NON-CONNECTION METHOD FOR LINEAR CONNECTIONS IN THE LARGE.

By

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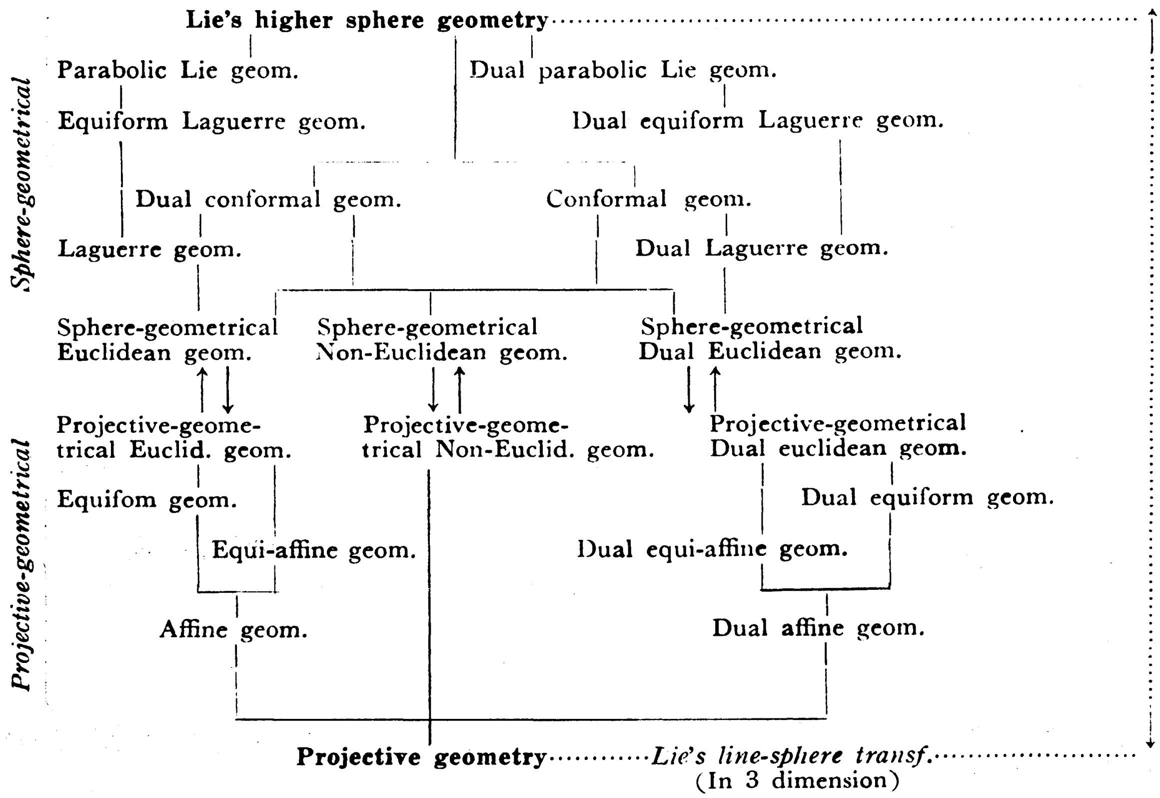
Introduction.

In [1⋯9, 19⋯25], the present author has extended the branches of geometry tabulated below to the case, where the group parameters are appropriate functions of coordinates and realized all the extended geometries so obtained in the differentiable manifolds:



In [19] and [24], he began to establish theories of

extended linear connections in the large | extended principal fibre bundles

with structure groups extended as are explained above. These results are situated among others as are indicated by * in the following system:

F. Klein's "Erlanger Programm"

J. A. Schouten's program
recommended by E. Cartan
1926

Non-holonomic systems
Horak, Vranceanu, Schouten
1926-1928
Teleparallelism
E. Cartan, Weitzenböck
1926 1928

S. S. Chern, C. Ehresmann,
A. Lichnerowicz
Global theory of connections
1951-1953

A. G. Walker, T. J. Willmore
Global connections in
relation to distributions
1949-1958

T. Tahasu's extensions*
of classical geometries
by group parameter
extensions
1956-1961

T. Ōtsuki, A. Arangnol
General connections
1957-1961

T. Takasu
Global theory of connections
with structure groups obtained
by group parameter extensions
1960-

T. Takasu's program*
Realizations of the extended
geometries in the global
differentiable manifolds
1951-1961

Newton

A. Einstein's scecial relativity
1906

A. Einstein's general relativity
1916
A. Einstein's generalized gravitation
theory
1955

T. Takasu's relativity theory
as an extended equiform Laguerre geometry ♯)
realized in the ordinary space
1951-1959

Physically *naturally* led

Future

In this paper, we will show that our extended geometries yield us ***non-connection methods for connection spaces*** (***differentiable manifolds***) ***in the large with***

extended | ***ordinary***

♯) That is under the author's *extended* Lorentz transformation group.

structure groups in such a way that the respective connection spaces are obtained from the respective classical spaces tabulated above firstly.

Thus e. g. the

Riemannian	affine connection

space is obtained from the classical

Euclidean	affine

space by our ***extended***

Euclidean transformation.	***affine transformation.***

The present paper is a ***detailed*** and ***revised*** exposition of [24], in which rather ***laborious proofs*** for "Some Preliminary Formulas" (Art. 7) are omitted and the ***tensor character considerations of the equations*** (9.3), (9.4) ***of structure*** (Art. 9) are lacking, and [9].

As our first step, we treat in the following lines the cases of the ***extended Euclidean space*** and the ***extended affine space.***

§ 1. Preliminaries.

1. Differentiable Manifolds. In order to fix our conceptions, we will first ricapitulate some definitions, etc.

Let R^n be an n-dimensional Cartesian space provided with the real coordinates (x^λ), $(\lambda=1, 2, \cdots, n)$. We call the topological representation of an open subset U_α of an $n-$dimensional manifold M (considered below) on an open subset $x(U_\alpha)$ of R^n a ***system of local coordinates*** (or a local *chart*, [12]) of M. U_α is called the domain of the coordinates system. To each point x of $U_\alpha \subset M$, there corresponds a point P of R^n, which is represented by (x^λ). The (x^λ) are called the ***coordinates of P in the chart*** under consideration.

A ***differentiable manifold M*** of the class C^v, (v=positive integer or $v=\infty$ or $v=\omega$) is an n-dimensional manifold, to which a system A (*atlas*) of charts satisfying the following conditions are associated [12]:

A_1. $M=\bigcup\limits_{\alpha} U_\alpha$,

A_2. If $x \in U_1 \cap U_2$, (U_1, U_2: two domains of charts of A), and $(x^\lambda), (y^\lambda)$ are the local coordinates having U_1 and U_2 as the domains respectively, then

$$y^\lambda = y^\lambda(x^\nu) \qquad\Big|\qquad x^\lambda = x^\lambda(y^\nu)$$

are functions of class C^v such that

$$\frac{\partial(y)}{\partial(x)} \neq 0. \qquad\Big|\qquad \frac{\partial(x)}{\partial(y)} \neq 0.$$

Two atlas A and B are said to be ***equivalent***, when their reunion is also an atlas of class C^v.

In order that two atlas A and B of one and the same differentiable manifold M may be equivalent, it is necessary and sufficient that A, B satisfy the axiom A_2.

Two equivalent atlas are said to define ***one and the same structure of differentiable manifold of class C^v on M.***

A system of local coordinates of M is said to be ***compatible*** with the structure of differentiable manifold (or to be admissible) when the reunion of its domain with an atlas defining M as differentiable manifold is also an atlas of the same class.

2. II-Geodesic Curves. Set

(2.1) $$\omega^l \overset{\text{def}}{=} \omega^l_\mu(x^\nu)\,dx^\mu, \quad (l, m, \ldots, \lambda, \mu, \ldots = 1, 2, \ldots, n),$$

where the Pfaffians ω^l are assumed to be ***not exact*** (i. e. ***anholonomic***) in general and to be linearly independent, so that the condition

(2.2) $$|\omega^l_\mu(x^\nu)| \neq 0 \text{ in } M$$

is satisfied.

Since (2.1) is written in an ***invariant form***, ω^l are ***global*** in $\bigcup_\alpha U_\alpha$.

For the syatem $\omega^l_\mu(x^\nu)$, we introduce $\Omega^\lambda_l(x^\nu)$ by the condition:

(2.3) $$\Omega^\lambda_l\omega^l_\mu = \delta^\lambda_\mu \qquad \longleftrightarrow \qquad \Omega^\lambda_m\omega^l_\lambda = \delta^l_m,$$

where the δ's are Kronecker deltas. Owing to (2.2), (Ω^λ_l) is the reciprocal matrix of (ω^l_μ) divided by $|\omega^l_\mu|$.

We define the connection parameters $\Lambda^\lambda_{\mu\nu}$ and Λ^l_{mn} by

(2.4) $$\begin{cases} \Lambda^\lambda_{\mu\nu} \overset{\text{def}}{=} \Omega^\lambda_l \dfrac{\partial\omega^l_\mu}{\partial x^\nu} \equiv -\,\omega^l_\mu \dfrac{\partial\Omega^\lambda_l}{\partial x^\nu}, \\[2ex] \Lambda_{mn} \overset{\text{def}}{=} \omega^l_\lambda \dfrac{\partial\Omega^\lambda_m}{\omega^n} \equiv -\,\Omega^\lambda_m \dfrac{\partial\omega_\lambda}{\omega^n}, \end{cases}$$

the last identity arising from (2.3). Thereby the operator $\dfrac{\partial}{\omega^n}$ is defined as follows:

(2.5) $$\frac{\partial}{\omega^n} \overset{\text{de}}{=} \Omega^\mu_n \frac{\partial}{\partial x^\mu}.$$

Indeed,

$$\lim_{dx^\mu \to 0} \frac{\psi(x^\mu + dx^\mu) - \psi(x^\mu)}{\omega^l} = \lim_{dx^\mu \to 0} \frac{\dfrac{\partial\psi}{\partial x^\mu}\,dx^\mu}{\omega^l_\mu\,dx^\mu} = \lim_{dx^\mu \to 0} \frac{\dfrac{\partial\psi}{\partial x^\mu}\Omega^\mu_q\omega^q}{\omega^l_\mu\Omega^\mu_p\,\omega^p}$$

$$= \lim_{dx^\mu \to 0} \frac{\Omega^\mu_q \dfrac{\partial\psi}{\partial x^\mu}\omega^q}{\delta_p\omega^p} = \lim_{\omega^l \to 0} \frac{\Omega^\mu_q\dfrac{\partial\psi}{\partial x^\mu}\omega^q}{\omega^l} = \lim_{\omega^l \to 0} \frac{\omega^q}{\omega^l}\Omega^\mu_q\frac{\partial\psi}{\partial x^\mu} = \delta^q_l\Omega^\mu_q\frac{\partial\psi}{\partial x^\mu} = \Omega^\mu_l\frac{\partial\psi}{\partial x^\mu}.$$

Consider a parametrized curve $x^\lambda = x^\lambda(t)$, where t is assumed to be an invariant (e. g. the ordinary affine length).

A straight forward calculation shows us the following identity:

$$\frac{d}{dt}\frac{\omega^l}{dt} \equiv \omega^l_\lambda \left(\frac{d^2x^\lambda}{dt^2} + \Lambda^\lambda_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right). \tag{2.6}$$

We consider the combined manifold $(\{x^\lambda\}, \{\omega^l_\mu(x^\nu)\})$ forming a principal fibre bundle, the $\{\omega^l_\mu(x^\nu)\}$ ***making the structure group.*** Although the group elements contain the local coordinates, the ***function forms*** depending on the ordinary group parameters $(c_1, c_2, \cdots, c_r)$ make the group elements independent of the local coordinates in a certain sense.

From (2.6), we obtain

(2.7) (i) $\dfrac{d}{dt}\dfrac{\omega^l}{dt} = 0,$ which is global. | (ii) $\dfrac{d^2x^\lambda}{dt^2} + \Lambda^\lambda_{\mu\nu}\dfrac{dx^\mu}{dt}\dfrac{dx^\nu}{dt} = 0,$ which is local.

The differential equations

$$\text{(ii)} = \frac{d^2x^\lambda}{dt^2} + \bar{\Lambda}^\lambda_{\nu\mu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = 0, \qquad (\bar{\Lambda}^\lambda_{\mu\nu} = \tfrac{1}{2}(\Lambda^\lambda_{\mu\nu} + \Lambda^\lambda_{\nu\mu}))$$

define the ***autoparallel curves*** (i. e. ***paths***) ***of the teleparallelism*** of $\omega^l_\mu(x^\nu)$ and $\Omega^\lambda_l(x^\nu)$. Indeed, from

$$\frac{d}{dt}\Omega^\lambda_l + \Lambda^\lambda_{\mu\nu}\frac{dx^\mu}{dt}\Omega^\nu_l = 0, \qquad \frac{d}{dt}\omega^l_\nu - \Lambda^\lambda_{\mu\nu}\frac{dx^\mu}{dt}\omega^l_\lambda = 0,$$

we can easily deduce $(2.4)_1$. ***The*** (i) ***is convenient for the study of the global properties.*** Indeed, the identity (2.6):

$$\Omega^\lambda_l\frac{d}{dt}\frac{\omega^l}{dt} \equiv \frac{d^2x^\lambda}{dt^2} + \Lambda^\lambda_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}, \tag{2.6)'}$$

transforms the global path (i) ***piece-wise onto the local path*** (ii) ***by the inverse transformation***

$$dx^\lambda = \Omega^\lambda_l(x^\nu)\,\omega^l \tag{2.1)'}$$

of (2.1).

The differential equations (i) are integrated readily:

$$\omega^l = a^l dt, \quad (a^l: \text{const.}), \tag{2.8}$$

$$\int \frac{\omega^l}{dt}\, dt = a^l t + c^l, \quad (c^l: \text{const.}), \tag{2.9}$$

the integration (2.9) being guided by the simple clear form $a^l dt$ of (2.8).

We set

$$\xi^l \overset{\text{def}}{=} a^l t + c^l, \quad (a^l: \text{const.}),$$

so that

$$\xi^l \overset{\text{def}}{=\!=} \int \frac{\omega^l}{dt} dt = a^l t + c^l. \tag{2.10}$$

Hereby we have to convince ourselves with the fact that the function $\int \frac{\omega^l}{dt} dt$ is itself the very function $a^l t + c^l$. Otherwise, it must have followed from (2.10) that $t = \text{const.}$, what is now not the case.

From (2.10), we see that *the curves represented by* (2.7), (i) *or by* (2.10) *behave as for meet and join like straight lines.* We will call these curves II-*geodesic curves* (read: geodesic curves of the second kind).

