ON BOOLEAN ALGEBRAS WHICH HAVE THE M_α-PROPERTY

By

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1. Intoroduction

The general theory of α-atomic Boolean algebras has been developed by *R. S. Pierce* [1]. In this paper, I introduced the concept of the M_α-property in a Boolean algebra. That is, let α be an infinite cardinal number and let A be a Boolean algebra, then A is said to have the M_α-property provided if $P=\{a_\xi : \xi<\alpha\}$ is any subset of A such that every finite subset of P has non-zero meet, then then there is a non-zero element a in A satisfying $a\subset a_\xi$ for $\xi<\alpha$. The existence of such a Boolean algebra will be proved.

It is clear that if A is a Boolean algebra which has the M_α-property, then the minimal β-extension. A^β of A is α-atomic. Therefore, we can apply the results of *R. S. Pierce* for α-atomic Boolean algebra to A^β. *E. C. Smith* and *A. Tarski* has proved the theorem in their paper [2] such that if β is a singular, strong limit cardinal and A is an β-complete Boolean algebra which is (α, β)-distributive for every cardinal $\alpha<\beta$, then A is (β, β)-distributive. Moreover, I modified this theorem and applied it to a Boolean algebra which has the M_α-property for every cardinal $\alpha<\beta$, Thus I proved the following theorem.

Suppose that β is an arbitrary infinite cardinal number and that A is a Boolean algebra which has the M_α-property for every cardinal $\alpha<\beta$. Then A is β-representable.

2. Preliminaries

The set-theoretical operations are represented by rounded symbols: ϵ, $\cup$, $\cap$ and $\subseteqq$ respectively denote membership, union, intersection and inclusion. If A and B are sets, $B-A$ is the set of all elements of B which are not in A; the complement (in a fixed set) of A is designated A^c. The empty set is denoted by ϕ.

The following definitions and results concerning the ordinal numbers and the cardinal numbers are due to *Alexander Abian* [3].

A set β is called an ordinal nnmber (or simply an ordinal) if β can be well ordered so that for element α of β the initial segment $I(\alpha)$ of β is equal to α, i. e., $I(\alpha)=\alpha$ for every $\alpha\epsilon\beta$. For every two ordinal numbers α and β, one and only one of the following three cases holds (i) $\alpha=\beta$ (ii) α is equal to an initial segment of β (iii) β is equal to an initial segment of α. We define $\alpha\leqq\beta$ if α is equal to β or α is equal to an initial segment of β. If $\alpha \leqq \beta$ and $\alpha \neq \beta$, we say that α is less than β and as usual we denote $\alpha < \beta$. Every ordinal number β is equal to the set of all ordinals less than β. We denote this set $W(\beta)$. Let us call an ordinal β immediate successor of ordinal α if $\alpha < \beta$; and if an ordinal γ is such that $\alpha < \gamma$, then $\beta \leqq \gamma$. Every ordinal number α has the immediate successor. The immediate successor of α is denoted by $\alpha + 1$. An ordinal number α is said to be immediate predecessor of an ordinal β if $\alpha < \beta$; and if an ordinal γ is such that $\gamma < \beta$, then $\gamma \leqq \alpha$.

Two sets A, B are called equipollent, in symbol $A \cong B$, if there exists a one-to-one correspondence between them. An ordinal number α is called a cardinal number (or simply a cardinal), if for every ordinal number β, $\alpha \cong \beta$ implies $\alpha \leqq \beta$. We say such a cardinal number an initial number. Every set A is equipollent to an unique cardinal number α. We denote $\overline{\overline{A}} = \alpha$. Every infinite cardinal number has no immediate predecessor. We say that a cardinal number β is the immediate successor of a cardinal α if $\alpha < \beta$ and, if for no cardinal γ is it the case that $\alpha < \gamma < \beta$. Every cardinal number α has the unique immediate successor. It is denoted by α^{+}.

If A and B are non-empty sets, then A^{B} will denote the set of all functions of B into A. For every two cardinal numbers α and β the β-th power of α, denoted by $\alpha^{(\beta)}$, is defined as $\alpha^{(\beta)} = \overline{\overline{\alpha^{\beta}}}$.

For every X of ordinal (cardinal) numbers, the union $\cup X$ of X is an ordinal (cardinal) number. Moreover, $\cup X$ is the least upper bound of X. A cardinal number β is called singular if it can be represented as the least upper bound of a set S of cardinals, each of S is less than β and $\overline{\overline{S}} < \beta$. All other cardinals are called regular.

