SOME RESULTS FOR QUEUEING SYSTEM $M/M/s\,(N+s)$

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§ 1. Introduction

In this paper we shall consider some equilibrium behaviors of the queueing system $M/M/s\,(N+s)$ in which (i) there are s servers, (ii) the customers arrive at random at a mean arrival rate λ and are served in the order of arrival, (iii) in each server the service time has the same exponential distribution with a mean service rate μ, (iv) the size of waiting room is N and when a customer arrive to find the waiting room full he depart never to return. It has been proved by Finch (1959) that in the queueing system $M/G/1\,(N+1)$ the successive inter-departure intervals are not independent in the limit even when the service time is exponential. Using a procedure similar to Finch's, it will be shown that in the system $M/M/s\,(N+s)$ the successive inter-incoming intervals are also not independent. Next we shall treat the equilibrium distribution of the number of customers which income the system in a given time interval $(0, t)$. The Laplace transform of the generating function of the obove distribution will be obtained by the method given by Homma (1957).

On the output process and the overflow process, the similar results will be given. From these results the equilibrium distribution of the number of service completions in an arbitrary time interval will be noted to be the same as the incoming distribution. It seems to be difficult to obtain the explicite formulas of the above distributions, but the above results will be used to make numerical tables. The tables are not completed yet and we hope they will soon be done.

§ 2. The dependence of successive inter-incoming intervals.

Let n_r be the number of customers present in the system $M/M/s\,(N+s)$ just after the r-th incoming occures and let

$$t_{ij}=P(n_{r+1}=j \mid n_r=i), \quad P^{(r)}(j)=P(n_r=j). \tag{2.1}$$

Now tn the above system, we shall consider the limiting distribution

$$p_j=\lim_{r\to\infty} P^{(r)}(j), \qquad (j=1, 2, \cdots, N+s). \tag{2.2}$$

Then it is seen that the transition matrix $\boldsymbol{T}=\| t_{ij} \|$ is

$$
(2.3)\qquad
\begin{array}{r|llllll}
\nearrow & 1 & 2 & \cdots & s+1 & \cdots & N+s \\
\hline
1 & t_{11} & t_{12} & & & & \\
2 & t_{21} & t_{22} & t_{23} & & & \\
\vdots & & & \ddots & & 0 & \\
s & t_{s1} & \cdots & \cdots & t_{s,s+1} & & \\
\vdots & & & & & \ddots & \\
N+s-1 & t_{N+s-1,1} & \cdots & \cdots & \cdots & \cdots & t_{N+s-1,N+s} \\
N+s & t_{N+s,1} & \cdots & \cdots & \cdots & \cdots & t_{N+s,N+s}
\end{array}
$$

where $\qquad t_{i,j}=0, \qquad (1 \leqq i \leqq N+s-2,\ i+2 \leqq j).$

By some calculations, we have

$$
(2.4)\qquad
\begin{aligned}
& t_{11}=\frac{1}{1+\rho}, \qquad t_{12}=\frac{\rho}{1+\rho}, \\
& t_{ij}=\begin{cases} \dfrac{i}{i+\rho}\, t_{i-1,j} & (2 \leqq i \leqq s, \qquad 1 \leqq j \leqq i), \\[2ex] \dfrac{s}{s+\rho}\, t_{i-1,j}, & (s+1 \leqq i \leqq N+s-1, \qquad 1 \leqq j \leqq i), \end{cases} \\
& t_{i,i+1}=\begin{cases} \dfrac{\rho}{i+\rho}, & (1 \leqq i \leqq s), \\[2ex] \dfrac{\rho}{s+\rho}, & (s+1 \leqq i \leqq N+s-1), \end{cases} \\
& t_{N+s,j}=t_{N+s-1,j} \qquad (1 \leqq j \leqq N+s),
\end{aligned}
$$

where $\qquad \rho=\lambda/\mu.$

From (2.4), all elements of the transition matrix $\boldsymbol{T}$ are determined.

The existence of the limiting distribution $\{p_j\}$ follows from the fact that the process is a finite, irreducible, aperiodic Markov chain.

The equations which determine the limiting distribution are

$$
(2.5)\qquad
\begin{aligned}
p_1 &= t_{11}\, p_1 + t_{21}\, p_2 + \cdots\cdots + t_{N+s,1}\, p_{N+s}, \\
p_2 &= t_{12}\, p_1 + t_{22}\, p_2 + \cdots\cdots + t_{N+s,2}\, p_{N+s}, \\
&\ \vdots \\
p_{N+s-1} &= t_{N+s-2,N+s-1}\, p_{N+s-2} + t_{N+s-1,N+s-1}\, p_{N+s-1} + t_{N+s,N+s-1}\, p_{N+s}, \\
p_{N+s} &= t_{N+s-1,N+s}\, p_{N+s-1} + t_{N+s,N+s}\, p_{N+s}.
\end{aligned}
$$

From (2.4) and (2.5), we have

(2.6)
$$p_i=\begin{cases}\dfrac{\rho}{i-1}p_{i-1}, & (1\leqq i\leqq s),\\[2mm] \dfrac{\rho}{s}p_{i-1}, & (s\leqq i\leqq N+s).\end{cases}$$

Using (2.6) and $\sum_{i=1}^{N+s}p_i=1$, we have

(2.7)
$$p_i=\begin{cases}\dfrac{\rho^{i-1}}{(i-1)!}p_1, & (1\leqq i\leqq s),\\[2mm] \dfrac{\rho^s}{s!}\left(\dfrac{\rho}{s}\right)^{i-s-1}p_1, & (s+1\leqq i\leqq N+s),\end{cases}$$

where

$$p_1=1\Big/\left\{\sum_{i=1}^{s}\frac{\rho^{i-1}}{(i-1)!}+\sum_{i=s+1}^{N+s}\frac{\rho^s}{s!}\left(\frac{\rho}{s}\right)^{i-s-1}\right\}.$$

Next we shall prove that two successive inter-incoming intervals are not independent in the limit. Since the discussion follows the line indicated by Finch, the proof will be simplified.