The definition (2.10) of ξ^l means that *we adopt such curves as ξ^l-axes.*

The formula (2.10) tells us that $(\xi^l - c^l)$ *is the projection of t by a^l.*

Although ω^l are anholonomic in general, me may write ω^l in the form of curve-differential:

$$d\xi^l = \omega^l = a^l_\mu(x^\nu)\, dx^\mu(t) = a^l dt \tag{2.11}$$

for II-*geodesic line-elements,* where

$$a^l_\mu(x^\nu) \overset{\text{def}}{=\!=} \omega^l_\mu(x_\nu), \qquad (|a^l_\mu(x_\nu)| \neq 0 \text{ in } M). \tag{2.12}$$

The line-element $d\xi^l(t)$ corresponds to a line-element of an *arbitrary* II-geodesic curve (2.10). Thereby μ runs $1, 2, \cdots\cdots, n$ independently as l runs $1, 2, \cdots, n$. Thus in (2.11), *we may omit t and write down as follows*:

$$d\xi^l = a^l_\mu(x^\nu)\, dx^\mu. \tag{2.13}$$

That the anholonomic Pfaffian $a^l_\mu(x^\nu)\, dx^\mu$ can be expressed in the form of the differential $d\xi^l$ is an unexpected result encountered by the readers.

The first equation (i) of (2.7) may now be rewritten as follows:

$$\frac{d^2\xi^l}{dt^2} = 0. \tag{2.14}$$

Multiplying (2.8) with Ω^λ_m, we see that *the relations*

$$\frac{dx^\lambda}{dt} = a^m \Omega^\lambda_m \tag{2.15}$$

hold along the II-*geodesic line-elements.*

We will call the (ξ^l) the II-*geodesic parallel coordinates corresponding to* $a^l_\mu(x^\nu)$ *referred to the* II-*geodesic coordinate axes.* The (ξ^l) are *global for* $\cup U_\alpha$ *having been obtained by pasting the ordinary paths* (2.7), (ii) *for U_α etc.*

3. Extension of the Affine Transformation Group by Extending the Group Parameters to Functions of Coordinates. When the differentiable manifold M is the classical affine space, so that the (x^ν) are the ordinary parallel coordinates, the $\bigcup_\alpha U_\alpha$ reduces to a single subset U_α, which is the classical affine space.

In general case, the II-geodesic parallel coordinates (ξ^l) can stand for (x^ν), so that the $\bigcup_\alpha U_\alpha$ consists of a single subset U_α and in place of (2.13), we come to consider

$$d\bar{\xi}^l = a^l_m(\xi^p)\,d\xi^m, \quad (|a^l_m(\xi^p)| \neq 0 \text{ in } M) \tag{3.1}$$

for II-geodesic line-elements corresponding to $a^l_m(\xi^p)$. We take II-*geodesic curves as tangents to curves to be studied.*

We eonsider a transformation

$$\bar{\xi}^l = a^l_m(\xi^p)\,\xi^m + a^l_0, \quad (|a^l_m(\xi^p)| \neq 0 \text{ in } M). \tag{3.2}$$

We will call the transformations (3.2), which transform II-geodesic curves $\xi_m(t)$ into II-geodesic curves $\bar{\xi}^l(t)$ corresponding to $a^l_m(\xi^p)$, *extended affine transformation.* By such a transformation, II-geodesic curves

$$\frac{d^2\xi^l}{dt^2} = 0 \tag{3.3}$$

are transformed into II-geodesic curves

$$\frac{d^2\bar{\xi}^l}{dt^2} = 0. \tag{3.4}$$

Now, by (3.1), we have

$$\frac{d^2\bar{\xi}^l}{dt^2} = \frac{d}{dt} a^l_m(\xi^p)\frac{d\xi^m}{dt} + a^l_m(\xi^p)\frac{d^2\xi^m}{dt^2}.$$

Hence, by the demands (3.3) and (3.4), we must have

$$da^l_m(\xi^p)\,d\xi^m = 0 \tag{3.5}$$

for the II-*geodesic line-elements.*

Integrating (3.1) along the II-geodesic ξ^l-axis, we obtain

$$\bar{\xi}^l = a^l_m(\xi^p)\,\xi^m - \int \xi^m \frac{da^l_m(\xi^p)}{dt}\,dt.$$

Now

$$\int \xi^m \frac{da^l_m(\xi^p)}{dt}\,dt = \int \frac{da^l_m(\xi^p)}{dt}\,dt \int d\xi^m = \int\int\left\{\frac{da^l_m(\xi^p)}{dt}\,dt\,d\xi^m\right\} \tag{*}$$
$$= \text{const.}$$

by (3.5), the condition for that the repeated integral may be converted into the double integral (that the integrand is continuous) being evidently satisfied. Hence, for the a^l_o in (3.2), we have

$$a^l_o = \text{const.} \tag{3.6}$$

From (3.2) and (3.5), we see that

$$da^l_m(\xi^p)\,\xi^m = 0 \tag{3.7}$$

for the II-*geodesic line-elements.*

The totality of the extended affine transformations

$$\bar{\xi}^m = a^m_k(\xi^q)\,\xi^k + a^m_o, \quad (a^m_o = \text{const.}, \ |a^m_k(\xi^q)| \neq 0), \tag{3.8}$$

whose inverse transformation is

$$(3.9)\qquad \xi^k=\Omega^k_m(\bar{\xi}^l)\,\bar{\xi}^m+\Omega^k_o,\quad (\Omega^k_o=\text{const.},\quad |\Omega^k_m(\bar{\xi}^l)|\neq 0)$$

with

$$(3.10)\qquad a^m_k\,\Omega^l_m=\delta^l_k \quad \longleftrightarrow \quad a^l_m\Omega^m_k=\delta^l_k,$$

forms a group, $\mathfrak{G}$, say. For, the combination of (3.8) with

$$(3.11)\qquad \bar{\bar{\xi}}^l=\bar{a}^l_m(\bar{\xi}^p)\,\bar{\xi}^m+\bar{a}^l_o,\quad (\bar{a}^l_o=\text{const.},\quad |\bar{a}^l_m(\bar{\xi}^q)|\neq 0),$$

gives

$$(3.12)\qquad \bar{\bar{\xi}}=b^l_k(\xi^r)\,\xi^k+b^l_o,\quad (b^l_o=\text{const.},\quad |b^l_k(\xi^r)|\neq 0),$$

where

$$(3.13)\qquad b^l_k(\xi^r)=\bar{a}^l_m(a^p_h(\xi^r)\,\xi^h+a^q_o)\,a^m_k(\xi^r),$$

$$(3.14)\qquad b^l_o=\bar{b}^l_m(\xi^r)\,a^m_o+\bar{a}^l_o,$$

$$(3.15)\qquad \bar{b}^l_m(\xi^r)=\bar{a}^l_m(a^q_k(\xi^r)\,\xi^k+a^q_o).$$

We shall see that

$$(3.16)\qquad b^l_o=\bar{b}^l_m(\xi^r)\,a^m_o+\bar{a}^l_o=\text{const.}$$

owing to the summation with respect to m, for which it suffices to prove that

$$(3.17)\qquad a^m_o d\bar{b}^l_m(\xi^r)=0$$

on summation with respect to m. For (3.11), the condition (3.7) for that the $\bar{\xi}^l$-axes may be II-geodesic curves corresponding to $\bar{a}^l_m(\bar{\xi}^p)$ becomes

$$(3.18)\qquad \bar{\xi}^m d\bar{a}^l_m(\bar{\xi}^p)=0.$$

We shall show that (3.17) follows from (3.18). Indeed (3.18) becomes

$$\{a^m_k(\xi^q)\,\xi^k+\bar{a}^m_o\}d\bar{a}^l_m(\xi^p)=\{a^m_k(\xi^q)\,\xi^k+a^m_o\}\,d\bar{b}^l_m(\xi^r)=0,$$

so that

$$\begin{aligned}(3.19)\qquad a^m_o d\bar{a}^l_m(\bar{\xi}^r)&=a^m_o d\bar{b}^l_m(\xi^r)\\ &=-a^m_k(\xi^r)\,d\bar{b}^l_m(\xi^r)\,\xi^k\\ &=-a^m_k(\xi^r)\,d\bar{b}^l_m(\xi^r)\,\xi^k-\{\xi^k da^m_k(\xi^q)\}\bar{b}^l_m(\xi^r)\end{aligned}$$

by the differential equation

$$(3.20)\qquad \xi^k da^m_k(\xi^q)=0$$

for the II-geodesic curves corresponding to $a^m_k(\xi^q)$.

Thus we have

$$(3.21)\qquad a^m_o\bar{a}^l_m(\bar{\xi}^p)=-\xi^k d\{a^m_k(\xi^r)\,\bar{b}^l_m(\xi^r)\}=-\xi^k d\{a^m_k(\xi^r)\,\bar{a}^l_m(\xi_r)\}=-\xi^k db^l_m(\xi^r)=0$$

by the differential equation

$$(3.22)\qquad \xi^k db^l_k(\xi^r)=0$$

for the II-geodesic curves corresponding to $b^l_k(\xi^r)$. The (3:21) shows us (3.17).

We shall call the group $\mathfrak{G}$ the *extended affine group. The most general extended affine group $\mathfrak{G}$ contains the ordinary affine group $\mathfrak{E}$* (in an abstract sense) *as a subgroup.*

The totality of the elements of $\mathfrak{G}$, which are free from $\mathfrak{E}$, together with the unit transformation, forms a subgroup $\mathfrak{H}$ of $\mathfrak{G}$, so that we have

(3.23) $$\mathfrak{G}=\mathfrak{C}\mathfrak{H}+\mathfrak{H}\mathfrak{C}.$$

4. Extended Equi-affine Group. The totality of the elements of the extended affine group such that

(4.1) $$|a^l_m(\xi^p)|=1$$

forms a subgroup of $\mathfrak{G}$, which we will call the ***extended equi-affine group***.

The general n-volume

(4.2) $$(\xi^l, \xi^l+d_1\xi^l, \cdots, \xi^l+d_n\xi^l)=|d_1\xi^l\, d_2\xi^2 \cdots d_n\xi^n|$$

is an invariant under the extended equi-affine group.

5. An Invariant Parameter of Curves under the Extended Equi-affine Group. Denoting the derivatives with respect to

t by dashes	s by dots,

we introduce an invariant parameter s of curves under the extended equi-affine group by the demand:

(5.1) $$|\dot{\xi}\, \ddot{\xi} \cdots \overset{(n)}{\xi}|_s=|\xi'\, \xi'' \cdots \xi^{(n)}|_t \left(\frac{dt}{ds}\right)^{n(n+1)/2}.$$

Owing to the konown peculiar form of the left-hand side, we may condition s by the demand:

(5.2) $$|\dot{\xi}\, \ddot{\xi} \cdots \overset{(n)}{\xi}|=1.$$

The (5.2) may be rewritten as follows:

$$|\dot{\xi}\, \ddot{\xi} \cdots \overset{(n)}{\xi}|=\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ \xi^1 & \dot{\xi}^1 & \ddot{\xi}^1 & \cdots & \overset{(n)}{\xi^1} \\ \xi^2 & \dot{\xi}^2 & \ddot{\xi}^2 & \cdots & \overset{(n)}{\xi^2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \xi^n & \dot{\xi}^n & \ddot{\xi}^n & \cdots & \overset{(n)}{\xi^n} \end{vmatrix}=1,$$

which tells us that $ds^{n(n+1)/2}$ represents $n!$ times the generalized ***n-volume*** of $n+1$ consecutive points:

(5.3) $$ds=|d\xi\, d^2\xi \cdots d^n\xi|^{2/n(n+1)}.$$

When (ξ^l) are ordinary parallel coordinates, s is the ***ordinary affine-length.*** Thus ***the ordinary affine-length is an invariant under the extended equi-affine group.***

6. Extended Affine Principal Fibre Bundles. As was stated in Art. 2, we consider an extended affine principal fibre bundle on a differentiable manifold $M=\bigcup_\alpha U_\alpha$ such that the structure group is

(6.1) $$(a^l_\mu(x^\nu)) \quad \text{or} \quad (a^l_\mu(x^\nu), \Omega^\lambda_l(x^\nu)).$$

We shall call a ***frame*** the object formed by a point $P\in M$ and linearly independent tangent vectors at P. The principal fibre bundle $\mathfrak{B}$ is the space of all frames over M; its dimension is n^2+n. To a local coordinate system x^λ in M, there correspond a system of local coordinates $x^\lambda, \Omega^\lambda_i(x^\nu)$ in $\mathfrak{B}$ defined by the condition that *vectors of the frame are* (cf. (2.5)) *given by*

$$e_i=\Omega^\lambda_i\frac{\partial}{\partial x^\lambda}=\frac{\partial}{\partial \xi^i}=\frac{\partial}{\omega^i}. \tag{6.2}$$

Since

$$|\Omega^\lambda_i(x^\nu)|\neq 0, \tag{6.3}$$

these vectors are linearly independent and we have relations (2.3).

Suppose we restrict ourselves to $U_\alpha \cup U_\beta$, where U_α and U_β correspond to x^λ and $x^{*\lambda}$ respectively. Denote the local coordinate system in $\mathfrak{B}$ corresponding to $x^{*\lambda}$ by $x^{*\lambda}, \Omega^{*\lambda}_i(x^{*\nu})$ and introduce the $\omega^{*i}_\mu(x^\nu)$ by the law (2.3). Then we have

$$\Omega^{*\lambda}_i=\frac{\partial x^{*\lambda}}{\partial x^\mu}\Omega^\mu_i, \qquad \omega^{*i}_\lambda=\omega^i_\mu\frac{\partial x^\mu}{\partial x^{*\lambda}}, \tag{6.4}$$

which implies in particular

$$\omega^{*i}_\mu dx^{*\mu}=\omega^i_\mu dx^\mu. \tag{6.5}$$

It follows that the differential form $\omega^i=\omega^i_\mu(x^\nu)\,dx^\mu$ have as representatives the two sides of the above equation, is independent of the choice of the local coordinates and is thus defined in the large in $\mathfrak{B}$.

Suppose now an (extended) linear connection to be given in M. In the expression

$$DX^\lambda \overset{\text{def}}{=} X^\lambda_{;\mu}\,dx^\mu \overset{\text{def}}{=} dX^\lambda+\omega^\lambda_\mu X^\mu, \tag{6.6}$$

we regard the X^λ as independent variables and apply it to each of the vectors of our frame. Then

$$D\Omega^\lambda_i=d\Omega^\lambda_i+\omega^\lambda_\mu\Omega^\mu_i \tag{6.7}$$

are linear differential forms in $x^\lambda, \Omega^\lambda_i(x^\nu)$. From

$$DX^{*\lambda}=p^\lambda_\mu DX^\mu, \qquad \left(p^\lambda_\mu=\frac{\partial x^{*\lambda}}{\partial x^\mu}\right), \tag{6.8}$$

we get

$$D\Omega^{*\lambda}_i=\frac{\partial x^{*\lambda}}{\partial x^\mu}D\Omega^\mu_i. \tag{6.9}$$

This, togetner with (6.4), gives

$$\omega^{*l}_\mu D\Omega^{*\mu}_m=\omega^l_\mu D\Omega^\mu_m. \tag{6.10}$$

It follows that the two members of the equation are representatives, in the coordinates $x^{*\lambda}$ and x^λ respectively, of differential forms in $\mathfrak{B}$. We shall denote them by θ^l_m:

$$\theta^l_m \overset{\text{def}}{=} \omega^l_\mu D\Omega^\mu_m. \tag{6.11}$$

Notice that, whereas the ω^l are defined in $\mathfrak{B}$ by means of the differential structure in M only, the θ^l_m are defined only when M has an (extended) linear connection. It is clear that these n^2+n linear differential forms ω^l and θ^l_m are linearly independent.