For every indexed family $\{\alpha_i : i \in I\}$ of cardinal numbers, the sum of all cardinal numbers belonging to this family is denoted by $\sum_{i \in I}{}^{*} \alpha_i$ and is defined as: $\overline{\overline{\bigcup_{i \in I} (\alpha_i \times \{i\})}}$. Accordingly, $\sum_{i \in I}{}^{*} \alpha_i = \overline{\overline{\bigcup_{i \in I} (\alpha_i \times \{i\})}}$ where $\alpha_i \times \{i\}$ is the Cartesian product of α_i and $\{i\}$. For every two families $\{\alpha_i : i \in I\}$ and $\{\beta_i ; i \in I\}$ of cardinal numbers α_i, and β_i, $\alpha_i \leqq \beta_i$ for every $i \in I$ implies $\sum_{i \in I}{}^{*} \alpha_i \leqq \sum_{i \in I}{}^{*} \beta_i$. For an indexed family $\{\alpha_i : i \in I\}$ of cardinal numbers, if $\overline{\overline{I}} = \beta$, and $\alpha_i = \alpha$ for every $i \in I$, then we have $\sum_{i \in I}{}^{*} \alpha_i = \alpha\beta$, where $\alpha\beta = \overline{\overline{u \times v}}$ with $\alpha \cong u$ and $\beta \cong v$. If $\{A_\xi : \xi < \alpha\}$ is any family of sets, pairwise disjoint or not, then $\overline{\overline{\bigcup_{\xi < \alpha} A_\xi}} \leqq \sum_{\xi < \alpha}{}^{*} \overline{\overline{A}}_\xi$. Finally, for every non-zero cardinal α and every infinite cardinal number β, $\alpha \leqq \beta$ implies $\alpha\beta = \beta$.

We shall denote the fundamental Boolean operations, join, meet and inclusion by $+$, $\cdot$ and $\subset$. The generalizations of join and meet denoted by Σ and Π, respectively. If a is an element of a Boolean algebra A, $\bar{a}$ denotes the complement of a in A. The null and universal elements of a Boolean algebra will be denoted by 0 and 1, respectively, as well as the ordinary numbers zero and one. A Boolean algebra A is called α-complete if and only if whenever $\boldsymbol{B} \subseteq A$ and $\overline{\overline{\boldsymbol{B}}} \leqq \alpha$, $\Sigma \boldsymbol{B}$ (or $\sum_{b \in \boldsymbol{B}} b$) exists in A.

By a field of sets we shall understand any non-empty class $\boldsymbol{F}$ of subsets of a fixed set X such that (i) if sets A, B are in $\boldsymbol{F}$, then their union is in $\boldsymbol{F}$. (ii) if a set A is in $\boldsymbol{F}$, then its complement in the fixed set X is in $\boldsymbol{F}$. Clearly, every field of sets is a Boolean algebra, the Boolean operations $+, \cdot, -$ being the set-theoretical union, intersection and complementation, respectively.

3. The existence of a Boolean algebra which has the M_α-property

A set $\boldsymbol{D}$ of elements of a Boolean algebra $\boldsymbol{A}$ is said to be dense (in $\boldsymbol{A}$) if, for every non-zero element $a \in \boldsymbol{A}$, there exists an element $b \in \boldsymbol{D}$ such that $0 \neq b \subset a$.

Let α be an infinite cardinal number. A partially ordered set P will be called α-compact if P is closed under finite meets contains a zero element and satisfies the condition that $M \subseteq P, \overline{\overline{M}} \leqq \alpha$ and no finite subset of M has zero meet, then M has a non-zero lower bound in P. A Boolean algebra $\boldsymbol{A}$ will be called α-atomic if $\boldsymbol{A}$ contains a dense subset which is α-compact.

Definition. *A Boolean algebra $\boldsymbol{A}$ is said to have the M_α-property if $\boldsymbol{A}$ itself is α-compact.*

We shall show that the existence of a Boolean algebra which has the M_α-property.

Let Y be an infinite set with $\overline{\overline{Y}} = \beta > \omega$ and $\boldsymbol{B}$ be the field (i.e. Boolean algebra) composed of all finite subsets of Y and of all cofinite subsets of Y. Let y be any point which does not belong to Y, and $X = Y \cup \{y\}$. The mapping

$$\varphi(A) = \begin{cases} A & \text{if } A \in \boldsymbol{B} \text{ is finite} \\ A \cup \{y\} & \text{if } A \in \boldsymbol{B} \text{ is cofinite} \end{cases}$$

is an isomorphism of $\boldsymbol{B}$ onto a field $\boldsymbol{F}$ of subsets of X.

