x'_k+b'_k z_k, \qquad x_k$ and $x'_k\in L^{(k,i(k))}$

By (a-4), we have

(11) $\quad \|x_k\|=\|P^{(k,i(k))}P^{(k,i(k)+1)}f\|\leqslant 4\|f\|$

$\quad \|x'_k\|\leqslant 4\|Af\|$

We now prove the following statements:

(I) *For every sequence* $k_m\nearrow\infty$, $\underline{\lim}\, L^{(k_m,i(k_m))}\neq B$.

(II) *For some sequence* $k'_m\nearrow\infty$, $\underline{\lim}\, L^{(k'_m,i(k'_m)+1)}\neq 0$.

(III) *If for every sequnce* $k_m\nearrow\infty$, $\underline{\lim}\, L^{(k_m,i(k_m))}=0$, *then for every sequence* $k'_m\nearrow\infty$, $\underline{\lim}\, L^{(k'_m,i(k'_m)+1)}\neq B$.

Proof of (I). If $\underline{\lim}\, L^{(k_m,i(k_m))}=B$, then by (a-2) $P^{(k_m,i(k_m))}f\longrightarrow f$. Which contradicts to (8).

Proof of (II). Suppose the contrary, then the bounded sequence $\{P^{(k,i(k)+1)}f\}$ is transformed into a sequence $\{p(A)P^{(k,i(k)+1)}f\}$ converging to 0. By lemma 2, since $p(A)f=p(A)(f-P^{(k,i(k)+1)}f)+p(A)P^{(k,i(k)+1)}f$

We get $\quad \|p(A)f\| \leqslant \lim \|p(A)(f-P^{(k,i(k)+1)}f)\|$

$$\leqslant \lim\inf \|p(A)\| \cdot \|f-P^{(k,i(k)+1)}f\|$$

Which, by (8), gives $\|p(A)f\| \leqslant a\|p(A)\| \cdot \|f\|$ is contradiction to (7).

Proof of (III). Suppose that for every $k_m \nearrow \infty$, $\underline{lim}\ L^{(k_m, i(k_m)+1)} = B$.

By (b-2), we have $P^{(k'_m, i(k'_m)+1)} f \longrightarrow f$ and $P^{(k'_m, i(k'_m)+1)} Af \longrightarrow Af$.

Then by (10) we have

$$f = \lim\ (x_{k'_m} + b_{k'_m} z_{k'_m})$$

$$Af = \lim\ (x'_{k'_m} + b'_{k'_m} z_{k'_m})$$

Hence $\quad p(A)f = \lim\ (p(A)x_{k'_m} + p(A)b_{k'_m} z_{k,m})$

$$p(A)Af = \lim\ (p(A)x'_{k'_m} + p(A)b'_{k'_m} z_{k'_m})$$

By (11) and lemma 2, it follows that

$$p(A)f = \lim\ p(A)b_{k'_m} z_{k'_m}$$

$$p(A)Af = \lim\ p(A)b'_{k'_m} z_{k'_m}$$

Hence $b'_{k'_m}/b_{k'_m}$ converges to some number c, and $f = cAf$ which contradicts to (3).

We complete the proof of the theorem as follow: If there is any sequence $k_m \nearrow \infty$, such that $S = \underline{lim}\ L^{(k_m, i(k_m))} \neq (0)$, then in view of statement (I) and corollary 1, S is a proper invariant subspace. If there is no such sequence $\{k_m\}$, then by statement (II), we choose a sequence $k'_m \nearrow \infty$, so that

$$S' = \underline{lim}\ L^{(k'_m, i(k'_m)+1)} \neq 0$$

By statement (III) and corollary 1, S' is then a proper invariant subspace.

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National Tsing Hua University
Hsinchu, Taiwan, Republic of China.