7. Some Preliminary Formulas. The formulas (2.13) run:

$$d\xi^l=\omega^l=\omega^l_\mu(x^\sigma)\,dx^\mu, \tag{7.1}$$

whose inverse transformation is

$$dx^\lambda=\Omega^\lambda_l(x^\sigma(\xi^\tau))\,d\xi^l. \tag{7.2}$$

The formulas for the transformation of the connection parameters $P^\lambda_{\mu\nu}$ are

$$P'^{\lambda}_{\mu\nu}\frac{\partial x^\omega}{\partial x'^\lambda}=P^\omega_{\sigma\tau}\frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x^\tau}{\partial x'^\nu}+\frac{\partial^2 x^\omega}{\partial x'^\mu\partial x'^\nu}. \tag{7.3}$$

For the transformations (7.1) and (7.2), we have

$$L^h_{mn}\Omega^\lambda_h=L^\lambda_{\mu\nu}\Omega^\mu_m\Omega^\nu_n+\frac{\partial\Omega^\lambda_m}{\partial\xi^n}, \tag{7.4}$$

which follows also from (6.11), and

$$L^l_{mn}=L^\lambda_{\mu\nu}\omega^l_\lambda\Omega^\mu_m\Omega^\nu_n+\Lambda^l_{mn},\qquad\left(\Lambda^l_{mn}\overset{\text{def}}{=}\omega_\lambda\frac{\partial\Omega^\lambda_m}{\partial\xi^n}\equiv-\Omega^\lambda_m\frac{\partial\omega^l_\lambda}{\partial\xi^n}\right). \tag{7.4$'$}$$

$$L^\lambda_{\mu\nu}=L^l_{mn}\Omega^\lambda\omega^m_\mu\omega^n_\nu+\Lambda^\lambda_{\mu\nu},\qquad\left(\Lambda^\lambda_{\mu\nu}\overset{\text{def}}{=}\Omega^\lambda_l\frac{\partial\omega^l_\mu}{\partial x^\nu}\equiv-\omega^l_\mu\frac{\partial\Omega^\lambda}{\partial x^\nu}\right). \tag{7.4$''$}$$

In case $L^\lambda_{\mu\nu}=\Lambda^\lambda_{\mu\nu}$, (7.4)$'$ gives

$$\tilde{L}^l_{mn}=\Lambda^\lambda_{\mu\nu}\omega_\lambda\Omega^\mu_m\Omega^\nu_n+\Lambda^l_{mn}$$

$$=\Omega^\lambda_p\frac{\partial\omega^p_\mu}{\partial x^\nu}\omega^l_\lambda\Omega^\mu_m\Omega^\nu_n+\Lambda^l_{mn}=\Omega^\mu_m\frac{\partial\omega^l_\mu}{\omega^n}+\Lambda^l_{mn}=-\omega^l_\mu\frac{\partial\Omega^\mu_m}{\omega^n}+\Lambda^l_{mn}=0.$$

$$\tilde{L}^l_{mn}=0, \tag{7.4$'''$}$$

which follows also from (7.4)$''$ on putting $\Lambda^\lambda_{\mu\nu}\to L^\lambda_{\mu\nu}$.

From (7.4)$'$ and (7.4)$''$, we obtain

$$\Omega^\lambda_p\theta^p_l=\Omega^\mu_l(\theta^\lambda_\mu-\Theta^\lambda_\mu), \tag{7.5}$$

$$\theta^p_l=\Omega^\mu_l\omega^p_\lambda(\theta^\lambda_\mu-\Theta^\lambda_\mu) \tag{7.6}$$

in concordance with (6.11), where

$$\theta^\lambda_\mu\overset{\text{def}}{=}L^\lambda_{\mu\nu}dx^\nu, \tag{7.7}$$

$$\Theta^\lambda_\mu\overset{\text{def}}{=}\Lambda^\lambda_{\mu\nu}dx^\nu=\Omega^\lambda_l d\omega^l_\mu. \tag{7.7$'$}$$

From (7.7)$'$ follows:

$$d\omega^h_\lambda=\omega^h_\sigma\Lambda^\sigma_{\lambda\mu}dx^\mu=\omega^h_\sigma\Theta^\sigma_\lambda=-\omega^r_\lambda\Lambda^h_{rs}\omega^s. \tag{7.8}$$

The first half of (7.8) follows also from the parallelity of ω^l_λ in $\Lambda^\lambda_{\mu\nu}$ and the last part is seen as follows:

$$-\omega^r_\lambda \Lambda^h_{rs}\omega^s = -\omega^r_\lambda \omega^h_\sigma \frac{\partial \Omega^\sigma_r}{\omega^s}\omega^s = -\omega^r_\lambda \omega^h_\sigma d\Omega^\sigma_r$$

$$=\omega^h_\sigma \Omega^\sigma_\tau d\omega^\tau_\lambda = \omega^h_\sigma \Lambda^\sigma_{\lambda\mu} dx^\mu.$$

From (7.8), it follows that

$$(7.9) \qquad \Lambda^l_{mn} = -\omega^l_\lambda \Omega^\mu_m \Omega^\nu_n \Lambda^\lambda_{\mu\nu}, \qquad \tilde{\Theta}^l_m \overset{\text{def}}{=} \Lambda^l_{mn}\omega^n = -\omega^l_\lambda \Omega^\mu_m \Theta^\lambda_\mu.$$

The parallelity of Ω^λ_l in $\Lambda^\lambda_{\mu\nu}$ tells us:

$$(7.10) \qquad d\Omega^\mu_l = -\Lambda^\mu_{\omega\nu}\Omega^\omega_l dx^\nu,$$

which becomes further to

$$(7.11) \qquad d\Omega^\mu_l = -\Lambda^\mu_{\omega\nu}\Omega^\omega_l dx^\nu = -\Omega^\omega_l \Theta^\mu_\omega = \Omega^\mu_h \theta^h_l - \Omega^\nu_l \theta^\mu_\nu$$

by virtue of (7.4)$''$.

From (7.6), (7.8) and (7.11), we obtain

$$(7.12) \qquad d\theta^h_l = \{(\Omega^\nu_\kappa \theta^\kappa_l - \Theta^\nu_\kappa \Omega^\kappa_l)\,\omega^h_\tau + \Omega^\nu_l \omega^h_\sigma \Theta^\sigma_\tau\}\,(\theta^\tau_\nu - \Theta^\tau_\nu) + \Omega^\nu_l \omega^h_\lambda d\,(\theta^\lambda_\nu - \Theta^\lambda_\nu).$$

$$\Theta^\lambda_\mu = \Omega^\lambda_l \frac{\partial \omega^l_\mu}{\partial x^\nu} dx^\nu,$$

$$d\Theta^\lambda_\mu = d\left(\Omega^\lambda_l \frac{\partial \omega^l_\mu}{\partial x^\nu}\right)\wedge dx^\nu = d\Omega^\lambda_l \frac{\partial \omega^l_\mu}{\partial x^\nu}\wedge dx^\nu + \Omega^\lambda_l d\frac{\partial \omega^l_\mu}{\partial x^\nu}\wedge dx^\nu$$

$$= -\Omega^\omega_l \Theta^\lambda_\omega \frac{\partial \omega^l_\mu}{\partial x^\nu}\wedge dx^\nu + \Omega^\lambda_l d\frac{\partial \omega^l_\mu}{\partial x^\nu}\wedge dx^\nu$$

$$= -\Theta^\lambda_\omega \Lambda^\omega_{\mu\nu}\wedge dx^\nu + \Omega^\lambda_l d\frac{\partial \omega^l_\mu}{\partial x^\nu}\wedge dx^\nu,$$

where d denotes the exterior differential, so that

$$d\Theta^\lambda_\mu = -\Theta^\lambda_\omega \wedge \Theta^\omega_\mu + \Omega^\lambda_l d\frac{\partial \omega^l_\mu}{\partial x^\nu}\wedge dx^\nu.$$

Hence we have

$$(7.13) \qquad d\Theta^\lambda_\mu + \Theta^\lambda_\nu \wedge \Theta^\nu_\mu = \Omega^\lambda_l dd\omega^l_\mu$$

$$= \tfrac{1}{2}\left[\left(\frac{\partial \Lambda^\lambda_{\mu\beta}}{\partial x^\alpha} - \frac{\partial \Lambda^\lambda_{\mu\alpha}}{\partial x^\beta}\right) - (\Lambda^\lambda_{\kappa\beta}\Lambda^\kappa_{\mu\alpha} - \Lambda^\lambda_{\kappa\alpha}\Lambda^\kappa_{\mu\beta})\right] dx^\alpha \wedge dx^\beta$$

$$= \tfrac{1}{2} R^{\,\lambda}_{.\mu\alpha\beta} dx^\alpha \wedge dx^\beta, \text{ say.}$$

Similarly, we have

$$(7.14) \qquad d\theta^\lambda_\mu + \theta^\lambda_\nu \wedge \theta^\nu_\mu = \tfrac{1}{2}\left[\left(\frac{\partial L^\lambda_{\mu\beta}}{\partial x^\alpha} - \frac{\partial L^\lambda_{\mu\alpha}}{\partial x^\beta}\right) - (L^\lambda_{\kappa\beta}L^\kappa_{\mu\alpha} - L^\lambda_{\kappa\alpha}L^\kappa_{\mu\beta})\right] dx^\alpha \wedge dx^\beta$$

$$= \tfrac{1}{2} R^{\,\lambda}_{.\mu\alpha\beta} dx^\alpha \wedge dx^\beta, \text{ say}$$

and $\qquad d\theta^l_m + \theta^l_h \wedge \theta^h_m$

$$(7.15) \qquad = \tfrac{1}{2}\left[\left(\frac{\partial L^l_{mb}}{\partial \xi^a} - \frac{\partial L^l_{ma}}{\partial \xi^b}\right) - (L^l_{kb}L^k_{ma} - L^l_{ka}L^k_{mb})\right] \omega^a \wedge \omega^b$$

$$= \tfrac{1}{2} R^{\,l}_{.mab}\omega^a \wedge \omega^b, \text{ say.}$$

Further, we have

$$\Lambda^l_{hk} du^h du^k = -d\omega^l_\mu dx^\mu = \omega^l_\lambda d\Omega^\lambda_h d\xi^h. \tag{7.16}$$

For,

$$\Lambda^l_{hk}\omega^h\omega^k = \omega^l_\lambda \frac{\partial \Omega^\lambda_h}{\omega^k}\omega^h\omega^k = \omega^l_\lambda d\Omega^\lambda_h \omega^h = \omega^l_\lambda d\Omega^\lambda_h d\xi^h = -\Omega^\lambda_h d\omega^l_\lambda \omega^h = -d\omega^l_\lambda dx^\lambda.$$

8. Recapitulation of Some Theorems of H. Friesecke and J. A Schouten.

In order to make a closer examination of various mathematical conditions easier, we will recapitulate some theorems of H. Friesecke and J. A Schouten ([29], [30], [31]).

Two vectors at a point are said to have the *same direction*, if corresponding components are proportional. Accordingly, if a set of functions $\lambda^\lambda(x^\nu)$ satisfy

$$\frac{d\lambda^\lambda}{dt} + L^\lambda_{\mu\nu}\lambda^\mu \frac{dx^\nu}{dt} = 0, \tag{8.1}$$

the vectors of components

$$\bar{\lambda}^\lambda = \varphi\lambda^\lambda, \tag{8.2}$$

where φ is any function of t, for a curve C, should be interpreted as parallel with respeet to the given curve C. From (8.1) and (8.2), we have

$$\frac{d\bar{\lambda}^\lambda}{dt} + L^\lambda_{\mu\nu}\bar{\lambda}^\mu \frac{dx^\nu}{dt} = \bar{\lambda}^\lambda f(t), \tag{8.3}$$

where

$$f(t) = \frac{d\log\varphi(t)}{dt}. \tag{8.4}$$

Conversely, if we have any set of functions $\bar{\lambda}^\lambda$ of t, which satisfy (8.3), they are components of a family of contravariant vectors parallel with respect to C; and by means of (8.2) and (8.4), we find the vectors satisfying (8.1).

From (8.3), we have, on eliminating $f(t)$ and omitting the bars,

$$\lambda^\sigma\left(\frac{d\lambda^\lambda}{dt} + L^\lambda_{\mu\nu}\lambda^\mu \frac{dx^\nu}{dt}\right) - \lambda^\lambda\left(\frac{d\lambda^\sigma}{dt} + L^\sigma_{\mu\nu}\lambda^\mu \frac{dx^\nu}{dt}\right) = 0 \tag{8.5}$$

as the conditions of parallelism which hold for (8.2) whatever be φ.

As a particular example of the foregoing, we consider the curves, whose tangents are parallel with respect to the curves. From (8.5), it follows that the equations of these curves are

$$\frac{dx^\sigma}{dt}\left(\frac{d^2x^\lambda}{dt^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right) - \frac{dx^\lambda}{dt}\left(\frac{d^2x^\sigma}{dt^2} + \Gamma^\sigma_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right) = 0. \tag{8.6}$$

where

$$\Gamma^\lambda_{\mu\nu} = \tfrac{1}{2}(L^\lambda_{\mu\nu} + L^\lambda_{\nu\mu}), \tag{8.7}$$

and that, conversely, any curve defined by these equations possesses the above property.

From the form of (8.6), it is evident that all connected spaces, for which the symmetrical parts of the parameters $L^\lambda_{\mu\nu}$ are the same but the skew symmetrical parts of $L^\lambda_{\mu\nu}$ are arbitrary have the same paths.

In the following, however, it will be shown that this is not a necessary condition.