Suppose that $\mathscr{T}$ is the family which consists of all unions of members of $\boldsymbol{F}$. Then $\mathscr{T}$ is a topology in X and $\boldsymbol{F}$ is an open basis for X. Of course, every set $B \in \boldsymbol{F}$ is open. It is also closed in this topology $\mathscr{T}$ since $X-B$ belongs to $\boldsymbol{F}$. $\boldsymbol{F}$ being reduced, the space X is totally disconnected.

To prove that X is compact, we suppose that $\boldsymbol{C}$ is an open covering of X. We can assume that each set B in $\boldsymbol{C}$ belongs to $\boldsymbol{F}$, because each set B in $\boldsymbol{C}$ is the union of members of $\boldsymbol{F}$. Then there is at least one $B \in \boldsymbol{C}$ such that $y \in B$. Hence there exists a cofinite set $A \in \boldsymbol{B}$ such that $B = A \cup \{y\}$. Moreover B^c is finite. Therefore we can find a finite sequence $B_1, \cdots\cdots, B_n \in \boldsymbol{C}$ such that $X = B_1 \cup \cdots\cdots \cup B_n$.

Now we shall prove that a set $B \subseteq X$ is open-closed, then $B \in \boldsymbol{F}$. Indeed, B is the union of a family $\boldsymbol{K}$ of sets in $\boldsymbol{F}$ since B is open. Since B is a closed subset of the compact space X, there exists a finite sequence $B_1, \cdots\cdots, B_n \in \boldsymbol{K} \subseteq \boldsymbol{F}$ such that

$B=B_1\cup\cdots\cdots\cup B_n$. Hence $B\epsilon\boldsymbol{F}$. Consequently, the field $\boldsymbol{F}$ consists of all open-closed subsets of X.

Since the Boolean algebra $\boldsymbol{B}$ is isomorphic to the field $\boldsymbol{F}$ of all open-closed subsets of the compact totally disconnected space X, X is the Stone space of $\boldsymbol{B}$.

Theorem 1. *The Boolean algebra* $\boldsymbol{B}$ *has the* M_α*-property for every cardinal* $\alpha<\beta$ *where* $\omega\leqq\alpha$.

Proof. To prove that $\boldsymbol{B}$ has the M_α-property, it suffices to show that for every subset $\boldsymbol{M}=\{A_\xi:\xi<\alpha\}$ of $\boldsymbol{B}$ which has the finite intersection property, there is non-zero element $A\epsilon\boldsymbol{B}$ such that $A\subseteqq A_\xi$ for every $\xi<\alpha$. Since $\{A_\xi:\xi<\alpha\}$ has the finite intersection property, the subset $\{\varphi(A_\xi):\xi<\alpha\}$ of $\boldsymbol{F}$ has the same property. Moreover, X being compact, we obtain $\bigcap\limits_{\xi<\alpha}\varphi(A_\xi)\neq\phi$.

Case I. If there is at least one finite set A_ξ in $\boldsymbol{M}$, then there is a point $x\epsilon X$ distinct from y such that $x\epsilon\bigcap\limits_{\xi<\alpha}\varphi(A_\xi)$. This means that the singleton $\{x\}\subseteqq\varphi(A_\xi)$ for every $\xi<\alpha$. On the other hand, by the property of φ that $\varphi(\{x\})=\{x\}, \varphi(\{x\})\subseteqq\varphi(A_\xi)$ for every $\xi<\alpha$. Consequently, $\phi\neq\{x\}\subseteqq A_\xi$ for every $\xi<\alpha$ and $\{x\}\epsilon\boldsymbol{B}$.

Case II. Let us assume that there is no finite set A_ξ in $\boldsymbol{M}$. Suppose now that $\bigcap\limits_{\xi<\alpha}\varphi(A_\xi)=\{y\}$. Then, by the de Morgan law, $\bigcup\limits_{\xi<\alpha}\varphi(A_\xi^c)=Y$ where $A_\xi^c=Y-A_\xi$. Each A_ξ^c being finite set, $\bigcup\limits_{\xi<\alpha}A_\xi^c=Y$. Hence we have $\beta=\overline{\overline{Y}}\leqq\sum\limits_{\xi<\alpha}{}^*\overline{\overline{A_\xi^c}}\leqq\omega\cdot\alpha=\alpha<\beta$. This leads to a contradiction. Therefore $\bigcap\limits_{\xi<\alpha}\varphi(A_\xi)$ contains a point x of X distinct from y. By means of a similar argument, one can obtain the element $\{x\}\epsilon\boldsymbol{B}$ such that $\phi\neq\{x\}\subseteqq A_\xi$ for every $\xi<\alpha$.