Now we consider the system $M/M/s\,(N+s)$. Let l_r be the time interval between r-th and $(r+1)$-th incoming, and let $H_{r+1}(t,j)$ be the joint frequency function for n_{r+1} and l_r, that is

(2.8) $$H_{r+1}(t,j)\,dt=P(n_{r+1}=j,\ t<l_r<t+dt),\qquad (1\leqq j\leqq N+s).$$

The existence of the limiting distribution $p_j=\lim\limits_{r\to\infty}P(n_r=j)$ implies the existence of the limiting distribution

(2.9)
$$H(t,N+s)=\lim_{r\to\infty}H_r(t,N+s),$$
$$H(t)=\lim_{r\to\infty}\Big(\sum_{j=1}^{N+s}H_r(t,j)\Big),$$

and in the case $N>0$ these are given by

(2.10)
$$\begin{aligned}H(t,N+s)&=p_{N+s-1}e^{-s\mu t}\lambda e^{-\lambda t}+p_{N+s}\int_0^t s\mu e^{-s\mu x}e^{-s\mu(t-x)}e^{-\lambda(t-x)}\lambda dx\\&=\lambda p_{N+s-1}e^{-s\mu t-\lambda t}+p_{N+s}s\mu\,(e^{-s\mu t}-e^{-s\mu t-\lambda t})\\&=p_{N+s}s\mu e^{-s\mu t}\end{aligned}$$

$$H(t)=(1-p_{N+s})\lambda e^{-\lambda t}+p_{N+s}\int_0^t s\mu e^{-s\mu x}e^{-\lambda(t-x)}\lambda dx$$

$$(2.11) \qquad = \begin{cases} (1-p_{N+s})\,\lambda e^{-\lambda t}+p_{N+s}\dfrac{s\mu\lambda}{\lambda-s\mu}\,(e^{-s\mu t}-e^{-\lambda t}), & (\lambda\neq s\mu), \\ (1-p_{N+s})\,\lambda p^{-\lambda t}+p_{N+s}\,\lambda^2\, te^{-\lambda t}, & (\lambda=s\mu). \end{cases}$$

l_r and n_r will be independent in the limit $r\to\infty$ if and only if

$$(2.12) \qquad H(t, j)=H(t)\,p_j, \qquad (j=1, 2, \cdots, N+s).$$

In order to prove that (2.12) is not the case for $j=1, 2, \cdots, N+s$, it is sufficient to prove that it is not the case for $j=N+s$. From (2.10) and (2.11) we can show the following result

$$(2.13) \qquad H(t, N+s) \neq H(t)\,p_{N+s}, \qquad (N>0).$$

This shows that for the queueing system $M/M/s\,(N+s)$, $N>0$, the system size found by a incoming customer and the duration of the interval since the previous incoming are not independent in the limit. Also from the above result we can easily prove that two successive incoming intervals are not independent in the limit.

Next we shall consider the above dependence in the case $N=0$. Since $P(n_r=1)=1$ in the case $s=1$, it will be evident that l_r and n_r are independent. In the case $N=0, s>1$ we have

$$(2.14) \qquad H(t, s)= \begin{cases} p_{s-1}\,\lambda e^{-[\lambda+(s-1)\mu]t}+p_s\dfrac{s\mu\lambda}{\mu-\lambda}\,\{e^{-[\lambda+(s-1)\mu]t}-e^{-s\mu t}\}, & (\mu\neq\lambda), \\ p_{s-1}\,\lambda e^{-s\lambda t}+p_s\, s\,\lambda^2\, t\, e^{-s\lambda t} \qquad , & (\mu=\lambda), \end{cases}$$

$$(2.15) \qquad H(t)= \begin{cases} (1-p_s)\,\lambda e^{-\lambda t}+p_s\dfrac{s\mu\lambda}{s\mu-\lambda}\,(e^{-\lambda t}-e^{-s\mu t}), & (\lambda\neq s\mu), \\ (1-p_s)\,\lambda e^{-\lambda t}+p_s\,\lambda^2\, t\, e^{-\lambda t} \qquad , & (\lambda=s\mu). \end{cases}$$

From these results, we can also show that

$$(2.16) \qquad H(t, s)\neq H(t)\,p_s$$

Thus it follows that for the queueing system $M/M/s\,(s)$, $(s>1)$, two successive inter-incoming intervals are not independent in the limit.

§ 3. The distribution of the number of incoming customers.

Let $I_{m,k}(t)$ be the conditional probability that in the queueing system $M/M/s\,(N+s)$ the number of incoming customers in an arbitrary time interval $(0, t)$ is k when the system-size at $t=0$ is m. Also let $p_{ij}(t)$ denote the probability that in $(0, t)$ (i) i customers arrive, (ii) j customers are denied without remitted and (iii) there are no survices completed. Then we have the following results.

In the case $0 < m < s, \quad N+s-m \leqq k,$

$$
\begin{aligned}
I_{m,k}(t) = & \sum_{j=0}^{\infty} \int_0^t p_{N+s-m+j,\, j}(x)\, s\mu\, I_{N+s-1,\, k-(N+s-m)}(t-x)\, dx \\
& + \sum_{j=1}^{N} \int_0^t p_{N+s-m-j,\, 0}(x)\, s\mu\, I_{N+s-j-1,\, k-(N+s-m-j)}(t-x)\, dx \\
& + \sum_{j=N+1}^{N+s-m} \int_0^t p_{N+s-m-j,\, 0}(x)\, (N+s-j)\, \mu I_{N+s-j-1,\, k-(N+s-m-j)}(t-x)\, dx \qquad (3.1) \\
& + \begin{cases} \displaystyle\sum_{j=0}^{\infty} p_{k+j,\, j}(t) & (k=N+s-m), \\ 0 & (k>N+s-m). \end{cases}
\end{aligned}
$$