Let $L^\lambda_{\mu\nu}$ and $\bar{L}_{\mu\nu}$ be the parameters of two different connections. We inquire whether it is possible that parallel directions along every curve in the space are the same for the two connections. To this end we make use of the equations of parallelism in the form (8.5). Subtracting these equations from the corresponding ones in the L's, we have

(8.8) $$(\delta^\tau_\rho a^\lambda_{\mu\nu} - \delta^\lambda_\rho a^\tau_{\mu\nu})\lambda^\rho\lambda^\mu\frac{dx^\nu}{dt}=0,$$

where

(8.9) $$a^\lambda_{\mu\nu} \overset{\text{def}}{=} \bar{L}^\lambda_{\mu\nu} - L^\lambda_{\mu\nu}.$$

From the principle (7.3), it is seen that $a^\lambda_{\mu\nu}$ are the components of a tensor. Since these equations must hold for any curve and for vectors parallel to any vector with respect to this curve, we must have

(8.10) $$\delta^\tau_\rho a^\lambda_{\mu\nu} + \delta^\tau_\mu a^\lambda_{\rho\nu} - \delta^\lambda_\rho a^\tau_{\mu\nu} - \delta^\lambda_\mu a^\tau_{\rho\nu} = 0.$$

Contracting for τ and ρ, we have

(8.11) $$a^\lambda_{\mu\nu} = 2\delta^\lambda_\mu \psi_\nu,$$

where ψ_ν is the vector defined by

(8.12) $$2n\psi_\nu = a^\sigma_{\sigma\nu}.$$

Conversely; if we take

(8.13) $$\bar{L}^\lambda_{\mu\nu} = L^\lambda_{\mu\nu} + 2\delta^\lambda_\mu \psi_\nu,$$

where ψ_ν is an arbitrary non-null vector, the above conditions are satisfied. Hence we have

Theorem 1°. (H. Friesecke, [26]) *Equations* (8.12), ***in which*** ψ_ν ***is an arbitrary covariant non-null vector, defines the most general change of connection, which preserves parallelism.***

From the form of (8.13), it is seen that both sets of connection parameters cannot be symmetric in the subscripts. (As for an actual example of the case, where one set does not posses this property, see Eisenhart [29], Art. 14.) Hence we have

Theorem 2°. ***It is not possible to have two symmetric connections, with respect to which parallel directions along any curve in the space are the same for both connections.***

If

$$\Gamma^{\lambda}_{\mu\nu} \overset{\text{def}}{=} \tfrac{1}{2}(L^{\lambda}_{\mu\nu}+L^{\lambda}_{\nu\mu}), \quad \bar{\Omega}^{\lambda}_{\mu\nu} \overset{\text{def}}{=} \tfrac{1}{2}(L^{\lambda}_{\mu\nu}-L^{\lambda}_{\nu\mu}),$$

we have

$$\bar{\Gamma}^{\lambda}_{\mu\nu}=\Gamma^{\lambda}_{\mu\nu}+\delta^{\lambda}_{\mu}\psi_{\nu}+\delta^{\lambda}_{\nu}\psi_{\mu}, \tag{8.14}$$

and

$$\bar{\Omega}^{\lambda}_{\mu\nu}=\Omega^{\lambda}_{\mu\nu}+\delta^{\lambda}_{\mu}\psi_{\nu}-\delta^{\lambda}_{\nu}\psi_{\mu}. \tag{8.15}$$

From the definition (8.6) of the paths of a connected manifold, it follows that the paths are the same for all connections related as in (8.12). This can be shown directly by means of (8.14) as follows:

$$\bar{L}^{\lambda}_{\mu\nu}=L^{\lambda}_{\mu\nu}+2\delta^{\lambda}_{\mu}\psi_{\nu}, \qquad \bar{L}^{\lambda}_{\nu\mu}=L^{\lambda}_{\nu\mu}+2\delta^{\lambda}_{\nu}\psi_{\mu},$$

$$\bar{\Gamma}^{\lambda}_{\mu\nu}=\tfrac{1}{2}(\bar{L}^{\lambda}_{\mu\nu}+\bar{L}^{\lambda}_{\nu\mu})=\tfrac{1}{2}(L^{\lambda}_{\mu\nu}+L^{\lambda}_{\nu\mu})+\delta^{\lambda}_{\mu}\psi_{\nu}+\delta^{\lambda}_{\nu}\psi_{\nu}.$$

Conversely, if we apply to equations (8.6) the same reasoning as was adopted to (8.5), we can show that expressions of the form (8.14) give the most general relation connecting the Γ's so that the equations (8.6) are unaltered. Hence we have

Theorem 3^0. (H. Friesecke, [26]) ***Equations*** (8.14) ***and an arbitrary choice of*** $\Omega^{\lambda}_{\mu\nu}$, ***define the most general change in connection, which preserves the paths.***

The necessary and sufficient condition (8.6) for that a curve $x^{\lambda}=x^{\lambda}(t)$ may be a path, tells us that the tangent vector $\frac{dx^{\lambda}}{dt}$ at a point remains tangent if it is displaced parallel along the curve. The condition (8.6) may be rewritten as follows:

$$\frac{\delta}{dt}\frac{dx^{\lambda}}{dt}=\frac{d^2x^{\lambda}}{dt^2}+\Gamma^{\lambda}_{\mu\nu}\frac{dx^{\mu}}{dt}\frac{dx^{\nu}}{dt}=\alpha(t)\frac{dx^{\lambda}}{dt}, \tag{8.16}$$

from which we see that the paths only depend on $\Gamma^{\lambda}_{(\mu\nu)}$. If a new curve parameter z:

$$z=z(t); \qquad t=t(z) \tag{8.17}$$

is introduced, the equation (8.16) takes the form

$$\frac{d^2x^{\lambda}}{dz^2}+\Gamma^{\lambda}_{\mu\nu}\frac{dx^{\mu}}{dz}\frac{dx^{\nu}}{dz}=-\frac{dx^{\lambda}}{dz}\left(\frac{d^2z}{dt^2}-\alpha\frac{dz}{dt}\right)\left(\frac{dz}{dt}\right)^{-2} \tag{8.18}$$

and the right-hand sid of these equations reduces to zero, if z is a solution of the ordinary differential equation of second order:

$$\frac{d^2z}{dt^2}-\alpha(t)\frac{dz}{dt}=0. \tag{8.19}$$

If this is the case, z is called ***an affine parameter*** on the path. The general solution of (8.19) has the form

$$z=C_1\int e^{\int\alpha dt}dt+C_2, \qquad (C_1\neq 0: C_1, C_2;\ \text{const.}) \tag{8.20}$$

and accordingly the affine parameter is fixed to within the place of the null-point and a constant factor. This implies that two segments on the same path have an invariant ratio of "length" that can be measured by means of one parameter chosen arbitrarily from all possible affine parameters.

In an ordinary Riemannian space V^n, the length s on a real path is always an affine parameter (cf. (5.3)), because $\frac{dx^\lambda}{ds}$ is a unit-vector and the covariant differential of a unit-vector is always perpendicular to the vector. Hence the equation of a path in V^n takes the form

$$\frac{d^2x^\lambda}{ds^2}+\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}=0. \tag{8.21}$$

We ask whether it is possible to transform a connection $\Gamma^\lambda_{\mu\nu}$ in such a way that all paths remain paths. If

$$\Gamma^\lambda_{\mu\nu}\rightarrow \Gamma^\lambda_{\mu\nu}+P_{\mu\nu}^{\cdot\cdot\lambda}, \tag{8.22}$$

$P_{\mu\nu}^{\cdot\cdot\lambda}$ is necessarily a tensor and we may take $P_{[\mu\nu]}^{\cdot\cdot\lambda}=0$, because the alternating part does not affect the paths. According to (8.16), we must have

$$P_{\mu\nu}^{\cdot\cdot\lambda}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}=\beta(t)\frac{dx^\lambda}{dt} \tag{8.23}$$

for every choice of $\frac{dx^\lambda}{dt}$ with a function $\beta(t)$ that depends on this choice. But this is possible only when $P_{\mu\nu}^{\cdot\cdot\lambda}$ has the form ([30], p. 132; [28], p. 9):

$$P_{\mu\nu}^{\cdot\cdot\lambda}=\psi_\mu\delta^\lambda_\nu+\psi_\nu\delta^\lambda_\mu \tag{8.24}$$

by (8.14). If (8.24) is satisfied, the new affine parameter $'z$ has the form

$$'z=C_1\int e^{2\psi_\mu dx^\mu}dz+C_2,\quad (C_1\neq 0\,;C_1,C_2:\text{ const.}), \tag{8.25}$$

whence follows that

Theorem 4^0. ([31], p. 156) *z is an affine parameter for the transformed connection when and only when*

$$\psi_\mu dx^\mu=0 \tag{8.26}$$

at all the points of the path considered i.e. when at all points ψ_μ is tangent to the path.

This is seen as well from (8.14) as from

$$(\psi_\mu\delta^\lambda_\nu+\psi_\nu\delta^\lambda_\mu)\,dx^\mu dx^\nu\equiv(\psi_\mu dx^\mu+\psi_\nu dx^\nu)\,dx^\lambda=0. \tag{8.27}$$

Theorem 5^0. ([31], p. 156) *z remains affine parameter for all paths when and only when*

$$\psi_\lambda=0. \tag{8.28}$$

Hence the

Theorem 6^0. ([31], p. 156) *A symmetric connection is wholly determined by its paths and the affine parameters on them.*

9. A System of Profitable Hypercomplex Units. For subsequent use, we will introduce a system of profitable hypercomplex units γ_p by the condition:

$$(9.1)\qquad \gamma_p\gamma_q+\gamma_q\gamma_p=2\delta_{pq}, \qquad (p, q=1, 2\ldots, n),$$

where δ_{pq} are Kronecker deltas. Hereby it should be noticed that

$$(9.2)\qquad (\gamma_p\gamma_q)^2=\gamma_p\gamma_q\gamma_p\gamma_q=-\gamma_p\gamma_p\gamma_q\gamma_q=-1, \ (p\neq q)$$

and that generally

$$\begin{aligned}(9.3)\qquad (\gamma_p A^p)(\gamma_q B^q) &= A^p B^p+\underset{(p\neq q)}{\gamma_p\gamma_q}A^p B^q\\ &= A^p B^p+\tfrac{1}{2}(\underset{(p\neq q)}{\gamma_p\gamma_q}A^p B^q+\underset{(p\neq q)}{\gamma_q\gamma_p}A^q B^p)\\ &= A^p B^p+\tfrac{1}{2}(\underset{(p\neq q)}{\gamma_p\gamma_q}A^p B^q+\underset{(p\neq q)}{\gamma_q\gamma_p}A^q B^p)\\ &= A^p B^p+\tfrac{1}{2}\underset{(p\neq q)}{(\gamma_p\gamma_q)}(A^p B^q-A^q B^p).\end{aligned}$$

Let us set for this as follows:

$$(9.4)\qquad (\gamma_p A^p)(\gamma_q B^q)=A^p B^p+\mathfrak{D}A^p\wedge B^p.$$

Subsequently we will utilize this formalism.

§2. A Duality in the Equations of Structure for Linear Connections in the Large.

10. A Duality in the Equations of Structure for Linear Connections in the Large. Let us now establish the following theorem implying a *duality* in the linear connections in the large.

Theorem. *In order that n linearly independent linear differential forms*

$$(10.1)\qquad \omega^l=\omega^l_\mu(x^\nu)\,dx^\mu$$

and

$$(10.2)\qquad \theta^l_m=L^l_{mn}(x^\nu)\,\omega^n, \quad [\text{cf. } (6.11) \text{ and } (7.6)]$$

defined on the n^2+n dimensional principal fibre bundle $\mathfrak{B}$ consisting of $\{\xi^l\}$ and $\{\omega^l_\mu\}$ may define an extended linear connection on a differentiable manifold M, it is necessary and sufficient that the conditions

$$\begin{aligned}(10.3)\qquad d\omega^l+\theta^l_h\omega^h &= \omega^l_\lambda(d^2x^\lambda+\Lambda^l_{\mu\nu}dx^\mu dx^\nu)+\tfrac{1}{2}S^l_{m\omega}\omega^m\omega^n\\ &= \omega^l_\lambda(d^2x^\lambda+\theta^\lambda_\mu dx^\mu)\\ &= \omega^l_\lambda(d^2x^\lambda+\tfrac{1}{2}S^\lambda_{\mu\nu}dx^\mu dx^\nu),\end{aligned}$$

$$\begin{aligned}(10.4)\qquad &(d\theta^k_l+\theta^k_h\theta^h_l)-(d\Theta^k_l+\Theta^k_h\Theta^h_l)-(\theta^k_h-\Theta^k_h)\tilde{\Theta}^h_l\\ &=\omega^k_\lambda\Omega^\mu_l\{(d\theta^\lambda_\mu+\theta^\lambda_\nu\theta^\nu_\mu)-(d\Theta^\lambda_\mu+\Theta^\lambda_\nu\Theta^\nu_\mu)-(\theta^\lambda_\nu-\Theta^\lambda_\nu)\Theta^\nu_\mu\}, \ [\Theta^h_l=0],\end{aligned}$$

$$\begin{aligned}(10.4)'\qquad &(Q^k_{.lpq}-L^k_{lh}\Lambda^h_{pq}-L^k_{hp}\Lambda^h_{lq})\,\omega^p_\alpha\omega^q_\beta\\ &=\omega^k_\lambda\Omega^\mu_l\{\tfrac{1}{2}(Q^\lambda_{.\mu\alpha\beta}-Q^\lambda_{.\mu\alpha\beta})-(L^\lambda_{\nu\mu}-\Lambda^\lambda_{\nu\mu})\Lambda^\nu_{\alpha\beta}\}\end{aligned}$$

$$\begin{aligned}d\omega^l+\theta^l_h\wedge\omega^h &= d\omega^l+\tfrac{1}{2}T^l_{mn}\omega^m\wedge\omega^n\\ &= \omega^l_\lambda(ddx^\lambda+\theta^\lambda_\mu\wedge dx^\mu), \qquad [ddx^\lambda=0]\\ &= \omega^l_\lambda(ddx^\lambda+\tfrac{1}{2}T^\lambda_{\mu\nu}dx^\mu\wedge dx^\nu),\end{aligned}$$

$$\begin{aligned}(d\theta^k_l+\theta^k_h\wedge\theta^h_l) &= \omega^k_\lambda\Omega^\mu_l\{(d\theta^\lambda_\mu+\theta^\lambda_\nu\wedge\theta^\nu_\mu)-(d\Theta^\lambda_\mu+\Theta^\lambda_\nu\wedge\Theta^\nu_\mu)\}\\ &= \omega^k_\lambda\Omega^\mu_l(d\theta^\lambda_\mu+\theta^\lambda_\nu\wedge\theta^\nu_\mu),\end{aligned}$$

$$\begin{aligned}R^k_{.lpq}\omega^p_\mu\omega^q_\nu &= \omega^k_\lambda\Omega^\mu_l(R^\lambda_{.\mu\alpha\beta}-R^\lambda_{.\mu\alpha\beta})\\ &= \omega^k_\lambda\Omega^\mu_l R^\lambda_{.\mu\alpha\beta}, \qquad [R^\lambda_{.\mu\alpha\beta}=0],\end{aligned}$$

are satisfied, where the **d** *denotes the exterior differentials and*

(10.5) $S^{l}_{mn} \overset{\text{def}}{=} \omega^{l}_{\lambda}\Omega^{\mu}_{m}\Omega^{\nu}_{n}\{S^{\lambda}_{\mu\nu}-(\Lambda^{\lambda}_{\nu\mu}+\Lambda^{\lambda}_{\mu\nu})\}$, $\quad T^{l}_{mn} \overset{\text{def}}{=} \omega^{l}_{\lambda}\Omega^{\mu}_{m}\Omega^{\nu}_{n}\{T^{\lambda}_{\mu\nu}-(\Lambda^{\lambda}_{\nu\mu}-\Lambda^{\lambda}_{\mu\nu})\}$,