4. The distributivity

A Boolean algebra $\boldsymbol{A}$ is (α,β)-distributive if the following is satisfied: given any subset $\{a_{\xi,\eta}:\xi<\alpha,\eta<\beta\}$ of $\boldsymbol{A}$ such that all the joins $\sum\limits_{\eta<\beta}a_{\xi,\eta}$ for $\xi<\alpha$, their meet $\prod\limits_{\xi<\alpha}\sum\limits_{\eta<\beta}a_{\xi,\eta}$ and all the meets $\prod\limits_{\xi<\alpha}a_{\xi,f(\xi)}$ for $f\epsilon\beta^\alpha$ exist, then the join $\sum\limits_{f\epsilon\beta^\alpha}\prod\limits_{\xi>\alpha}a_{\xi,f(\xi)}$ also exists and we have

$$\prod_{\xi<\alpha}\sum_{\eta<\beta}a_{\xi,\eta}=\sum_{f\epsilon\beta^\alpha}\prod_{\xi<\alpha}a_{\xi,f(\xi)}.$$

If a Boolean algebra $\boldsymbol{A}$ is (α,β)-distributive for every cardinal number β, we say that $\boldsymbol{A}$ is (α,∞)-distributive.

Actually, in order to demonstrate that a Boolean algebra $\boldsymbol{A}$ is (α,β)-distributive, it is sufficient to show that if $\{a_{\xi,\eta}:\xi<\alpha,\eta<\beta\}$ is any subset of $\boldsymbol{A}$ such that all the joins $\sum\limits_{\eta<\beta}a_{\xi,\eta}$ for $\xi<\alpha$ exist and their meet $\prod\limits_{\xi<\alpha}\sum\limits_{\eta<\beta}a_{\xi,\eta}$ exists and is not zero, then there is an $f\epsilon\beta^\alpha$ such that $\prod\limits_{\xi<\alpha}a_{\xi,f(\xi)}$is false; i. e. either $\prod\limits_{\xi<\alpha}a_{\xi,f(\xi)}$ does not exist or is not zero.

Theorem 2. *Suppose that β is a singular cardinal number and that $\boldsymbol{A}$ is an β-complete Boolean algebra which is (α, ∞)-distributive for every cardinal $\alpha<\beta$. Then $\boldsymbol{A}$ is (β, ∞)-distributive.*

Proof. Let γ be an arbitrary cardinal number and $\{ a_{\xi,\eta} : \xi<\beta, \eta<\gamma\}$ be any subset of $\boldsymbol{A}$ such that

$$(1) \qquad \prod_{\xi<\beta}\sum_{\eta<\gamma} a_{\xi,\eta} \neq 0$$

β being singular, we can find a set $S=\{\beta_\xi : \xi<\alpha\}$ of cardinal numbers β_ξ such that $\beta_\xi<\beta$ for every $\xi<\alpha<\beta$ and $\beta=\bigcup_{\xi<\alpha}\beta_\xi$. Since β is the least upper bound of S and has no immediate predecessor,

(2) for any $\eta<\beta$ there is a ξ satisfying $\eta<\beta_\xi<\beta$.

Let

$$(3) \qquad \boldsymbol{D}_\xi=\{ x : x= \prod_{\eta<\beta_\xi} a_{\eta, f(\eta)} \text{ and } f\epsilon\gamma^{\beta_\xi}\} \text{ for } \xi<\alpha.$$

Moreover for each $\xi<\alpha$, let $\rho_\xi=\gamma^{(\beta_\xi)}$, and find a bijective function $\boldsymbol{F}_\xi$ (or one-to-one onto map) on γ^{β_ξ} onto ρ_ξ. For every $\xi<\alpha$ let b_ξ be a function ρ_ξ such that

$$b_\xi(\boldsymbol{F}_\xi)(f)= \prod_{\eta<\beta_\xi} a_{\eta, f(\eta)}$$

for each $f\epsilon\gamma^{\beta_\xi}$. Let $b_\xi(\eta)=b_{\xi,\eta}$ for $\xi<\alpha$ and $\eta<\rho_\xi$.