In the case $0 < m < s, \quad s-m \leqq k < N+s-m,$

$$
\begin{aligned}
I_{m,k}(t) = & \sum_{j=N+s-m-k}^{N} \int_0^t p_{N+s-m-j,\, 0}(x)\, s\mu\, I_{N+s-j-1,\, k-(N+s-m-j)}(t-x)\, dx. \\
& + \sum_{j=1}^{s-m} \int_0^t p_{s-m-j,\, 0}(x)\, (s-j)\, \mu\, I_{s-j-1,\, k-(s-m-j)}(t-x)\, dx \qquad (3.2) \\
& + p_{k,0}(t)
\end{aligned}
$$

In the case $0 < m < s, \quad 0 < k < s-m,$

$$
I_{m,k}(t) = \sum_{j=s-m-k}^{s-m} \int_0^t p_{s-m-j,\, 0}(x)\, (s-j)\, \mu\, I_{s-j-1,\, k-(s-m-j)}(t-x)\, dx + p_{k,0}(t), \qquad (3.3)
$$

In the case $0 < m < s, \quad k=0,$

$$
I_{m,0}(t) = e^{-\lambda t}. \qquad (3.4)
$$

If we denote the generating function of $\{I_{m,k}(t)\}$ by $\varphi_m(t, z)$, from (3.1) ~ (3.4) and some calculations, we have

$$
\begin{aligned}
\varphi_m(t, z) = & \sum_{k=0}^{\infty} I_{m,k}(t)\, z^k \qquad (3.5) \\
= & \sum_{j=0}^{\infty} z^{N+s-m} \int_0^t p_{N+s-m+j,\, j}(x)\, s\mu\varphi_{N+s-1}(t-x, z)\, dx \\
& + \sum_{j=1}^{N} z^{N+s-m-j} \int_0^t p_{N+s-m-j,\, 0}(x)\, s\mu\, \varphi_{N+s-j-1}(t-x, z)\, dx \\
& + \sum_{j=1}^{s-m} \int_0^t p_{s-m-j,\, 0}(x)\, (s-j)\, \mu\varphi_{s-j-1}(t-x, z)\, dx
\end{aligned}
$$

$$+\sum_{k=1}^{N+s-m-1} p_{k,0}(t)\, z^k + z^{N+s-m} \sum_{j=0}^{\infty} p_{N+s-m+j,\, j}(t)$$

$$+ e^{-(\lambda+m\mu)t}$$

In order to obtain the Laplace transform of $\varphi_m(t, z)$, the following three Laplace transforms $A_m(\theta)$, $B_{m,k}(\theta)$ and $C_{m,k}(\theta)$ will be given by the method seen in Homma's paper (1957).

$$A_m(\theta) \equiv \int_0^\infty \Big(\sum_{j=0}^{\infty} p_{N+s-m+j,\, j}(t) \Big) e^{-\theta\mu t}\, dt$$

(3.6)
$$= \int_0^\infty \Big[\sum_{j=0}^{\infty} e^{-s\mu t-\lambda t} \rho^{s-m} \lambda \int_0^t \frac{(e^{\mu x}-1)^{s-m}}{(s-m)!} \cdot \frac{\{\lambda(t-x)\}^{N+j-1}}{(N+j-1)!} dx \Big] e^{-\theta\mu t}\, dt$$

$$= \rho^{s-m} \Big(\frac{\rho}{\theta+s+\rho} \Big)^N \frac{1}{(\theta+m+\rho)(\theta+m+1+\rho)\cdots(\theta+s-1+\rho)} \cdot \frac{1}{\mu(\theta+s)},$$

$$B_{m,k}(\theta) \equiv \int_0^\infty p_{k,0}(t)\, e^{-\theta\mu t} dt = \int_0^\infty \Big\{ e^{-(m+k)\mu t-\lambda t} \rho^k \frac{(e^{\mu t}-1)^k}{k!} \Big\} e^{-\theta\mu t} dt$$

(3.7)
$$= \frac{\rho^k}{(\theta+m+\rho)(\theta+m+1+\rho)\cdots(\theta+m+k+\rho)} \cdot \frac{1}{\mu}, \qquad (0 \leqq k \leqq s-m-1),$$

$$C_{m,k}(\theta) \equiv \int_0^\infty p_{k,0}(t)\, e^{-\theta\mu t} = \int_0^\infty \Big\{ e^{-s\mu t-\lambda t} \rho^{s-m} \lambda \int_0^t \frac{(e^{\mu x}-1)^{s-m} \{\lambda(t-x)\}^{k-s+m-1}}{(s-m)!\,(k-s+m-1)!} dx \Big\} e^{-\theta\mu t} dt$$

(3.8)
$$= \rho^{s-m} \Big(\frac{\rho}{\theta+s+\rho} \Big)^{k-s+m} \frac{1}{(\theta+m+\rho)(\theta+m+1+\rho)\cdots(\theta+s+\rho)} \cdot \frac{1}{\mu}, \quad (s-m \leqq k \leqq N+s-m-1),$$

where $R(\theta) > 0$.

Using (3.6), (3.7), (3.8) and Laplace transforms of the both sides of (3.5), we have

$$\Phi_m(\theta, z) \equiv \int_0^\infty (\varphi_m(t, z))\, e^{-\theta\mu t} dt$$

$$= z^{N+s-m} A_m(\theta)\, \Phi_{N+s-1}(\theta, z)\, s\mu + \sum_{k=s-m}^{N+s-m-1} z^k C_{m,k}(\theta)\, \Phi_{k+m-1}(\theta, z)$$

(3.9)
$$+ \sum_{k=0}^{s-m-1} z^k (m+k)\, \mu\, B_{m,k}(\theta)\, \Phi_{m+k-1}(\theta, z) + \sum_{k=1}^{s-m-1} z^k B_{m,k}(\theta) + \sum_{k=s-m}^{N+s-m-1} z^k C_{m,k}(\theta)$$

$$+ z^{N+s-m} A_m(\theta) + \frac{1}{(\theta+m+\rho)\,\mu},$$

where let $\sum_{k=1}^{s-m-1}{}' z^k B_{m,k}(\theta)=0$ when $m=s-1$.

Next we consider the case $s\leqq m<N+s$.