(10.6)
$$Q^{k}_{.lmn} \overset{\text{def}}{=} \frac{\partial L^{k}_{ln}}{\omega^{m}}+\frac{\partial L^{k}_{lm}}{\omega^{n}}+(L^{s}_{lm}L^{k}_{sn}+L^{s}_{ln}L^{k}_{sm})=\omega^{k}_{\sigma}\Omega^{\lambda}_{l}\Omega^{\mu}_{m}\Omega^{\nu}_{n}\{(Q^{\sigma}_{.\lambda\mu\nu}-Q^{\sigma}_{.\lambda\mu\nu})-4\Lambda^{\tau}_{\mu\lambda}\Gamma^{\sigma}_{\nu\tau})\},$$
$$R^{\nu}_{.lmn} \overset{\text{def}}{=} \frac{\partial L^{k}_{ln}}{\omega^{m}}-\frac{\partial L^{k}_{lm}}{\omega^{n}}-(L^{s}_{lm}L^{k}_{sn}-L^{s}_{ln}L^{k}_{sm})=\omega^{k}_{\sigma}\Omega^{\lambda}_{l}\Omega^{\mu}_{m}\Omega^{\nu}_{n}R^{\sigma}_{.\lambda\mu\nu},$$

(10.7) $S^{\lambda}_{\mu\nu} \overset{\text{def}}{=} L^{\lambda}_{\nu\mu}+L^{\lambda}_{\mu\nu}$, $\quad T^{\lambda}_{\mu\nu} \overset{\text{def}}{=} L^{\lambda}_{\nu\mu}-L^{\lambda}_{\mu\nu}$,

(10.8)
$$\Omega^{\sigma}_{.\lambda\mu\nu} \overset{\text{def}}{=} \frac{\partial L^{\sigma}_{\lambda\nu}}{\partial x^{\mu}}+\frac{\partial L^{\sigma}_{\lambda\mu}}{\partial x^{\nu}}+(L^{\varepsilon}_{\lambda\mu}L^{\sigma}_{\varepsilon\nu}+L^{\varepsilon}_{\lambda\nu}L^{\sigma}_{\varepsilon\mu}).\qquad R^{\sigma}_{.\lambda\mu\nu} \overset{\text{def}}{=} \frac{\partial L^{\sigma}_{\lambda\nu}}{\partial x^{\mu}}-\frac{\partial L^{\sigma}_{\lambda\mu}}{\partial x^{\nu}}-(L^{\varepsilon}_{\lambda\mu}L^{\sigma}_{\varepsilon\nu}-L^{\varepsilon}_{\lambda\nu}L^{\sigma}_{\varepsilon\mu}).$$

(10.9)
$$\Theta^{\sigma}_{.\lambda\mu\nu} \overset{\text{def}}{=} \frac{\partial \Lambda^{\sigma}_{\lambda\nu}}{\partial x^{\mu}}+\frac{\partial \Lambda^{\sigma}_{\lambda\mu}}{\partial x^{\nu}}+(\Lambda^{\varepsilon}_{\lambda\mu}\Lambda^{\sigma}_{\varepsilon\nu}+\Lambda^{\varepsilon}_{\lambda\nu}\Lambda^{\sigma}_{\varepsilon\mu}).\qquad R^{\sigma}_{.\lambda\mu\nu} \overset{\text{def}}{=} \frac{\partial \Lambda^{\sigma}_{\lambda\nu}}{\partial x^{\mu}}-\frac{\partial \Lambda^{\sigma}_{\lambda\mu}}{\partial x^{\nu}}-(\Lambda^{\varepsilon}_{\lambda\mu}\Lambda^{\sigma}_{\varepsilon\nu}-\Lambda^{\varepsilon}_{\lambda\nu}\Lambda^{\sigma}_{\varepsilon\mu}).$$

Proof. *Necessity.* We have $\gamma_{p}\omega^{p}=\gamma_{p}\omega^{p}_{\sigma}dx^{\sigma}$ and (7.4)$'$:
$$\gamma_{l}(L^{h}_{kl}-\Lambda^{h}_{kl})=L^{\lambda}_{\mu\nu}\Omega^{\mu}_{k}\gamma_{l}\Omega^{\nu}_{l}\omega^{h}_{\lambda}.$$
Multiplying these in the form: $(\gamma_{q}L^{h}_{kq})(\gamma_{p}\omega^{p})$, we obtain
$$L^{h}_{kp}\omega^{p}+\underset{(p\neq q)}{\gamma_{q}\gamma_{p}}(L^{h}_{kq}\omega^{p})=\Lambda^{h}_{kp}\omega^{p}+\underset{(p\neq q)}{\gamma_{q}\gamma_{p}}(\Lambda^{h}_{kq}\omega^{p})+L^{\lambda}_{\mu\nu}\Omega^{\mu}_{k}dx^{\nu}\omega^{h}_{\lambda}+\underset{(p\neq q)}{\gamma_{q}\gamma_{p}}(L^{\lambda}_{\mu\nu}\Omega^{\mu}_{k}\Omega^{\nu}_{q}\omega^{h}_{\lambda}\omega^{p}_{\sigma}dx^{\sigma}),$$
$$\theta^{h}_{k}+\underset{(p\neq q)}{\gamma_{q}\gamma_{p}}L^{h}_{kq}\omega^{p}=\omega^{h}_{\lambda}\left(\frac{\partial\Omega^{\lambda}_{k}}{\omega^{p}}\omega^{p}+\theta^{\lambda}_{\mu}\Omega^{\mu}_{k}\right)+\underset{(p\neq q)}{\gamma_{q}\gamma_{p}}\left[\omega^{h}_{\lambda}\left(\frac{\partial\Omega^{\lambda}_{k}}{\omega^{q}}+L^{\lambda}_{\mu\nu}\Omega^{\mu}_{k}\Omega^{\nu}_{q}\right)\omega^{p}\right],$$
$$\theta^{h}_{k}+\underset{(p\neq q)}{\gamma_{q}\gamma_{p}}L^{h}_{kq}\omega^{p}=\omega^{h}_{\lambda}(d\Omega^{\lambda}_{k}+\theta^{\lambda}_{\mu}\Omega^{\mu}_{k})+\underset{(pq\neq)}{\gamma_{q}\gamma_{p}}\left[\omega^{h}_{\lambda}\left(\frac{\partial\Omega^{\lambda}_{k}}{\omega^{q}}+L^{\lambda}_{\mu\nu}\Omega^{\mu}_{k}\Omega^{\nu}_{q}\right)\omega^{p}\right], \tag{10.10}$$
whence follows:
$$\Omega^{\lambda}_{h}(\theta^{h}_{k}+\underset{(p\neq q)}{\gamma_{q}\gamma_{p}}L^{h}_{kq}\omega^{p})=d\Omega^{\lambda}_{k}+\theta^{\lambda}_{\mu}\Omega^{\mu}_{k}+\underset{(p\neq q)}{\gamma_{q}\gamma_{p}}\left(\frac{\partial\Omega^{\lambda}_{k}}{\omega^{q}}+L^{\lambda}_{\mu\nu}\Omega^{\mu}_{k}\Omega^{\nu}_{q}\right)\omega^{p},$$
so that
$$d\Omega^{\lambda}_{k}+\underset{(p\neq q)}{\gamma_{q}\gamma_{p}}\left(\frac{\partial\Omega^{\lambda}_{k}}{\omega^{q}}\omega^{p}\right)=(\Omega^{\lambda}_{h}\theta^{h}_{k}-\theta^{\lambda}_{\mu}\Omega^{\mu}_{k})+\underset{(p\neq q)}{\gamma_{q}\gamma_{p}}(\Omega^{\lambda}_{h}L^{h}_{qk}-L^{\lambda}_{\mu\nu}\Omega^{\mu}_{k}\Omega^{\nu}_{q})\omega^{p}. \tag{10.11}$$
Now we have
$$d\tilde{x}^{\lambda} \overset{\text{def}}{=} (\gamma_{q}\Omega^{\lambda}_{q})(\gamma_{p}\omega^{p})=dx^{\lambda}+\underset{(p\neq q)}{\gamma_{q}\gamma_{p}}\Omega^{\lambda}_{q}\omega^{p}. \tag{10.12}$$
Hence
$$d\,d\tilde{x}^{\lambda}=d(\gamma_{q}\Omega^{\lambda}_{q}\gamma_{p}\omega^{p})=d^{2}x^{\lambda}+\underset{(p\neq q)}{\gamma_{q}\gamma_{p}}d(\Omega^{\lambda}_{q}\omega^{p})$$
$$=d\Omega^{\lambda}_{p}\omega^{p}+\Omega^{\lambda}_{p}d\omega^{p}+\underset{(p\neq q)}{\gamma_{q}\gamma_{p}}(d\Omega^{\lambda}_{q}\omega^{p}+\Omega^{\lambda}_{q}d\omega^{p})$$

$$=(\Omega^\lambda_h\theta^h_p-\Omega^\mu_p\theta^\lambda_\mu)\,\omega^p+\Omega^\lambda_p d\omega^p+\underset{(p\neq q)}{\gamma_q\gamma_p}\,[(\Omega^\lambda_h\theta^h_q-\Omega^\mu_q\theta^\lambda_\mu)\,\omega^p+\Omega^\lambda_q d\omega^p]$$

$$=\Omega^\lambda_p\,(d\omega^p+\theta^p_l\omega^l)-dx^\mu\theta^\lambda_\mu+\underset{(p\neq q)}{\gamma_q\gamma_p}\,(\Omega^\lambda_q d\omega^p+\Omega^\lambda_h\theta^h_q\omega^p-\Omega^\mu_q\omega^p\theta^\lambda_\mu)$$

on the other hand. Therefore

$$\Omega^\lambda_p\,(d\omega^p+\theta^p_l\omega^l)+\underset{(p\neq q)}{\gamma_q\gamma_p}\,(\Omega^\lambda_q d\omega^p+\Omega^\lambda_h\theta^h_q\omega^p)$$
$$=d^2x^\lambda+dx^\mu\theta^\lambda_\mu+\underset{(p\neq q)}{\gamma_q\gamma_p}\,\{\Omega^\mu_q\theta^\lambda_\mu\omega^p+d\,(\Omega^\lambda_q\omega^p)\}. \tag{10.13}$$

Now

$$dx^\mu\theta^\lambda_\mu+\underset{(p\neq q)}{\gamma_q\gamma_p}\,(\Omega^\mu_q\theta^\lambda_\mu\omega^p)=\theta^\lambda_\mu dx^\mu+\underset{(p\neq q)}{\gamma_q\gamma_p}\,(\Omega^\mu_q\theta^\lambda_\mu\omega^p_\omega dx^\omega).$$

Hence

$$\Omega^\lambda_p\,(d\omega^p+\theta^p_h\omega^h)+\underset{(p\neq q)}{\gamma_q\gamma_p}\,(\Omega^\lambda_q d\omega^p+\Omega^\lambda_h\theta^h_q\omega^p)$$
$$=d^2x^\lambda+L^\lambda_{\mu\nu}dx^\mu dx^\nu+\underset{(p\neq q)}{\gamma_q\gamma_p}\,\{\Omega^\mu_q\theta^\lambda_\mu\omega^p_\omega dx^\omega+d\,(\Omega^\lambda_q\omega^p)\},\ (d\,(\Omega^\lambda_p\omega^p)=0), \tag{10.14}$$

from which we read out:

(10.15)

$$\begin{aligned}\Omega^\lambda_p\,(d\omega^p+\theta^p_h\omega^h)&=d^2x^\lambda+\theta^\lambda_\mu dx^\mu\\&=d^2x^\lambda+L^\lambda_{\mu\nu}dx^\mu dx^\nu\\&=d^2x^\lambda+\tfrac{1}{2}S^\lambda_{\mu\nu}dx^\mu dx^\nu.\end{aligned}
\qquad\Bigg|\qquad
\begin{aligned}\Omega^\lambda_p\,(d\omega^p+\theta^p_h\wedge\omega^h)&=ddx^\lambda+\theta^\lambda_\mu\wedge dx^\mu\\&=L^\lambda_{\mu\nu}dx^\nu\wedge dx^\mu,\quad(ddx^\lambda=0)\\&=\tfrac{1}{2}T^\lambda_{\mu\nu}dx^\mu\wedge dx^\nu.\end{aligned}$$

This formula maps the local paths

$$\frac{d^2x^\lambda}{dt^2}+L^\lambda_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}=0,\qquad\Bigg|\qquad\frac{(ddx^\lambda)}{dt^2}+L^\lambda_{\mu\nu}\frac{dx^\mu}{dt}\wedge\frac{dx^\nu}{dt}=0,$$

of the domain U_x of (x^σ) piece-wise continuingly into the global (II-geodesic) curves in $\bigcup\limits_\alpha U_\alpha$:

$$\frac{d}{dt}\frac{\omega^p}{dt}+L^p_{mn}\frac{\omega^m}{dt}\frac{\omega^n}{dt}=0.\qquad\Bigg|\qquad\frac{d\omega^p}{dt^2}+L^p_{mn}\frac{\omega^m}{dt}\wedge\frac{\omega^n}{dt}=0.$$