Let $\rho=\bigcup_{\xi<\alpha}\rho_\xi$ and if $\rho_\xi<\rho$ for some $\xi<\alpha$, we define $b_{\xi,\eta}=0$ for each $\rho_\xi\leqq\eta<\rho$. Then, by the (α, ∞)-distributivity of $\boldsymbol{A}$

$$(4) \qquad \prod_{\xi<\alpha}\sum \boldsymbol{D}_\xi= \prod_{\xi<\alpha}\sum_{f\epsilon\gamma^{\beta_\xi}}\{b_\xi(\boldsymbol{F}_\xi(f))\}= \prod_{\xi<\alpha}\sum_{\eta<\rho_\xi} b_{\xi,\eta}$$
$$= \prod_{\xi<\alpha}\sum_{\eta<\rho} b_{\xi,\eta}= \sum_{g\epsilon\rho^\alpha}\prod_{\xi<\alpha} b_{\xi, g(\xi)}$$

Since fot each $\xi<\alpha$ we have

$$\prod_{\eta<\beta_\xi}\sum_{\lambda<\gamma} a_{\eta,\lambda}\supset \prod_{\eta<\beta}\sum_{\lambda<\gamma} a_{\eta,\lambda},$$

by (1), (4) and the (β_ξ, ∞)-distributivity of $\boldsymbol{A}$,

$$0\neq \prod_{\eta<\beta}\sum_{\lambda<\gamma} a_{\eta,\lambda} \subset \prod_{\xi<\alpha}\prod_{\eta<\beta_\xi}\sum_{\lambda<\gamma} a_{\eta,\lambda}= \prod_{\xi<\alpha}\sum_{f\epsilon\gamma^{\beta_\xi}}\prod_{\eta<\beta_\xi} a_{\eta, f(\eta)}$$
$$= \prod_{\xi<\alpha}\sum_{f\epsilon\gamma^{\beta_\xi}} b_\xi(\boldsymbol{F}_\xi(f))=\prod_{\xi<\alpha}\sum \boldsymbol{D}_\xi,$$

so that by (4) there is a $g\epsilon\rho^\alpha$ such that

$$(5) \qquad \prod_{\xi<\alpha} b_{\xi, g(\xi)}\neq 0$$

If for some $\rho_\xi \leqq g(\xi)$ theu $b_{\xi, g(\xi)}=0$. Thus $g(\xi)<\rho_\xi$ for every $\xi<\alpha$. By the definition of $\boldsymbol{F}_\xi$ we have for each $\xi<\alpha$, $g(\xi)=\boldsymbol{F}_\xi(f)$ for some $f\epsilon\gamma^{\beta_\xi}$. Since g is at this time fixed, this f depend only upon ξ. Accordingly, we denote it f_ξ, that is, $g(\xi)=\boldsymbol{F}_\xi(f_\xi)$.

Now by (2), we can define an $h \in \gamma^{\beta}$ by the condition that for each $\eta<\beta$, $h(\eta)=f_{\xi}(\eta)$ where ξ is so chosen that β_{ξ} is the least member of $\{\beta_{\xi}: \eta<\beta_{\xi}<\beta, \xi<\alpha\}$. By the definition of b_{ξ} for each $\eta<\beta$, it follows that

$$a_{\gamma, h(\eta)}=a_{\gamma, f_{\xi}(\eta)} \supset \prod_{\lambda<\beta_{\xi}} a_{\lambda, f_{\xi}(\lambda)}=b_{\xi}(F_{\xi}(f_{\xi}))=b_{\xi, F_{\xi} f(\xi)}$$

$$=b_{\xi, g(\xi)} \subset \prod_{\xi<\alpha} b_{\xi, g(\xi)}$$

thus by (5) we obtain

$$\prod_{\eta<\beta} a_{\gamma, h(\eta)} \supset \prod_{\xi<\alpha} b_{\xi, g(\xi)} \neq 0,$$

which means that A is (β, γ)-distributive. γ being an arbitrary cardinal number, A is (β, ∞)-distributive. The proof is complete.

The following two theorems and corollary are due to *R. S. Pierce* [1].

Theorem 3. *Let A be an α-complete, α-atmoic Boolean algebra. Then A has the following property:*

(P) *if $\{A_{\xi}: \xi<\nu\}$ is a family of coverings of A such that $\nu \leqq \alpha^{+}$ and ν is cardinal and if $b \neq 0$ in A, then there is a choice function φ on ν such that $\varphi(\xi) \in A_{\xi}$ with property that if $T \subseteqq W(\nu)$ and $\overline{\overline{T}}<\alpha^{+}$, Then*

$$b \cdot \prod_{\xi \in T} \varphi(\xi) \neq 0$$

Theorem 4. *Suppose that A is an α-complete Boolean algebra which satisfies the property (P) of Theorem 3. Then A is (α, ∞)-distributive.*