In the case $s\leqq m<N+s$, $N+s-m\leqq k$,

$$
\begin{aligned}
I_{m,k}(t)=&\int_0^t \sum_{j=0}^{N+s-m} e^{-s\mu x}s\mu \frac{e^{-\lambda x}(\lambda x)^j}{j\,!} I_{m+j-1,k-j}(t-x)\,dx \\
&+\int_0^t \sum_{j=N+s-m+1}^{\infty} e^{-s\mu x}s\mu \frac{e^{-\lambda x}(\lambda x)^j}{j\,!} I_{N+s-1,k-(N+s-m)}(t-x)\,dx \\
&+\begin{cases} \displaystyle\sum_{j=N+s-m}^{\infty} e^{-s\mu t}\frac{e^{-\lambda t}(\lambda t)^j}{j\,!} & (k=N+s-m), \\ 0 & (k>N+s-m). \end{cases}
\end{aligned}
\tag{3.10}
$$

In the case $s\leqq m<N+s$, $0<k<N+s-m$,

$$
\begin{aligned}
I_{m,k}(t)=&\int_0^t \sum_{j=0}^{k} e^{-s\mu t}s\mu \frac{e^{-\lambda x}(\lambda x)^j}{j\,!} I_{m+j-1,k-j}(t-x)\,dx \\
&+\frac{e^{-s\mu t}e^{-\lambda t}(\lambda t)^k}{k\,!}.
\end{aligned}
\tag{3.11}
$$

In the case $s\leqq m<N+s$, $k=0$,

$$
I_{m,0}(t)=e^{-\lambda t}. \tag{3.12}
$$

From (3.10) $\sim$ (3.12) we can obtain the following result for the Laplace transform $\Phi_m(\theta,z)\,(s\leqq m<N+s)$ quite as in the case $0<m<s$.

$$
\begin{aligned}
\Phi_m(\theta,z)=&\sum_{j=1}^{N+s-m-1}\frac{s(\rho z)^j}{(\theta+s+\rho)^{j+1}}\Phi_{m+j-1}(\theta,z)+\frac{s}{\theta+s+\rho}\Phi_{m-1}(\theta,z) \\
&+\frac{s(\rho z)^{N+s-m}}{(\theta+s+\rho)^{N+s-m}}\cdot\frac{1}{\theta+s}\Phi_{N+s-1}(\theta,z) \\
&+\frac{\rho^{N+s-m}}{(\theta+s+\rho)^{N+s-m}}\cdot\frac{1}{(\theta+s)\,\mu}z^{N+s-m}+\frac{1}{(\theta+s+\rho)\,\mu} \\
&+\sum_{k=1}^{N+s-m-1}\frac{(\rho z)^k}{(\theta+s+\rho)^{k+1}\,\mu} \qquad (s\leqq m<N+s).
\end{aligned}
\tag{3.13}
$$

Lastly we consider the cases $m=0$ and $m=N+s$.

$$
I_{0,k}(t)=\begin{cases}\displaystyle\int_0^t e^{-\lambda x}\lambda\, I_{1,k-1}(t-x)\,dx & (k\geqq 1), \\ e^{-\lambda t} & (k=0), \end{cases}
\tag{3.14}
$$

$$I_{N+s,\,k}(t)=\begin{cases}\displaystyle\int_0^t e^{-s\mu x}s\mu\, I_{N+s-1,\,k}(t-x)\,dx & (k\geqq 1),\\ \displaystyle\int_0^t e^{-s\mu x}s\mu\, I_{N+s-1,\,0}(t-x)\,dx+e^{-s\mu t} & (k=0).\end{cases}\tag{3.15}$$

From these results we have

$$\Phi_0(\theta,z)=\frac{1}{(\theta+\rho)\mu}+\frac{\rho z}{\theta+\rho}\Phi_1(\theta,z),\tag{3.16}$$

$$\Phi_{N+s}(\theta,s)=\frac{s}{\theta+s}\Phi_{N+s-1}(\theta,z)+\frac{1}{(\theta+s)\mu}.\tag{3.17}$$

Using (3. 6) $\sim$ (3.8) we obtain the following relations

$$\begin{cases}A_m\ (\theta)=A_{m+1}(\theta)\dfrac{\rho}{\theta+m+\rho},\\ B_{m,\,k}(\theta)=B_{m+1,\,k-1}(\theta)\dfrac{\rho}{\theta+m+\rho},\\ C_{m,\,k}(\theta)=C_{m+1,\,k-1}(\theta)\dfrac{\rho}{\theta+m+\rho}.\end{cases}\tag{3.18}$$

Thus from (3. 18) we can arrange (3.9) as follows:

$$\begin{aligned}\Phi_m(\theta,z)&=\frac{\rho z}{\theta+m+\rho}\Phi_{m+1}(\theta,z)+\frac{m}{\theta+m+\rho}\Phi_{m-1}(\theta,z)\\ &\quad+\frac{1}{(\theta+m+\rho)\mu}\qquad (0<m<s-1)\end{aligned}\tag{3.19}$$

Using (3.13), (3.16), (3.17) and (3.19), we can obtain the following equations which determine $\Phi_m(\theta,z)$ $(0\leqq m\leqq N+s,\quad N\geqq 1)$.