Further we have

$$d\,\{\gamma_q\gamma_p\,(\Omega^\lambda_q\theta^p_l)\}=(d\Omega^\lambda_p\theta^p_l+\Omega^\lambda_p d\theta^p_l)+\underset{(p\neq q)}{\gamma_q\gamma_p}\,(d\Omega^\lambda_q\theta^p_l+\Omega^\lambda_q d\theta^p_l)$$
$$=(\Omega^\lambda_h\theta^h_p-\theta^\lambda_\mu\Omega^\mu_p)\,\theta^p_l+\Omega^\lambda_p d\theta^p_l+\underset{(p\neq q)}{\gamma_q\gamma_p}\,\{(\Omega^\lambda_h\theta^h_q-\theta^\lambda_\mu\Omega^\mu_q)\,\theta^p_l+\Omega^\lambda_q d\theta^p_l\}$$
$$=\Omega^\lambda_p\,(d\theta^p_l+\theta^p_h\theta^h_l)-\Omega^\mu_p\theta^\lambda_\mu\theta^p_l+\underset{(p\neq q)}{\gamma_q\gamma_p}\,(\Omega^\lambda_q d\theta^p_l+\Omega^\lambda_h\theta^h_q\theta^p_l-\Omega^\mu_q\theta^\lambda_\mu\theta^p_l)$$

on one hand and

$$=d\Omega^\lambda_p\theta^p_l+\Omega^\lambda_p\,\{(d\Omega^\mu_l\omega^p_\mu+\Omega^\mu_l d\omega^p_\mu)\,(\theta^\omega_\mu-\Theta^\omega_\mu)+\Omega^\mu_l\omega^p_\omega d\,(\theta^\omega_\mu-\Theta^\omega_\mu)\}$$
$$+\mathfrak{D}\,[d\Omega^\lambda_p\wedge\theta^p_l+\Omega^\lambda_p\,\{(d\Omega^\mu_l\omega^p_\omega+\Omega^\mu_l d\omega^p_\omega)\wedge(\theta^\omega_\mu-\Theta^\omega_\mu)+\Omega^\mu_l\omega^p_\omega d\,(\theta^\omega_\mu-\Theta^\omega_\mu)\}$$
$$=d\Omega^\lambda_p\theta^p_l+\{d\Omega^\mu_l\,(\theta^\lambda_\mu-\Theta^\lambda_\mu)-\Omega^\mu_l\omega^p_\omega d\Omega^\lambda_p\,(\theta^\omega_\mu-\Theta^\omega_\mu)+\Omega^\mu_l d\,(\theta^\lambda_\mu-\Theta^\lambda_\mu)\}$$
$$+\mathfrak{D}\,[d\Omega^\lambda_p\wedge\theta^p_l+\{d\Omega^\mu_l\wedge(\theta^\lambda_\mu-\Theta^\lambda_\mu)-\Omega^\mu_l\omega^p_\omega d\Omega^\lambda_p\wedge(\theta^\omega_\mu-\Theta^\omega_\mu)+\Omega^\mu_l d\,(\theta^\lambda_\mu-\Theta^\lambda_\mu)\}]$$
$$=d\Omega^\mu_l\,(\theta^\lambda_\mu-\Theta^\lambda_\mu)+\Omega^\mu_l d\,(\theta^\lambda_\mu-\Theta^\lambda_\mu)+\mathfrak{D}\,[d\Omega^\mu_l\wedge(\theta^\lambda_\mu-\Theta^\lambda_\mu)+\Omega^\mu_l d\,(\theta^\lambda_\mu-\Theta^\lambda_\mu)]$$

by (7.5), (7.6), (7.11) and (7.12), the first and the third terms cancelling, on the other hand, so that

$$\begin{aligned}
&\Omega^\lambda_p\,(d\theta^p_l+\theta^p_h\theta^h_l)-\Omega^\mu_h\theta^\lambda_\mu\theta^h_l+\mathfrak{D}(\Omega^\lambda_p d\theta^p_l+\Omega^\lambda_h\theta^h_p\wedge\theta^p_l-\Omega^\mu_p\theta^\lambda_\mu\wedge\theta^p_l)\\
&\quad=d\Omega^\mu_l\,(\theta^\lambda_\mu-\Theta^\lambda_\mu)+\Omega^\mu_l d\,(\theta^\lambda_\mu-\Theta^\lambda_\mu)+\mathfrak{D}\,\{d\Omega^\nu_l\wedge(\theta^\lambda_\nu-\Theta^\lambda_\nu)+\Omega^\mu_l d\,(\theta^\lambda_\mu-\Theta^\lambda_\mu)\}\\
&\quad=-\Omega^\mu_l\Theta^\nu_\mu\,(\theta^\lambda_\nu-\Theta^\lambda_\nu)+\Omega^\mu_l d\,(\theta^\lambda_\mu-\Theta^\lambda_\mu)+\mathfrak{D}\,\{-\Omega^\mu_l\Theta^\nu_\mu\wedge(\theta^\lambda_\nu-\Theta^\lambda_\nu)+\Omega^\mu_l d\,(\theta^\lambda_\mu-\Theta^\lambda_\mu)\}
\end{aligned}$$

by (7.10). Thus we have

$$\begin{aligned}
&\Omega^\lambda_p\,(d\theta^p_l+\theta^p_h\theta^h_l)+\mathfrak{D}\{\Omega^\lambda_p\,(d\theta^p_l+\theta^p_h\wedge\theta^h_l)\\
&\quad=-\Omega^\mu_l\Theta^\nu_\mu\,(\theta^\lambda_\nu-\Theta^\lambda_\nu)+\theta^\lambda_\nu\Omega^\mu_l\,(\theta^\nu_\mu-\Theta^\nu_\mu)+\Omega^\mu_l d\,(\theta^\lambda_\mu-\Theta^\lambda_\mu)\\
&\qquad\qquad+\mathfrak{D}\{-\Omega^\mu_l\Theta^\nu_\mu\wedge(\theta^\lambda_\nu-\Theta^\lambda_\nu)+\theta^\lambda_\nu\wedge\Omega^\mu_l\,(\theta^\nu_\mu-\Theta^\nu_\mu)+\Omega^\mu_l d\,(\theta^\lambda_\mu-\Theta^\lambda_\mu)\}\\
&\quad=-2\Omega^\mu_l\Theta^\nu_\mu\theta^\lambda_\nu+\Omega^\mu_l\theta^\lambda_\nu\theta^\nu_\mu+\Omega^\mu_l\Theta^\nu_\mu\Theta^\lambda_\nu+\Omega^\mu_l d\,(\theta^\lambda_\mu-\Theta^\lambda_\mu)\\
&\qquad\qquad+\mathfrak{D}\Omega^\mu_l\,[\Theta^\lambda_\nu\wedge\Theta^\nu_\mu+\theta^\lambda_\nu\wedge\theta^\nu_\mu+d\,(\theta^\lambda_\mu-\Theta^\lambda_\mu)\}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
(10.17)\quad&\Omega^\lambda_p\,(d\theta^p_l+\theta^p_h\theta^h_l)+\mathfrak{D}\{\Omega^\lambda_p\,(d\theta^p_l+\theta^p_h\wedge\theta^h_l)\}\\
&\quad=\Omega^\mu_l\,\{-2\Theta^\nu_\mu\theta^\lambda_\nu+\theta^\lambda_\nu\theta^\nu_\mu+\Theta^\nu_\mu\Theta^\lambda_\nu+d\,(\theta^\lambda_\mu-\Theta^\lambda_\mu)\}\\
&\qquad\qquad+\mathfrak{D}\Omega^\mu_l\,\{\theta^\lambda_\nu\wedge\theta^\nu_\mu+\Theta^\nu_\mu\wedge\Theta^\lambda_\nu+d\,(\theta^\lambda_\mu-\Theta^\lambda_\mu)\}\\
&\quad=\Omega^\mu_l\,\{(d\theta^\lambda_\mu+\theta^\lambda_\nu\theta^\nu_\mu)-(d\Theta^\lambda_\mu-\Theta^\lambda_\nu\Theta^\nu_\mu)-2\Theta^\nu_\mu\theta^\lambda_\nu\}\\
&\qquad\qquad+\mathfrak{D}\Omega^\mu_l\,\{(d\theta^\lambda_\mu+\theta^\lambda_\nu\wedge\theta^\nu_\mu)-(d\Theta^\lambda_\mu+\Theta^\lambda_\nu\wedge\Theta^\nu_\mu)\}.
\end{aligned}$$

$$\begin{aligned}
(10.18)\quad&\Omega^\lambda_p\,(d\theta^p_l+\theta^p_h\theta^h_l)\\
&=\Omega^\mu_l\,\{(d\theta^\lambda_\mu+\theta^\lambda_\nu\theta^\nu_\mu)-(d\Theta^\lambda_\mu-\Theta^\lambda_\nu\Theta^\nu_\mu)\\
&\qquad\qquad-2\theta^\lambda_\nu\Theta^\nu_\mu\\
&=\Omega^\mu_l\,\{(d\theta^\lambda_\mu+\theta^\lambda_\nu\theta^\nu_\mu)-(d\Theta^\lambda_\mu+\Theta^\lambda_\nu\Theta^\nu_\mu)\\
&\qquad\qquad-2\,(\theta^\lambda_\nu-\Theta^\lambda_\nu)\,\Theta^\nu_\mu\}.
\end{aligned}$$

$$\begin{aligned}
&\Omega^\lambda_p\,(d\theta^p_l+\theta^p_h\wedge\theta^h_l)\\
&\quad=\Omega^\mu_l\,\{d\theta^\lambda_\mu+\theta^\lambda_\nu\wedge\theta^\nu_\mu)-(d\Theta^\mu_\mu+\Theta^\lambda_\nu\wedge\Theta^\nu_\mu)\}\\
&\quad=\Omega^\mu_l\,(d\theta^\lambda_\mu+\theta^\lambda_\nu\wedge\theta^\nu_\mu).
\end{aligned}$$

In order to bring these formulas to improved forms, we prepare two preliminary formulas. For the first place, we see $\omega^p=\omega^p_\nu dx^\nu$,

$$\omega^h_\lambda d^2x^\nu=d\omega^p-d\omega^p_\nu dx^\nu,$$

$$\omega^p_\nu d^2x^\nu=d\omega^p+\Lambda^p_{rs}\omega^r\omega^s,\quad [(7.8)]$$

$$L^k_{lp}\omega^p_\nu d^2x^\nu=\Omega^\mu_l\omega^k_\lambda\,(L^\lambda_{\mu\sigma}-\Lambda^\lambda_{\mu\sigma})\,d^2x^\sigma=L^k_{lr}\,(d\omega^p+\Lambda^p_{rs}\omega^r\omega^s),$$

$$(10.19)\qquad L^k_{lp}\omega^p_\nu d^2x^\nu=L^k_{lp}\,(d\omega^p+\tilde{\Theta}^p_r\omega^r).$$

For the second place, we seek for the expression for $\tilde{\Lambda}^l_{hk}$ in

$$(10.20)\qquad \frac{d}{dt}\,\frac{\omega^l}{dt}+\tilde{\Lambda}^l_{hk}\frac{\omega^h}{dt}\,\frac{\omega^k}{dt}=\omega^l_\lambda\Big(\frac{d^2x^\lambda}{dt^2}+\Lambda^\lambda_{\mu\nu}\frac{dx^\mu}{dt}\,\frac{dx^\nu}{dt}\Big).$$

For this purpose, we utilize the special line-elements, i. e. those of the II-geodesic curves ((2.14), (2.15)): $\omega^l=a^l dt,\ dx^\lambda=a^l\Omega^\lambda_l\,dt$. From (10.20), we have

$$\begin{aligned}
(10.21)\qquad \tilde{\Lambda}^l_{hk}\,a^h a^k&=\omega^l_\lambda\left(a^h\frac{d\Omega^\lambda_h}{dt}+\Lambda^\lambda_{\mu\nu}\Omega^\mu_h\Omega^\nu_k a^h a^k\right)\\
&=(\Lambda^l_{hk}+\omega^l_\lambda\Omega^\mu_h\Omega^\nu_k\Lambda^\lambda_{\mu\nu})\,a^h a^k
\end{aligned}$$

for all ratios of a^h. Thus we obtain

$$\tilde{\Lambda}^l_{hk} = \Lambda^l_{hk} + \omega^l_\lambda \Omega^\mu_h \Omega^\nu_k \Lambda^\lambda_{\mu\nu}, \quad \Theta^l_h = \tilde{\Theta}^l_h + \omega^l_\sigma \Omega^\mu_h \Theta^\sigma_\mu, \tag{10.22}$$

where

$$\Theta^l_h = 0, \quad \tilde{\Theta}^l_h = -\omega^l_\lambda \Omega^\mu_h \Theta^\lambda_\mu \tag{10.23}$$

as will be seen by comparing (10.20) with the identity (2.6).

For $\omega^k_\lambda \Omega^\mu_l (\theta^\lambda_\nu - \Theta^\lambda_\nu)\Theta^\nu_\mu$ in (10.4), we have

$$-\theta^k_p \tilde{\Theta}^p_l = -(\theta^k_p - \Theta^k_p)\tilde{\Theta}^p_l = \omega^k_\lambda \Omega^\mu_l (\theta^\lambda_\nu - \Theta^\lambda_\nu)\Theta^\nu_\mu$$

by virtue of (10.22), (10.23) and (7.6).

$$\begin{aligned} 0 &= d\Theta^l_k + \Theta^l_h \wedge \Theta^h_k \\ &= \omega^l_\lambda \Omega^\mu_h (d\Theta^\lambda_\mu + \Theta^\lambda_\nu \wedge \Theta^\nu_\mu) \\ &= \tfrac{1}{2}\omega^l_\lambda \Omega^\mu_k R^\lambda_{.\mu\alpha\beta} dx^\alpha \wedge dx^\beta. \end{aligned}$$

Thus (10.18) becomes

$$\begin{aligned} \Omega^\lambda_p \{(d\theta^p_l + \theta^p_h\theta^h_l) - (d\Theta^p_l + \Theta^p_h\Theta^h_l) & \\ - (d\theta^p_l - \Theta^p_h \Theta^h_l)\} & \\ = \Omega^\mu_l \{(d\theta^\lambda_\mu + \theta^\lambda_\nu\theta^\nu_\mu) - (d\Theta^\lambda_\mu + \Theta^\lambda_\nu\Theta^\nu_\mu) - (\theta^\lambda_\nu - \Theta^\lambda_\nu)\Theta^\nu_\mu\}, & \end{aligned} \tag{10.24}$$

$$\begin{aligned} &\Omega^\lambda_p (d\theta^p_l + \theta^p_h \wedge \theta^h_l) \\ &= \Omega^\mu_l \{(d\theta^\lambda_\mu + \theta^\lambda_\nu \wedge \theta^\nu_\mu) - (d\Theta^\lambda_\mu + \Theta^\lambda_\nu \wedge \Theta^\nu_\mu)\} \\ &= \Omega^\mu_l (d\theta^\lambda_\mu + \theta^\lambda_\nu \wedge \theta^\nu_\mu), \end{aligned}$$

or

$$\begin{aligned} &\Omega^\lambda_k (Q^k_{.lpq} - L^k_{lh}\Lambda^h_{pq} - L^k_{hp}\Lambda^h_{lq})\omega^p_\alpha\omega^q_\beta \\ &= \Omega^\mu_l \{\tfrac{1}{2}(Q^\lambda_{.\mu\alpha\beta} - \Theta^\lambda_{.\mu\alpha\beta}) - (L^\lambda_{\nu\alpha} - \Lambda^\lambda_{\nu\alpha})\Lambda^\nu_{\mu\beta}\}. \end{aligned} \tag{10.25}$$

$$\Omega^\lambda_p R^p_{.lhk}\omega^h_\alpha\omega^k_\beta = \Omega^\mu_l R^\lambda_{.\mu\alpha\beta}.$$

Further, by (10.19), we haye

$$L^k_{lp}(d\omega^p + \tilde{\Theta}^p_r\omega^r) \equiv L^k_{lp}\omega^p_\mu d^2x^\mu = \omega^k_\lambda \Omega^\mu_l (L^\lambda_{\mu\sigma} - \Lambda^\lambda_{\mu\sigma}) d^2x^\sigma,$$

so that (10.4)′ follows from (10.4).