Proof. Let γ be an arbitrary cardinal number and let $\{a_{\xi, \eta}: \xi<\alpha, \eta<\gamma\}$ be a subset of A such that $\sum_{\eta<\gamma} a_{\xi, \eta}=1$ for every $\xi<\alpha$. Let $A_{\xi}=\{a_{\xi, \eta}: \eta<\gamma\}$. Then A_{ξ} becomes a covering of A. Since A satisfies the property (P), for any non-zero element a, there is a function $f \in \gamma^{\alpha}$ such that $a \cdot \prod_{\xi<\alpha} a_{\xi, f(\xi)} \neq 0$. This means that A is (α, γ)-distributive [See [4] 19.2 (d_2)]. γ being arbitrary, it follows that A is (α, ∞)-distributive.

Corollary. *Every α-complete, α-atomic Boolean algebra is (α, ∞)-distributive.*

If A is a Boolean algebra, then A^{β} will denote the minimal β-extension of A, i. e. A^{β} is an β-complete Boolean algebra, A is dense in A^{β} and β-generates A^{β}.

Theorem 5. *Suppose that β is a cardinal number and that A is a Boolean algebra which has the M_{α}-property for every cardinal $\alpha<\beta$. Let A^{β} be a minimal β-extension of A, then A^{β} is (α, ∞)-distributive for every cardinal $\alpha<\beta$.*

Proof. Since A is dense subalgebra of A^{β}, A^{β} is α-complete, α-atomic for every cardinal $\alpha<\beta$. By corollary, A^{β} is (α, ∞)-distributive for every cardinal $\alpha<\beta$.

Theorem 6. *Suppose that β is a singular cardinal number and that A is* *a Boolean algebra which has the M_α-property for every cardinal $\alpha<\beta$. Then $\boldsymbol{A}$ is (β, ∞)-distributive.*

Proof. Let $\boldsymbol{A}^\beta$ be a minimal β-extension of $\boldsymbol{A}$. Then, by Theorem 5, $\boldsymbol{A}^\beta$ is (α, ∞)-distributive for each cardinal $\alpha<\beta$. Since β is a singular cardinal, by Theoren 2, $\boldsymbol{A}^\beta$ is (β, ∞)-distributive. Moreover, $\boldsymbol{A}$ is a regular subalgebra of $\boldsymbol{A}^\beta$. Consequently, $\boldsymbol{A}$ is (β, ∞)-distributive.

5. Representability

Notice that if β is an infinite regular cardinal number and if $T \subseteqq W(\beta)$ and $\overline{\overline{T}}<\beta$, then there exists an ordinal number $\lambda<\beta$ such that $\tau<\lambda$ for every $\tau \in T$.

In fact, let us assume that there is no such an λ. Then there is at least one $\tau \in T$ for arbitrary $\lambda<\beta$ such that $\lambda \leqq \tau$. Since $\tau<\beta$ and every infinite cardinal number has no immediate predecessor, there exists an ordinal μ with $\tau<\mu<\beta$. By assumption, there is an ordinal $\nu \in T$ with $\mu \leqq \nu<\beta$. Thus we can find an ordinal number $\nu \in T$ for arbitrary $\lambda<\beta$ such $\lambda<\nu$. This means that $W(\beta)=\bigcup_{\xi \in T} W(\xi)$, what is the same, $\beta=\bigcup_{\xi \in T} \xi$. It is clear that $\beta>\xi$ for each $\xi \in T$. Therefore, it follows that $\beta>\overline{\overline{\xi}}$ for each $\xi \in T$. If a cardinal number λ has the property that $\lambda \geqq \overline{\overline{\xi}}$ for each $\xi \in T$, then $\lambda \geqq \xi$ for each $\xi \in T$. Since β is the least upper bound of $\{\xi : \xi \in T\}$, we have $\lambda \geqq \beta$, that is, $\beta=\bigcup_{\xi \in T} \overline{\overline{\xi}}$. This means that β is singular. This leads to contradiction.

Theorem 7. *Suppose that β is an infinte regular cardinal number and that $\boldsymbol{A}$ is a Boolean algebra which has the M_α-property for every cardinal $\alpha<\beta$. Let $\boldsymbol{A}^\beta$ be a minimal β-extension of $\boldsymbol{A}$, then $\boldsymbol{A}^\beta$ has the following property:*
(P') if $\{\boldsymbol{A}_\xi : \xi<\nu\}$ is a family of coverings of $\boldsymbol{A}^\beta$ such that a cardinal $\nu \leqq \beta$ and if $b \neq 0$ in $\boldsymbol{A}^\beta$, then there is a choice function φ on ν such that $\varphi(\xi) \in \boldsymbol{A}_\xi$ with the property that if $T \subseteqq W(\nu)$ and $\overline{\overline{T}}<\beta$, Then $b \cdot \prod_{\xi \in T} \varphi(\xi) \neq 0$.