$$\begin{cases}\Phi_0(\theta,z)=\dfrac{\rho z}{\theta+\rho}\Phi_1(\theta,z)+\dfrac{1}{(\theta+\rho)\mu},\\ \Phi_m(\theta,z)=\dfrac{\rho z}{\theta+m+\rho}\Phi_{m+1}(\theta,z)+\dfrac{m}{\theta+m+\rho}\Phi_{m-1}(\theta,z)+\dfrac{1}{(\theta+m+\rho)\mu}\quad(1\leqq m\leqq s-1),\\ \Phi_m(\theta,z)=\dfrac{\rho z}{\theta+s+\rho}\Phi_{m+1}(\theta,z)+\dfrac{s}{\theta+s+\rho}\Phi_{m-1}(\theta,z)+\dfrac{1}{(\theta+s+\rho)\mu}(s\leqq m\leqq N+s-1),\\ \Phi_{N+s}(\theta,z)=\dfrac{s}{\theta+s}\Phi_{N+s-1}(\theta,z)+\dfrac{1}{(\theta+s)\mu},\end{cases}\tag{320}$$

when $R(\theta)>0,\qquad |z|<1.$

In order to make insight of this solution, we examine the case $s=2, N=2$. Thus taking $s=2, N=2$ and putting $\Phi_m \equiv \Phi_m(\theta, z)$, we have the following solution

$$\Phi_m = \triangle_m / \triangle \qquad (0 \leqq m \leqq 4), \tag{3.21}$$

where

$$\triangle = \begin{vmatrix} \theta+\rho & -\rho z & 0 & 0 & 0 \\ -1 & \theta+1+\rho & -\rho z & 0 & 0 \\ 0 & -2 & \theta+2+\rho & -\rho z & 0 \\ 0 & 0 & -2 & \theta+2+\rho & -\rho z \\ 0 & 0 & 0 & -2 & \theta+2 \end{vmatrix}, \tag{3.22}$$

$\triangle_m$ is the determinant obtained by replacing each element in the $(m+1)$-th column of $\triangle$ by $1/\mu$ and it is evident that there is some $\delta, (0<\delta<1)$ and $\triangle \neq 0$ for $|z| < \delta$.

In the equilibrium state let p_m^* be the probability that the system-size at an epoch in the queueing system $M/M/s\,(N+s)$ is m. Then we have the following well known results

$$p_m^* = \begin{cases} \dfrac{\rho^m}{m\,!}\, p_0^* & (0 \leqq m \leqq s), \\[2ex] \dfrac{\rho^m}{s\,!\, s^{m-s}}\, p_0^* & (s+1 \leqq m \leqq N+s), \end{cases} \tag{3.23}$$

where
$$p_m^* = 1 \Big/ \left\{ \sum_{m=0}^{s} \frac{\rho^m}{m\,!} + \frac{s^s}{s\,!} \sum_{m=s+1}^{N+s} \left(\frac{\rho}{s}\right)^m \right\}.$$

Therefore in the above case the Laplace transform of the generating function of the distribution of incoming customers is

$$\sum_{m=0}^{4} p_m^*\, \Phi_m(\theta, z) = \frac{1}{\triangle} \sum_{m=0}^{4} p_m^*\, \triangle_m \tag{3.24}$$

Remarks. 1. In the case $N=0$, that is the system $M/M/s\,(s)$, the similar results will be obtained and we shall note that they are identical with those by putting $N=0$ and omiting the case $s \leqq m \leqq N+s-1$ in (3.20).

2. The existence of the solution of the integral equations (3.5) and its uniqueness will be shown by the line seen in Homma's paper (1957).

3. We shall formally give a method by which the mean of incoming customers in $(0, t)$ is obtained. Now we shall treat the case $s=2, N=2$. Differentiating both sides of (3.20) with respect to z and putting $z=1$, we have

$$
(3.25)\qquad \begin{cases}
\Phi_0'=\dfrac{\rho}{\theta+\rho}\cdot\dfrac{1}{\theta\mu}+\dfrac{\rho}{\theta+\rho}\,\Phi_1',\\[2mm]
\Phi_1'=\dfrac{\rho}{\theta+1+\rho}\cdot\dfrac{1}{\theta\mu}+\dfrac{\rho}{\theta+1+\rho}\,\Phi_2'+\dfrac{1}{\theta+1+\rho}\,\Phi_0',\\[2mm]
\Phi_2'=\dfrac{\rho}{\theta+2+\rho}\cdot\dfrac{1}{\theta\mu}+\dfrac{\rho}{\theta+2+\rho}\,\Phi_3'+\dfrac{2}{\theta+2+\rho}\,\Phi_1',\\[2mm]
\Phi_3'=\dfrac{\rho}{\theta+2+\rho}\cdot\dfrac{1}{\theta\mu}+\dfrac{\rho}{\theta+2+\rho}\,\Phi_4'+\dfrac{2}{\theta+2+\rho}\,\Phi_2',\\[2mm]
\Phi_4'=\dfrac{\rho}{\theta+2}\cdot\Phi_3',
\end{cases}
$$

where

$$
\Phi_m'\equiv\left[\frac{\partial\Phi_m(\theta,z)}{\partial z}\right]_{z=1}.
$$

Therefore we have

$$
(3.26)\qquad \Phi_m'=\triangle_m^*/\triangle^* \qquad (0\leqq m\leqq 4),
$$

where

$$
(3.27)\qquad \triangle^*=\begin{vmatrix}
\theta+\rho & -\rho & 0 & 0 & 0\\
-1 & \theta+1+\rho & -\rho & 0 & 0\\
0 & -2 & \theta+2+\rho & -\rho & 0\\
0 & 0 & -2 & \theta+2+\rho & -\rho\\
0 & 0 & 0 & -2 & \theta+2
\end{vmatrix},
$$

$\triangle_m^*$ is the determinant obtained by replacing the $(i+1)$-th column by a column by a column vector $(\rho/\theta\mu,\ \rho/\theta\mu,\cdots,\rho/\theta\mu, 0)$.