Sufficiency. It suffices to determine

$$\theta^\lambda_\mu = L^\lambda_{\mu\nu} dx^\nu,$$

when ω^l are given and θ^j_i and Θ^τ_λ satisfy (10.15) and (10.17):

$$\begin{aligned} \Omega^\lambda_p (d\theta^p_l + \theta^p_h\theta^h_l) &+ \mathfrak{D}\{\Omega^\lambda_p (d\theta^p_l + \theta^p_h \wedge \theta^h_l)\} \\ &= \Omega^\lambda_p \{(d\theta^\lambda_\mu + \theta^\lambda_\nu\theta^\nu_\mu) - (d\Theta^\lambda_\mu - \Theta^\lambda_\nu\Theta^\nu_\mu) - 2\Theta^\nu_\mu\theta^\lambda_\nu\} \\ &\quad + \mathfrak{D}\{(d\theta^\lambda_\mu + \theta^\lambda_\nu \wedge \theta^\nu_\mu) - (d\Theta^\lambda_\mu + \Theta^\lambda_\nu \wedge \Theta^\nu_\mu)\}, \end{aligned}$$

where $L^\lambda_{\mu\nu}$ is a function of x^σ independent of Ω^λ_l. Now

$$\begin{aligned} d\omega^l &= d\omega^l_\nu dx^\nu + \omega^l_\mu d^2x^\mu \\ &= \omega^l_\nu \Omega^\nu_j d\omega^j_\mu dx^\mu + \omega^l_\mu d^2x^\mu \\ &= -(\omega^l_\nu d\Omega^\nu_j)(\omega^j_\mu dx^\mu) + \omega^l_\mu d^2x^\mu \\ &= -\omega^l_\nu d\Omega^\nu_j \omega^j + \omega^l_\mu d^2x^\mu, \end{aligned}$$

$$\begin{aligned} d\omega^p &= d\omega^p_\mu \wedge dx^\mu \\ &= \omega^p_\nu \Omega^\nu_j d\omega^j_\mu \wedge dx^\mu \\ &= -(\omega^p_\nu d\Omega^\nu_j) \wedge (\omega^j_\mu dx^\mu) \\ &= -\omega^p_\nu d\Omega^\nu_j \wedge \omega^j, \end{aligned}$$

$$dx^\mu = \Omega^\mu_j \omega^j_\nu dx^\nu.$$

If we introduce these values into (10.3):

$$\begin{aligned} \Omega^\lambda_p (d\omega^p + \theta^p_h\omega^h) &+ \gamma_q\gamma_p{}_{(p\neq q)} (\Omega^\lambda_q d\omega^p + \Omega^\lambda_h \theta^h_q \omega^p) \\ &= d^2x^\lambda + L^\lambda_{\mu\nu} dx^\mu dx^\nu + \gamma_q\gamma_p{}_{(p\neq q)} (\Omega^\mu_q \theta^\lambda_\mu \omega^p_\nu dx^\nu), \end{aligned}$$

then we obtain

$$\Omega^\lambda_p(-\omega^p_\nu d\Omega^\nu_l\omega^l+\omega^p_\mu d^2x^\mu+\theta^p_h\omega^h)+\underset{(p\neq q)}{\gamma_q\gamma_p}(-\Omega^\lambda_q\omega^p_\nu d\Omega^\nu_j\omega^j+\Omega^\lambda_h\theta^h_q\omega^p)$$
$$=d^2x^\lambda+\theta^\lambda_\mu\Omega^\mu_l\omega^l_\nu dx^\nu+\underset{(p\neq q)}{\gamma_q\gamma_p}(\Omega^\mu_q\theta^\lambda_\mu\omega^p_\omega\Omega^\omega_l\omega^l_\nu dx^\nu)^*.$$

The left-hand side is

$$\Omega^\lambda_p(-\omega^p_\nu d\Omega^\nu_j\omega^j+\theta^p_h\omega^h)+d^2x^\lambda+\underset{(p\neq q)}{\gamma_q\gamma_p}(-\Omega^\lambda_q\omega^p_\nu d\Omega^\nu_j\omega^j+\Omega^\lambda_h\theta^h_q\omega^p)$$
$$=(-d\Omega^\lambda_p+\Omega^\lambda_h\theta^h_p)\,\omega^p+\mathfrak{D}(-d\Omega^\lambda_p+\Omega^\lambda_h\theta^h_p)\wedge\omega^p=\theta^\lambda_\mu dx^\mu+\mathfrak{D}(\theta^\lambda_\mu\wedge dx^\mu)$$
$$=\tfrac{1}{2}S^{\ \lambda}_{\mu\nu}dx^\mu dx^\nu+\mathfrak{D}(\tfrac{1}{2}T^\lambda_{\mu\nu}dx^\mu\wedge dx^\nu)$$
$$=\tfrac{1}{2}S^\lambda_{\mu\nu}\Omega^\mu_p\omega^p dx^\nu+\mathfrak{D}(\tfrac{1}{2}T^{\ \lambda}_{\mu\nu}\Omega^\mu_p\omega^p\wedge dx^\nu),$$

what shows us by * that

$$-d\Omega^\lambda_p+\Omega^\lambda_h\theta^h_p+\mathfrak{D}(-d\Omega^\lambda_p+\Omega^\lambda_h\theta^h_p)=\theta^\lambda_\mu\Omega^\mu_p+\mathfrak{D}(\Omega^\mu_p\theta^\lambda_\mu)$$
$$=\tfrac{1}{2}S^\lambda_{\mu\nu}\Omega^\mu_p dx^\nu-\mathfrak{D}(\tfrac{1}{2}T^{\ \lambda}_{\mu\nu}\Omega^\mu_p dx^\nu)$$

is a linear combination of ω^l. It suffices now to show that $L^\lambda_{\mu\nu}$ in

$$\theta^\lambda_\mu\overset{\text{def}}{=\!=}L^\lambda_{\mu\nu}dx^\nu,$$

which we will define thus by (7.11):

$$\Omega^\mu_h\theta^h_l-d\Omega^\mu_l=\Omega^\nu_l\theta^\mu_\nu,$$

are functions of x^λ only, independent of Ω^λ_l.

From (7.11), we have

$$\begin{array}{ll|l}
(10.27) & d\Omega^\lambda_h\theta^h_l+\Omega^\lambda_h d\theta^h_l & d\Omega^\lambda_h\wedge\theta^h_l+\Omega^\lambda_h d\theta^h_l \\
 & \quad =d\Omega^\mu_l\theta^k_h+\Omega^\mu_l d\theta^\lambda_\mu+d^2\Omega^\lambda_l. & \quad =d\Omega^\mu_l\wedge\theta^\lambda_\mu+\Omega^\mu_l d\theta^\lambda_\mu,\quad (dd\Omega^\mu_l=0).
\end{array}$$

If we take (10.18), (10.4) and

$$\begin{array}{l|l}
d\Omega^\lambda_h\theta^k_l=(\Omega^\lambda_k\theta^k_h-\theta^\lambda_\mu\Omega^\mu_h)\,\theta^h_l & d\Omega^\lambda_h\wedge\theta^h_l=(\Omega^\lambda_k\theta^k_h-\theta^\lambda_\mu\Omega^\mu_h)\wedge\theta^h_l
\end{array}$$

into account, then we obtain

$$\begin{array}{l|l}
(\Omega^\lambda_l\theta^l_k-\theta^\lambda_\mu\Omega^\mu_k)\,\theta^k_i & (\Omega^\lambda_l\theta^l_k-\theta^\lambda_\mu\Omega^\mu_k)\wedge\theta^k_i \\
\quad +\Omega^\lambda_k(Q^k_{.imn}\omega^m\omega^n-\theta^j_i\theta^k_j+L^k_{ij}d\omega^j) & \quad +\Omega^\lambda_k(R^k_{.imn}\omega^m\wedge\omega^n+\theta^j_i\wedge\theta^k_j) \\
\quad =(\Omega^\mu_l\theta^l_i-\theta^\mu_\nu\Omega^\nu_i)\,\theta^\lambda_\mu+\Omega^\mu_i d\theta^\lambda_\mu+d^2\Omega^\lambda_i, & \quad =(\Omega^\mu_l\theta^l_i-\theta^\mu_\nu\Omega^\nu_i)\wedge\theta^\lambda_\mu+\Omega^\mu_i d\theta^\lambda_\mu, \\
\Omega^\mu_i(d\theta^\lambda_\mu-\theta^\alpha_\mu\theta^\lambda_\alpha)+d^2\Omega^\lambda_i+2\Omega^\mu_l\theta^l_i\theta^\lambda_\mu & \Omega^\mu_i(d\theta^\lambda_\mu-\theta^\alpha_\mu\wedge\theta^\lambda_\alpha) \\
\quad =\Omega^\lambda_l(\tfrac{1}{2}Q^l_{.imn}\omega^m\omega^n+L^l_{ij}d\omega^j), & \quad =\tfrac{1}{2}\Omega^\nu_l R^l_{.imn}\omega^m\wedge\omega^n,
\end{array}$$

whence we see that

$$\begin{array}{l|l}
d\theta^\lambda_\mu-L^\lambda_{\mu\nu}d^2x^\nu & d\theta^\lambda_\mu
\end{array}$$

is a quadratic differential form in dx^λ on one hand, since the terms of d^2x^ν cancell among themselves by virtue of the relations (7.10) and (7.4)'':

$$L^\lambda_{\nu\mu}-\Lambda^\lambda_{\nu\mu}=\Omega^\lambda_l\omega^m_\mu\omega^n_\nu L^l_{mn}.$$

On the other hand, we have $\theta^\lambda_\mu=L^\lambda_{\mu\nu}dx^\nu$,

$$\begin{array}{l|l}
d\theta^\lambda_\mu-L^\lambda_{\mu\nu}d^2x^\nu=\dfrac{\partial L^\lambda_{\mu\lambda}}{\partial x^\kappa}dx^\kappa dx^\nu+\dfrac{\partial L^\lambda_{\mu\nu}}{\partial\Omega^\alpha_a}d\Omega^\alpha_a dx^\nu, & d\theta^\lambda_\mu=\dfrac{\partial L^\lambda_{\mu\nu}}{\partial x^\kappa}dx^\kappa\wedge dx^\nu+\dfrac{\partial L^\lambda_{\mu\nu}}{\partial\Omega^\alpha_a}d\Omega^\alpha_a\wedge dx^\nu,
\end{array}$$

where

$$\frac{\partial L^{\lambda}_{\mu\nu}}{\partial \Omega^{\alpha}_{a}} d\Omega^{\alpha}_{a} = \frac{\partial L^{\lambda}_{\mu\nu}}{\partial \Omega^{\alpha}_{a}} \frac{\partial \Omega^{\alpha}_{a}}{\partial c^{r}} dc^{r} = \frac{\partial L^{\lambda}_{\mu\nu}}{\partial c^{r}} dc^{r},$$

the c^r being the ordinary transformation parameters contained in ω^l_μ and Ω^λ_l. Therefore we must have $\frac{\partial L^{\lambda}_{\mu\nu}}{\partial c^{r}}=0$, so that $L^{\lambda}_{\mu\nu}$ does not contain c^r. If we had $\frac{\partial L^{\lambda}_{\mu\nu}}{\partial \Omega^{\alpha}_{a}} \neq 0$, then $L^{\lambda}_{\mu\nu}$ must have contained Ω^{α}_{a}, so that $L^{\lambda}_{\mu\nu}$ must have contained c_r, contrary to the last result, since $L^{\lambda}_{\mu\nu}$ could have contained c^r only through Ω^{α}_{a}. Hence we must have $\frac{\partial L^{\lambda}_{\mu\nu}}{\partial \Omega^{\alpha}_{a}}=0$,*) Q. E. D.

A duality in the linear connections in the large. A duality is read out from the last theorem, since theorems may be established for other linear connections in the large. ([1], [2], [3], [4], [5], [6], [8], [9], [19], [20], [21], [22]).

11. Bianchi Identities and Some Formulas.

Formulas.

(11.1) $$\omega^l_\lambda S^\lambda_{\mu\nu} dx^\mu dx^\nu = [(\Lambda^l_{nm}+\Lambda^l_{mn})+(L^l_{nm}+L^l_{mn})]\, \omega^m\omega^n,$$

(11.2) $$Q^{k}_{.lmn}\omega^m\omega^n+2L^k_{hl}d\omega^h = \Big[\frac{\partial L^k_{ln}}{\omega^m}+\frac{\partial L^k_{lm}}{\omega^n}+(L^k_{jm}L^j_{ln}+L^k_{jn}L^j_{lm}) + L^k_{ij}(\Lambda^j_{nm}+\Lambda^j_{mn})\Big]\omega^m\omega^n+2L^k_{hl}\omega^h_\mu d^2x^\mu;$$

(11.3) $$\Omega^\lambda_l S^l_{mn}\omega^m\omega^n = -(\Lambda^\lambda_{\mu\nu}+\Lambda^\lambda_{\nu\mu})+(L^\lambda_{\mu\nu}+L^\lambda_{\nu\mu})]\, dx^\mu dx^\nu,$$

(11.4) $$Q^{\kappa}_{.\lambda\mu\nu}dx^\mu dx^\nu+2L^\kappa_{\tau\lambda}d^2x^\tau = \Big[\frac{\partial L^\kappa_{\lambda\mu}}{\partial x^\nu}+\frac{\partial L^\kappa_{\lambda\nu}}{\partial x^\mu}+(L^\kappa_{\mu\tau}L^\tau_{\nu\lambda}+L^\kappa_{\nu\tau}L^\tau_{\mu\lambda}) + L^\kappa_{\lambda\pi}(\Lambda^\tau_{\nu\mu}+\Lambda^\tau_{\mu\nu})\Big]\, dx^\mu dx^\nu+2L^{n}_{\tau\ \lambda}\Omega^\tau_m d\omega^m.$$

$$\omega^l_\lambda T^\lambda_{\mu\nu}dx^\mu\wedge dx^\nu = [(\Lambda^l_{nm}-\Lambda^l_{mn})+(L^l_{nm}-L^l_{mn})]\, \omega^m\wedge\omega^n,$$

$$R^{k}_{.lmn}\omega^m\wedge\omega^n = \Big[\frac{\partial L^k_{ln}}{\omega^m}-\frac{\partial L^k_{lm}}{\omega^m}+(L^k_{jm}L^j_{ln}-L^k_{jn}L^j_{lm}) + L^k_{ij}(\Lambda^j_{nm}-\Lambda^j_{mn})\Big]\, \omega^m\wedge\omega^n;$$

$$\Omega^\lambda_l T^l_{mn}\omega^m\wedge\omega^n = [(\Lambda^\lambda_{\mu\nu}-\Lambda^\lambda_{\nu\mu})+(L^\lambda_{\mu\nu}-L^\lambda_{\nu\mu})]\, dx^\mu\wedge dx^\nu,$$

$$R^{\kappa}_{.\lambda\mu\nu}dx^\mu\wedge dx^\nu = \Big[\frac{\partial L^\kappa_{\lambda\mu}}{\partial x^\nu}-\frac{\partial L^\kappa_{\lambda\nu}}{\partial x^\mu}+(L^\kappa_{\mu\tau}L^\tau_{\nu\lambda}-L^\kappa_{\nu\tau}L^\tau_{\mu\lambda}) + L^\kappa_{\lambda\tau}(\Lambda^\tau_{\nu\mu}-\Lambda^\tau_{\mu\nu})\Big]\, dx^\mu\wedge dx^\nu.$$