Proof. We can assume that $\nu=\beta$. By transfinite inductive definition we can define functions $f: \beta \rightarrow \boldsymbol{A}$ and φ on β with $\varphi(\xi) \in \boldsymbol{A}_\xi$ having the following properties

(i) $\xi<\eta<\beta$ implies $0 \neq f(\eta) \subset f(\xi) \subset b$.

(ii) $f(\xi) \subset \varphi(\xi)$

These are constructed in the following way. Assume that $f(\xi)$ has been defined for every $\xi<\tau$, where $\tau<\beta$. By the M_α-property, $c=\prod_{\xi<\tau} f(\xi) \neq 0$. We assume that $c=1$, when $\tau=0$. Then we can find a $\varphi(0) \in \boldsymbol{A}_0$ such that $\varphi(0) \cdot b \neq 0$. Such an element $\varphi(0)$ exists, because $b=b \cdot 1=b \cdot \Sigma\, \boldsymbol{A}_0=\Sigma\,\{b \cdot a : a \in \boldsymbol{A}_0\}$. Since $\boldsymbol{A}$ is a dense subalgebra of $\boldsymbol{A}^\beta$, we can choose arbitrarily a $f(0) \in \boldsymbol{A}$ satisfying $0 \neq f(0) \subset \varphi(0) \cdot b$. Suppose that $\varphi(\xi), f(\xi)$ have been defined for every $\xi<\tau$, where $0<\tau<\beta$. Then we have $c \subset b$. Choose $\varphi(\tau) \in \boldsymbol{A}_\tau$ so that $\varphi(\tau) \cdot c \neq 0$. As before, some element of $\boldsymbol{A}_\tau$ will satisfy this requirement. Using the fact that $\boldsymbol{A}$ is dense, it is possible to find $f(\tau) \in \boldsymbol{A}_\tau$ such that $0 \neq f(\tau) \subset \varphi(\tau) \cdot c$. From this construction, it is evident that $f(\tau) \subset \varphi(\tau)$. If $\xi < \tau$, then we obtain $c = \prod_{\rho<\tau} f(\rho) \subset f(\xi)$. Accoridingly, it follows that $f(\xi) \supset c \supset c \cdot \varphi(\tau) \supset f(\tau)$, that is, $f(\tau) \subset f(\xi)$. Thus, the conditions (i) and (ii) are fulfilled.

Now if $T \subseteqq W(\beta)$ and $\overline{\overline{T}} < \beta$, then since β is infinite regular cardinal number, there exists $\lambda < \beta$ such that $\xi < \lambda$ for every $\xi \in T$.

$$b \cdot \prod_{\xi \in T} \varphi(\xi) \supset b \cdot \prod_{\xi<\lambda} \varphi(\xi) \supset b \cdot \prod_{\xi<\lambda} f(\xi)$$
$$\supset \prod_{\xi<\lambda} f(\xi) \supset f(\lambda) \neq 0,$$

what is the same, $b \cdot \prod_{\xi \in T} \varphi(\xi) \neq 0$.

A Boolean algebra $\boldsymbol{A}$ is said to have the property (P_β) where β is an infinite cardinal, if the following is satisfied: if $\{a_{\xi,\eta} : \xi, \eta < \beta\}$ is a subset of $\boldsymbol{A}$ such that all the joins $\sum_{\eta<\beta} a_{\xi,\eta}$ for $\xi < \beta$ exist and their meet $\prod_{\xi<\beta} \sum_{\eta<\beta} a_{\xi,\eta}$ exists and is not $\boldsymbol{0}$, then there is a function $f \in \beta^\beta$ such that $\prod_{\xi<\nu} a_{\xi, f(\xi)}$ is false for every $\nu < \beta$; i.e. either $\prod_{\xi<\nu} a_{\xi, f(\xi)}$ does not exist or else is not zero.

Theorem 8. *If an β-complete Boolean algebra $\boldsymbol{A}$ satisfies the following property:*

if $\{\boldsymbol{A}_\xi : \xi < \beta\}$ is a family of coverings of $\boldsymbol{A}$ such that if $b \neq 0$ in $\boldsymbol{A}$, then there is a choice function φ on β such $\varphi(\xi) \in \boldsymbol{A}_\xi$ with the property that if $T \subseteqq W(\beta)$ and $\overline{\overline{T}} < \beta$, then $b \cdot \prod \varphi(\xi) \neq 0$, then $\boldsymbol{A}$ has the property (P_β).