Then we have

$$
\triangle_0^*=\begin{vmatrix}
\rho/\theta\mu & -\rho & 0 & 0 & 0\\
\rho/\theta\mu & \theta+1+\rho & -\rho & 0 & 0\\
\rho/\theta\mu & -2 & \theta+2+\rho & -\rho & 0\\
\rho/\theta\mu & 0 & -2 & \theta+2+\rho & -\rho\\
0 & 0 & 0 & -2 & \theta+2
\end{vmatrix}
=\rho/\theta^2\mu\begin{vmatrix}
\theta & -\rho & 0 & 0 & 0\\
\theta & \theta+1+\rho & -\rho & 0 & 0\\
\theta & -2 & \theta+2+\rho & -\rho & 0\\
\theta & 0 & -2 & \theta+2+\rho & -\rho\\
0 & 0 & 0 & -2 & \theta+2
\end{vmatrix}
$$

$$
=\frac{\rho}{\theta^2\mu}\triangle^*-\frac{\rho}{\theta^2\mu}\begin{vmatrix}
0 & -\rho & 0 & 0 & 0\\
0 & \theta+1+\rho & -\rho & 0 & 0\\
0 & -2 & \theta+2+\rho & -\rho & 0\\
0 & 0 & -2 & \theta+2+\rho & -\rho\\
\theta & 0 & 0 & -2 & \theta+2
\end{vmatrix}.
$$

Thus it follows that

$$p_0^* \Phi_0' + p_1^* \Phi_1' + \cdots\cdots + p_4^* \Phi_4'$$

$$= \frac{\rho}{\theta^2\mu} + \frac{\rho}{\theta^2\mu\triangle^*} \begin{vmatrix} \theta+\rho & -\rho & 0 & 0 & 0 \\ -1 & \theta+1+\rho & -\rho & 0 & 0 \\ 0 & -2 & \theta+2+\rho & -\rho & 0 \\ 0 & 0 & -2 & \theta+2+\rho & -\rho \\ -p_0^*\theta & -p_1^*\theta & -p_2^*\theta & -p_3^*\theta & -p_4^*\theta \end{vmatrix}.$$

Multiplying the m-th row in the above determinant by p_{m-1}^*, $(1 \leqq m \leqq 4)$, adding them to the 5-th row and using the following results

$$\begin{cases} m p_m^* = \rho p_{m-1}^* & (m=1, 2), \\ s p_m^* = \rho p_{m-1}^* & (m=3, 4), \end{cases}$$

we have

$$\sum_{m=0}^{4} p_m^* \Phi_m' = \frac{\rho}{\theta^2\mu} - \frac{\rho}{\theta^2\mu} p_4^*.$$

Noting that

$$\int_0^\infty \lambda t\, e^{-\theta\mu t} dt = \frac{\rho}{\theta^2\mu},$$

it follows that in the equilibrium state the mean of incoming customers in an arbitrary time interval $(0, t)$ is

(3. 28) $$(1-p_4^*)\lambda t.$$

4. The above methods given in the special case $s=2, N=2$ will be similary used for the general case.

§ 4. Results for the output process and overflow process.

Let $O_{m,k}(t)$ be the conditional probability that in the system $M/M/s(N+s)$ k departures occure in an arbitrary time interval $(0, t)$ when the system-size is m at $t=0$. In the case $0 < m < s$,

$$O_{m,k}(t) = \sum_{j=0}^{\infty} \int_0^t p_{N+s-m+j,\, j}(x)\, s\mu\, O_{N+s-1,\, k-1}(t-x)\, dx$$

(4. 1) $$+ \sum_{j=1}^{N} \int_0^t p_{N+s-m-j,\, 0}(x)\, s\mu\, O_{N+s-j-1,\, k-1}(t-x)\, dx$$

$$+ \sum_{j=N+1}^{N+s-m} \int_0^t p_{N+s-m-j,\, 0}(x)\, (N+s-j)\,\mu\, O_{N+s-j-1,\, k-1}(t-x)\, dx, \qquad (k > 0),$$

(4.2) $$O_{m,0}(t)=\sum_{j=0}^{\infty} p_{N+s-m+j,\,j}(t)+\sum_{j=1}^{N} p_{N+s-m-j,\,0}(t)+\sum_{j=N+1}^{N+s-m} p_{N+s-m-j,\,0}(t).$$

In the case $s \leqq m \leqq N+s$,

(4.3) $$\begin{aligned} O_{m,k}(t)=&\int_0^t \sum_{j=0}^{N+s-m} \frac{e^{-s\mu x-\lambda x}(\lambda x)^j}{j\,!}\, s\mu\, O_{m+j-1,\,k-1}(t-x)\,dx \\ &+\int_0^t \sum_{j=N+s-m+1}^{\infty} \frac{e^{-s\mu x-\lambda x}(\lambda x)^j}{j\,!}\, s\mu\, O_{N+s-1,\,k-1}(t-x)\,dx \qquad (k>0), \end{aligned}$$

(4.4) $$O_{m,0}(t)=e^{-s\mu t}.$$

In the case $m=0$,

(4.5) $$O_{0,k}(t)=\int_0^t e^{-\lambda x}\lambda\, O_{1,k}(t-x)\,dx \qquad (k>0),$$

(4.6) $$O_{0,0}(t)=e^{-\lambda t}.$$

If we denote

(4.7) $$\Psi_m(\theta, z)=\int_0^\infty \Bigl(\sum_{k=0}^{\infty} O_{m,k}(t)\, z^k\Bigr) e^{-\theta\mu t}dt \qquad (R(\theta)>0, |z|<1),$$

we can obtain the following results by the argument similar to that in deriving (3.20).

(4.8) $$\begin{cases} \Psi_0\ (\theta, z)=\dfrac{\rho}{\theta+\rho}\Psi_1\ (\theta, z)+\dfrac{1}{(\theta+\rho)\,\mu}, \\ \Psi_m\ (\theta, z)=\dfrac{\rho}{\theta+m+\rho}\Psi_{m+1}\ (\theta, z)+\dfrac{mz}{\theta+m+\rho}\Psi_{m-1}(\theta, z)+\dfrac{1}{(\theta+m+\rho)\mu} & (0<m\leqq s), \\ \Psi_m\ (\theta, z)=\dfrac{\rho}{\theta+s+\rho}\Psi_{m+1}\ (\theta, z)+\dfrac{sz}{\theta+s+\rho}\Psi_{m-1}(\theta, z)+\dfrac{1}{(\theta+s+\rho)\,\mu} & (s+1\leqq m\leqq N+s-1), \\ \Psi_{N+s}(\theta, z)=\dfrac{sz}{\theta+s}\Psi_{N+s-1}(\theta, z)+\dfrac{1}{(\theta+s)\,\mu}. \end{cases}$$

From (4.8) the form of Laplace transform $\Psi_m(\theta, z)$ will be expressed by such determinants as (3.22). For example, in the syetem $M/M/2$ (4) we have

(4.9) $$\Psi_m(\theta, z)=\triangle'_m/\triangle' \qquad (0\leqq m\leqq 4),$$

where

(4.10) $$\triangle'=\begin{vmatrix} \theta+\rho & -\rho & 0 & 0 & 0 \\ -z & \theta+1+\rho & -\rho & 0 & 0 \\ 0 & -2z & \theta+2+\rho & -\rho & 0 \\ 0 & 0 & -2z & \theta+2+\rho & -\rho \\ 0 & 0 & 0 & -2z & \theta+2 \end{vmatrix},$$

and $\triangle'_m$ is the determinant obtained by replacing each element in the $(m+1)$-th column of $\triangle'$ by $1/\mu$.

Thus the Laplace transform of the equilibrium distribution of number of departures is given by

$$\sum_{m=0}^{4} p_m^* \Psi_m(\theta, z) = \frac{1}{\triangle'} \sum_{m=0}^{4} p_m^* \triangle'_m. \tag{4.11}$$

Remark. We shall note that (3.24) and (4.11) are identical.

By a mathematical induction we can prove that two determinants $\triangle$ and $\triangle'$ are identical for the general case.

Next by a mathematial induction used for the Seneral case we shall show that

$$\sum_{m=0}^{4} p_m^* \triangle_m = \sum_{m=0}^{4} p_m^* \triangle'_m. \tag{4.12}$$

Noting such relations as

$$\mu\triangle_0 = \begin{vmatrix} 1 & -\rho z & 0 & 0 & 0 \\ 1 & \theta+1+\rho & -\rho z & 0 & 0 \\ 1 & -2 & \theta+2+\rho & -\rho z & 0 \\ 1 & 0 & -2 & \theta+2+\rho & -\rho z \\ 0 & 0 & 0 & -2 & \theta+2 \end{vmatrix} + \begin{vmatrix} \theta+\rho & -\rho z & 0 & 0 & 0 \\ -1 & \theta+1+\rho & -\rho z & 0 & 0 \\ 0 & -2 & \theta+2+\rho & -\rho z & 0 \\ 0 & 0 & -2 & \theta+2+\rho & -\rho z \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix} \tag{4.13}$$

and using some assumptions for the mathematical induction, it will be sufficient to prove the following relation for the proof of (4.12).

$$\begin{aligned} &\begin{vmatrix} \theta+\rho & -\rho z & 0 & 0 & 0 \\ -1 & \theta+1+\rho & -\rho z & 0 & 0 \\ 0 & -2 & \theta+2+\rho & -\rho z & 0 \\ 0 & 0 & -2 & \theta+2+\rho & -\rho z \\ p_0^* & p_1^* & p_2^* & p_3^* & 0 \end{vmatrix} + \begin{vmatrix} \theta+\rho & -\rho z & 0 & 0 & 1 \\ -1 & \theta+1+\rho & -\rho z & 0 & 1 \\ 0 & -2 & \theta+2+\rho & -\rho z & 1 \\ 0 & 0 & -2 & \theta+2+\rho & 1 \\ 0 & 0 & 0 & -2p_4^* & 0 \end{vmatrix} \\ = &\begin{vmatrix} \theta+\rho & -\rho & 0 & 0 & 0 \\ -z & \theta+1+\rho & -\rho & 0 & 0 \\ 0 & -2z & \theta+2+\rho & -\rho & 0 \\ 0 & 0 & -2z & \theta+2+\rho & -\rho \\ p_0^* & p_1^* & p_2^* & p_3^* & 0 \end{vmatrix} + \begin{vmatrix} \theta+\rho & -\rho & 0 & 0 & 1 \\ -z & \theta+1+\rho & -\rho & 0 & 1 \\ 0 & -2z & \theta+2+\rho & -\rho & 1 \\ 0 & 0 & -2z & \theta+2+\rho & 1 \\ 0 & 0 & 0 & -2p_4^* & 0 \end{vmatrix} \end{aligned} \tag{4.14}$$

Using the following relations

$$p_0^* = k, \quad p_1^* = \rho k, \quad p_2^* = \frac{\rho^2}{2} k, \quad p_3^* = \frac{\rho^3}{4} k, \quad p_4^* = \frac{\rho^4}{8} k,$$

$$k=1/\left\{1+\rho+\frac{\rho^2}{2}+\frac{\rho^3}{4}+\frac{\rho^4}{8}\right\},$$

we have

1st determinant of the left side=2nd determinant of the right side,

1st determinant of the right side=2nd determinant of the left side.

The above method given for the proof of (4. 12) will be used for the general case. Thus in the system $M/M/s\ (N+s)$ the equilibrium distribution of number of departures and of incomings in $(0.\ t)$ are identical. This derives the well known result that in the equilibrium state the output process for $M/M/s\,(\infty)$ is again a Poisson process with the same parameter as the input process.

Next we shall treat the overflow process. Let $F_{m,k}(t)$ be the conditional probability that in the system $M/M/s\,(N+s)$, k customers overflow the system in $(0.\ t)$, when the system-size is m at $t=0$.