Proof. Suppose that $\{a_{\xi,\eta} : \xi, \eta < \beta\}$ is any subset of $\boldsymbol{A}$ such that $\prod_{\xi<\beta} \sum_{\eta<\beta} a_{\xi,\eta} = a \neq 0$. Let $a_{\xi,\beta} = \bar{a}$ for every $\xi < \beta$ and let $\boldsymbol{A}_\xi = \{a_{\xi,\eta} : \eta < \beta+1\}$. In this way every $\boldsymbol{A}_\xi$ becomes a covering of $\boldsymbol{A}$. Hence, by the property of $\boldsymbol{A}$, for this $a \neq 0$ in $\boldsymbol{A}$, there is a function $f \in (\beta+1)^\beta$ such that $a \cdot \prod_{\xi<\nu} a_{\xi, f(\xi)} \neq 0$ for every $\nu < \beta$. It is clear that $f(\xi) \neq \beta$ for every $\xi < \beta$. Consequently, there exists a function $f \in \beta^\beta$ such that $\prod_{\xi<\nu} a_{\xi, f(\xi)} \neq 0$ for every $\nu < \beta$. Hence it follows that $\boldsymbol{A}$ has the property $(P_\beta.)$

A Boolean algebra is said to be β-representable provided it is isomorphic to an β-regular subalgebra of quotient algebra $\boldsymbol{F}/\boldsymbol{I}$ where $\boldsymbol{F}$ is an β-field of sets and $\boldsymbol{I}$ is an β-ideal of $\boldsymbol{F}$. Thus an β-complete Boolean algebra is β-representable if and only if it is is isomorphic to a quotient algebra $\boldsymbol{F}/\boldsymbol{I}$ where $\boldsymbol{F}$ is an β-field of sets, and $\boldsymbol{I}$ is an β-ideal of $\boldsymbol{F}$.

Actually, in order to demonstrate that a Boolean algebra $\boldsymbol{A}$ is β-representable, it is sufficient to show that whenever $\{a_{\xi,\eta} : \xi, \eta < \beta\}$ is any subset of $\boldsymbol{A}$ such that all the joins $\sum_{\eta<\beta} a_{\xi,\eta}$ exist for $\xi<\beta$ and their meet $\prod_{\xi<\beta}\sum_{\eta<\beta} a_{\xi,\eta}$ exists and is not 0, then there is an $f\in\beta^\beta$ such that $\prod_{\xi\in T} a_{\xi,f(\xi)}\neq 0$ for every finite subset T of $W(\beta)$.

The following theorem was proved by *E. C. Smith* [5].

Theorem 9. *Every β-complete Boolean algebra which has the property (P_β) is β-representable.*

Theorem 10. *Suppose that β is an infinite regular cardinal number and that* $\boldsymbol{A}$ *is a Boolean algebra which has the M_α-property for every cardinal $\alpha<\beta$. Then* $\boldsymbol{A}$ *is β-representable.*

Proof. Let $\boldsymbol{A}^\beta$ be a minimal β-extension of $\boldsymbol{A}$, then by Theorem 7, $\boldsymbol{A}^\beta$ has the property (P'). Therefore, by Theorem 8, $\boldsymbol{A}^\beta$ has property (P_β). Accordingly, by Theorem 9, $\boldsymbol{A}^\beta$ is β representable. $\boldsymbol{A}$ is the regular subalgebra of $\boldsymbol{A}^\beta$, because $\boldsymbol{A}$ is the dense subalgebra of $\boldsymbol{A}^\beta$. Thus $\boldsymbol{A}$ is β-representable.

Theorem 11. *Suppose that β is an arbitrary infinite cardinal number and that* $\boldsymbol{A}$ *is a Boolean algebra which has the M_α-property for every cardinal $\alpha<\beta$. Then* $\boldsymbol{A}$ *is β-representable.*

Proof. If β is a regular cardinal number, then, by Theorem 10, it follows immediately that $\boldsymbol{A}$ is β-representable.

Next, if β is a singular cardinal number, then, by Theorem 6, it follows that $\boldsymbol{A}$ is (β, ∞)-distributive. Hence, $\boldsymbol{A}$ is (β, β)-distributive. Since every (β, β)-distributive Boolean algebra is β-representable, $\boldsymbol{A}$ is β-representable. The proof is complete.

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