In the case $0<m<s$,

$$(4.15)\quad \begin{aligned} F_{m,k}(t)=&\sum_{j=0}^{k}\int_0^t s\mu\, p_{N+s-m+j,\,j}(x)\,F_{N+s-1,\,k-j}(t-x)\,dx \\ &+\sum_{j=1}^{N}\int_0^t p_{N+s-m-j,\,0}(x)\,s\mu\,F_{N+s-j-1,\,k}(t-x)\,dx \\ &+\sum_{j=N+1}^{N+s-m}\int_0^t p_{N+s-m-j,\,0}(x)\,(N+s-j)\,\mu\,F_{N+s-j-1,\,k}(t-x)\,dx \\ &+p_{N+s-m+k,\,k}(t) \qquad (k\geqq 1), \end{aligned}$$

$$(4.16)\quad \begin{aligned} F_{m,0}(t)=&\sum_{j=0}^{N}\int_0^t p_{N+s-m-j,\,0}(x)\,s\mu\,F_{N+s-j-1,\,0}(t-x)\,dx \\ &+\sum_{j=N+1}^{N+s-m}\int_0^t p_{N+s-m-j,\,0}(x)\,(N+s-j)\,\mu\,F_{N+s-j-1,\,0}(t-x)\,dx \\ &+\sum_{j=0}^{N+s-m} p_{N+s-m-j,\,0}(t). \end{aligned}$$

In the case $s\leqq m\leqq N+s-1$,

$$(4.17)\quad \begin{aligned} F_{m,k}(t)=&\sum_{j=0}^{k}\int_0^t e^{-s\mu x}s\mu\,\frac{e^{-\lambda x}(\lambda x)^{N+s-m+j}}{(N+s-m+j)\,!}\,F_{N+s-1,\,k-j}(t-x)\,dx \\ &+\sum_{j=1}^{N+s-m}\int_0^t e^{-s\mu x}s\mu\,\frac{e^{-\lambda x}(\lambda x)^{N+s-m-j}}{(N+s-m-j)\,!}\,F_{N+s-j-1,\,k}(t-x)\,dx \end{aligned}$$

$$+e^{-s\mu t}\frac{e^{-\lambda t}(\lambda t)^{N+s-m+k}}{(N+s-m+k)!} \qquad (k\geqq 1).$$

$$F_{m,0}(t)=\sum_{j=0}^{N+s-m}\int_0^t e^{-s\mu x}s\mu\frac{e^{-\lambda x}(\lambda x)^{N+s-m-j}}{(N+s-m-j)!}F_{N+s-j-1,0}(t-x)\,dx$$

$$+\sum_{j=0}^{N+s-m}e^{-s\mu t}\frac{e^{-\lambda t}(\lambda t)^{N+s-m-j}}{(N+s-m-j)!}. \tag{4.18}$$

In the case $m=0$,

$$F_{0,k}(t)=\sum_{j=0}^{k}\int_0^t s\mu\, p_{N+s+j,j}(x)F_{N+s-1,k-j}(t-x)\,dx$$

$$+\sum_{j=1}^{N}\int_0^t s\mu\, p_{N+s-j,0}(x)F_{N+s-j-1,k}(t-x)\,dx \tag{4.19}$$

$$+\sum_{j=N+1}^{N+s-1}\int_0^t p_{N+s-j,0}(x)(N+s-j)\mu F_{N+s-j-1,k}(t-x)\,dx+p_{N+s+k,k}(t), \qquad (k\geqq 1),$$

$$F_{0,0}(t)=\sum_{j=0}^{N}\int_0^t p_{N+s-j,0}(x)s\mu F_{N+s-j-1,0}(t-x)\,dx$$

$$+\sum_{j=N+1}^{N+s-m-1}\int_0^t p_{N+s-j,0}(x)(N+s-j)\mu F_{N+s-j-1,0}(t-x)\,dx \tag{4.20}$$

$$+\sum_{j=0}^{N+s}p_{N+s-j,0}(t).$$

In the case $m=N+s$,

$$F_{N+s,k}(t)=\sum_{j=0}^{k}\int_0^t e^{-s\mu x}s\mu\frac{e^{-\lambda x}(\lambda x)^j}{j!}F_{N+s-1,k-j}(t-x)\,dx+e^{-s\mu t}\frac{e^{-\lambda t}(\lambda t)^k}{k!}. \tag{4.21}$$

Then the Laplace transform

$$\chi_m(\theta,z)=\int_0^\infty\Big(\sum_{k=0}^{\infty}F_{m,k}(t)z^k\Big)e^{-\theta\mu t}dt$$

will be given similarly.

The equations which determine the above Laplace transforms are

$$\begin{cases}\chi_0(\theta,z)=\dfrac{\rho}{\theta+\rho}\chi_1(\theta,z)+\dfrac{1}{(\theta+\rho)\mu}, \\[2ex] \chi_m(\theta,z)=\dfrac{\rho}{\theta+m+\rho}\chi_{m+1}(\theta,z)+\dfrac{m}{\theta+m+\rho}\chi_{m-1}(\theta,z)+\dfrac{1}{(\theta+m+\rho)\mu}, \quad (1\leqq m\leqq s), \\[2ex] \chi_m(\theta,z)=\dfrac{\rho}{\theta+s+\rho}\chi_{m+1}(\theta,z)+\dfrac{s}{\theta+s+\rho}\chi_{m-1}(\theta,z)+\dfrac{1}{(\theta+s+\rho)\mu} \quad (s+1\leqq m\leqq N+s-1), \\[2ex] \chi_{N+s}(\theta,z)=\dfrac{s}{\theta+s+\rho(1-z)}\chi_{N+s-1}(\theta,z)+\dfrac{1}{\{\theta+s+\rho(1-z)\}\mu}.\end{cases} \tag{4.22}$$

From the above relations we shall show the form of $\chi_m(\theta, z)$ for the system $M/M/2$ (4),

$$\chi_m(\theta, z) = \triangle''_m / \triangle'', \tag{4.23}$$

where

$$\triangle'' = \begin{vmatrix} \theta+\rho & -\rho & 0 & 0 & 0 \\ -1 & \theta+1+\rho & -\rho & 0 & 0 \\ 0 & -2 & \theta+2+\rho & -\rho & 0 \\ 0 & 0 & -2 & \theta+2+\rho & -\rho \\ 0 & 0 & 0 & -2 & \theta+2+\rho(1-z) \end{vmatrix}, \tag{4.24}$$

and $\triangle''_n$ is the determinant obtained by replacing each element in the $(m+1)$-th column of $\triangle''$ by $1/\mu$.

Using the method by which (3.28) has been derived from (3.26), it follows that in the equilibrium state the mean of overflowing customers in $(0, t)$ is $p^*_{N+s}\lambda t$. But, considering that the mean of arriving customers in $(0, t)$ is λt, the above result will be easily shown from (3.28).

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(Received September 10, 1966)