Proof. $\omega^l_\mu d^2x^\mu=\omega^l_\mu d(\Omega^\mu_m\omega^m)=\omega^l_\mu d\left(\frac{\partial\Omega^\mu_m}{\omega^n}\omega^m\omega^n+\Omega^\mu_m d\omega^m\right)=\Lambda^l_{mn}\omega^m\omega^n+d\omega^l$,

$$\begin{aligned} d\omega^l+\theta^l_k\omega^k &= \frac{\partial\omega^l_\mu}{\omega^n}\omega^n(\Omega^\mu_m\omega^m) + (L^l_{mn}\omega^m)\,\omega^n+\omega^l_\mu d^2x^\mu \\ &= \Omega^\mu_m\frac{\partial\omega^l_\mu}{\omega^n}\,\omega^m\omega^n+L^l_{mn}\omega^m\omega^n \\ &= \tfrac{1}{2}\Big[\Big(\Omega^\mu_m\frac{\partial\omega^l_\mu}{\omega^n}+\Omega^\mu_n\frac{\partial\omega^l_\mu}{\omega^m}\Big) + (L_{mn}+L^l_{nm})\Big]\,\omega^m\omega^n+\omega^l_\mu d^2x^\mu \\ &= \tfrac{1}{2}\,[-(\Lambda^l_{mn}+\Lambda^l_{nm})+(L^l_{mn}+L^l_{nm})]\omega^m\omega^n + \omega^l_\mu d^2x^\mu. \end{aligned}$$

$$\begin{aligned} d\omega^l+\theta^l_n\wedge\omega^n &= \frac{\partial\omega^l_\mu}{\omega^n}\omega^n\wedge(\Omega^\mu_m\omega^m) + (L^l_{nm}\omega^m)\wedge\omega^n \\ &= \Omega^\mu_m\frac{\partial\omega^l_\mu}{\omega^n}\omega^n\wedge\omega^m+L^l_{mn}\omega^m\wedge\omega^n \\ &= \tfrac{1}{2}\Big[\Big(\Omega^\mu_m\frac{\partial\omega^l_\mu}{\omega^n}-\Omega^\mu_n\frac{\partial\omega^l_\mu}{\omega^m}\Big) + (L^l_{mn}-L^l_{nm})\,\omega^m\wedge\omega^n \\ &= \tfrac{1}{2}[-(\Lambda^l_{mn}-\Lambda^l_{nm})+(L^l_{mn}-L^l_{nm})]\ \omega^m\wedge\omega^n. \end{aligned}$$

*) This contradicts in case $L^\lambda_{\mu\nu}=\{{}^{\lambda}_{\mu\nu}\}$ *only apparently* with (14.10). Indeed the expressibility (14.9) is valid only but for undergoing extended orthogonal transformations of the type $a^l_m(\xi^p, c^r)$.

$$d\theta^k_l+\theta^k_t\theta^t_l=d\,(L^k_{lm}\omega^m)+(L^j_{lm}\omega^m)\,(L^k_{jn}\omega^n)$$
$$=\tfrac{1}{2}\Big[\Big(\frac{\partial L^k_{lm}}{\omega^n}+\frac{\partial L^k_{ln}}{\omega^m}\Big)+(L^j_{ln}L^k_{jm}+L^j_{lm}L^k_{jn})$$
$$+L^k_{jl}\Big(\frac{\partial\omega^j_{\ \mu}}{\omega^m}\Omega^\mu_{\ n}+\frac{\partial\omega^l_{\ \mu}}{\omega^n}\Omega^\mu_{\ m}\Big)\Big]\,\omega^m\omega^n$$
$$+L^k_{lh}\omega^h_\mu d^2x^\mu,$$

$$d\theta^k_l+\theta^k_t\wedge\theta^t_l=d\,(L^k_{lm}\omega^m)-(L^j_{lm}\omega^m)\wedge(L^k_{jn}\omega^n)$$
$$=\tfrac{1}{2}\Big[\Big(\frac{\partial L^k_{lm}}{\omega^n}-\frac{\partial L^k_{ln}}{\omega^m}\Big)+(L^j_{ln}L^k_{jm}-L^j_{lm}L^k_{jn})$$
$$+L^k_{jl}\Big(\frac{\partial\omega^j_{\ \mu}}{\omega^m}\Omega^\mu_n-\frac{\partial\omega^j_{\ \mu}}{\omega^n}\Omega^\mu_{\ m}\Big)\Big]\,\omega^m\wedge\omega^n,$$

the (10.3) and the (10.4) being taken into account.

Similarly for the formulas with Greek indices.

Notation.

$$(11.5)\qquad \mathfrak{S}^l \overset{\text{def}}{=\!=} \tfrac{1}{2}S^l_{mn}\omega^m\omega^n, \qquad\Big|\qquad \mathfrak{T}^l \overset{\text{def}}{=\!=} \tfrac{1}{2}T^l_{mn}\omega^m\wedge\omega^n,$$
$$(11.6)\qquad \mathfrak{S}^l_k \overset{\text{def}}{=\!=} \tfrac{1}{2}Q^l_{.kmn}\omega^m\omega^n. \qquad\Big|\qquad \mathfrak{T}^l_k \overset{\text{def}}{=\!=} \tfrac{1}{2}R^l_{.kmn}\omega^m\wedge^n.$$

Bianchi Identities.

$$(11.7)\qquad d\mathfrak{S}^l+\mathfrak{S}^t\theta^l_t-\mathfrak{S}^l_t\omega^t=2\theta^l_h d\omega^h, \qquad\Big|\qquad d\mathfrak{T}^l+\mathfrak{T}^t\wedge\theta^l_t+\mathfrak{T}^l_t\wedge\omega^t=0,$$
$$(11.8)\qquad d\mathfrak{S}^l_k-\mathfrak{S}^t_k\theta^l_t-\mathfrak{S}^l_t\theta^t_k=-d(dL^l_{kh}\omega^h)+(\theta^j_kL^l_{hj}+\theta^l_jL^j_{hk})d\omega^h+d^2\theta^l_k-2\theta^j_k\theta^l_h\theta^h_j. \qquad\Big|\qquad d\mathfrak{T}^l_k+\mathfrak{T}^t_k\wedge\theta^l_t+\mathfrak{T}^l_t\wedge\theta^t_k=0.$$

Proof. Owing to (11.5) and (11.6), the equations of structure (10.3) and (10.4) become

$$(10.3)'\qquad d\omega^l+\theta^l_k\omega^k=d\omega^l+\mathfrak{S}^l, \qquad\Big|\qquad d\omega^l+\theta^l_k\wedge\omega^k=\mathfrak{T}^l,$$
$$(10.4)'\qquad d\theta^l_k+\theta^l_j\theta^j_{\ k}=L^l_{hk}d\omega^h+\mathfrak{S}^l_k. \qquad\Big|\qquad d\theta^l_k+\theta^l_j\wedge\theta^j_{\ k}=\mathfrak{T}^l_k.$$

Applying the operator

$$d \qquad\Big|\qquad d$$

to (10.3)′, we obtain

$$-d\omega^k\theta^l_k+\omega^i d\theta^l_k=d\mathfrak{S}^l,$$
$$-d\omega^k\theta^l_k+\omega^k\,(\mathfrak{S}^l_k+L^l_{hk}d\omega^h-\theta^j_{\ k}\theta^l_j$$
$$=d\mathfrak{S}^l,$$
$$2\,\theta^l_kd\omega^k+\mathfrak{S}^l_k\omega^k-\theta^l_j\mathfrak{S}^j=d\mathfrak{S}^l,$$

$$-d\omega^k\wedge\theta^l_k+\omega^k\wedge d\theta^l_k=d\mathfrak{T}^l,$$
$$-(\mathfrak{T}^k+\omega^j\wedge\theta^k_j)\wedge\theta^l_k+\omega^k\wedge(\mathfrak{T}^l_k+\theta^j_k\wedge\theta^l_j)$$
$$=d\mathfrak{T}^l,$$
$$-\mathfrak{T}^l_t\wedge\omega^t-\mathfrak{T}^t\wedge\theta^l_t=d\mathfrak{T}^l,$$

wnence follows (11.7).

Similarly from (10.4)′, we obtain

$$d^2\theta^l_k+d\theta^j_{\ k}\theta^l_j-\theta^j_{\ k}d\theta^l_j$$
$$=d\,(L^l_{hk}d\omega^h)+d\mathfrak{S}^l_k,$$
$$-(L^j_{\ hk}d\omega^h+\mathfrak{S}^j_{\ k}-\theta^m_k\theta^j_{\ m})\theta^l_j+\theta^j_{\ k}(L^l_{hj}d\omega^h+\mathfrak{S}^l_j-\theta^m_j\theta^l_m)$$
$$=d\,(L^l_{hk}d\omega^h)+d\mathfrak{S}^l_k,$$

$$-(d\theta^j_{\ k}\wedge\theta^l_j+\theta^j_{\ k}\wedge d\theta^l_j)$$
$$=d\mathfrak{T}^l_k,$$
$$-(\mathfrak{T}^l_k+\theta^m_k\wedge\theta^j_{\ m})\wedge\theta^l_j$$
$$+\theta^j_k\wedge(\mathfrak{T}^l_j+\theta^m_j\wedge\theta^l_m)=d\mathfrak{T}^l_k,$$

whence follows (11.8).

Cor. *For the linear connections under consideration,* (11.7) *and* (11.8) *become*

(11.9)
$$Q^k_{.lmn}+Q^k_{.mnl}+Q^k_{.nlm} = \frac{\partial S^k_{mn}}{\omega^l}+\frac{\partial S^k_{nl}}{\omega^m}+\frac{\partial S^k_{lm}}{\omega^n} +(S^j_{mn}L^k_{jl}+S^j_{nl}L^k_{jm}+S^j_{lm}L^k_{nj}),$$

$$R^k_{.lmn}+R^k_{.mnl}+R^k_{.nlm} = \frac{\partial T^k_{mn}}{\omega^l}+\frac{\partial T^k_{nl}}{\omega^m}+\frac{\partial T^k_{lm}}{\omega^n} +(T^j_{mn}L^k_{jl}+T^j_{nl}L^k_{jm}+T^j_{lm}L^k_{nj}),$$

(11.10)
$$\frac{\partial Q^k_{.lmn}}{\omega^s}+\frac{\partial Q^k_{.lns}}{\omega^m}+\frac{\partial Q^k_{.lsm}}{\omega^n} =(Q^k_{.jmn}L^j_{ls}+Q^k_{.jns}L^j_{lm}+Q^k_{.jsm}L^j_{ln}) -(Q^j_{.lmn}L^k_{js}+Q^j_{.lns}L^k_{jm}+Q^j_{.lsm}L^k_{jn})$$
$$+\frac{\partial^2\Gamma^l_{hn}}{\omega^s\omega^m}+\frac{\partial^2\Gamma^l_{ks}}{\omega^m\omega^n}+\frac{\partial^2\Gamma^l_{km}}{\omega^n\omega^s}-2\,(L^j_{ks}L^l_{tm}L^t_{jn}+L^j_{km}L^l_{tn}L^t_{js}+L^j_{kn}L^l_{ts}L^t_{jm}).$$

$$\frac{\partial R^k_{.lmn}}{\omega^s}+\frac{\partial R^k_{.lns}}{\omega^m}+\frac{\partial R^k_{.lsm}}{\omega^n} =(R^k_{.jmn}L^j_{ls}+R^k_{.jns}L^j_{lm}+R^k_{.jsm}L^j_{ln}) -(R^j_{.lmn}L^k_{js}+R^j_{.lns}L^k_{jm}+R^j_{.lsm}L^k_{jn}),$$

Proof. $d\mathfrak{S}^l+\mathfrak{S}^t\theta^l_t-\mathfrak{S}^l_t\omega^t=2\theta^l_h d\omega^h$,

$$d\mathfrak{S}^l=-\tfrac12 d\,(S^l_{mn}\omega^m\omega^n) =\tfrac12\frac{\partial S^l_{mn}}{\omega^p}\omega^p\omega^m\omega^n+S^l_{mn}d\omega^m\omega^n,$$

$$\mathfrak{S}^t\theta^l_t=\tfrac12\,(S^t_{mn}\omega^m\omega^n)\,(L^l_{tp}\omega^p) =\tfrac12\, S^t_{mn}L^l_{tp}\omega^p\omega^m\omega^n,$$

$$\mathfrak{S}^l_t\omega^t=\tfrac12\,(Q^l_{.pmn}\omega^m\omega^n)\,\omega^p =\tfrac12\, Q^l_{.pmn}\omega^p\omega^m\omega^n,$$

$$-2\theta^l_h\omega^h=-S^l_{mn}d\omega^m\omega^n.$$

$d\mathfrak{T}^l+\mathfrak{T}^t\wedge\theta^l_t+\mathfrak{T}^l_t\wedge\omega^t=0$,

$$d\mathfrak{T}^l=\tfrac12\, d\,(T^l_{mn}\omega^m\wedge\omega^n) =\tfrac12\frac{\partial T^l_{mn}}{\omega^p}\omega^p\wedge\omega^m\wedge\omega^n+T^l_{mn}d(\omega^m\wedge\omega^n)$$

$$=\tfrac12\frac{\partial T^l_{mn}}{\omega^p}\omega^p\wedge\omega^m\wedge\omega^n$$ by virtue of $$T^l_{mn}+T^l_{nm}=0,$$

$$\mathfrak{T}^t\wedge\theta^l_t=\tfrac12\,(T^l_{mn}\omega^m\wedge\omega^n)\wedge(L^l_{tp}\omega^p) =\tfrac12 T^t_{mn}L^l_{tp}\omega^p\wedge\omega^m\wedge\omega^n,$$

$$\mathfrak{T}^l_t\wedge\omega^t=\tfrac12(R^l_{.pmn}\omega^m\wedge\omega^n)\wedge\omega^p =\tfrac12 R^l_{.pmn}\omega^p\wedge\omega^m\wedge\omega^n,$$

Hence

$$\left(\frac{\partial S^l_{mn}}{\omega^p}+S^k_{mn}L^l_{kp}\right)\omega^p\omega^m\omega^n =Q^l_{.pmn}\omega^p\omega^m\omega^n,$$

$$Q^k_{.lmn}+Q^k_{.mnl}+Q^k_{.nlm} =\sum_{p<m<n}\left[\frac{\partial S^l_{mn}}{\omega^p}+\frac{\partial S^l_{np}}{\omega^m}+\frac{\partial S^l_{pm}}{\omega^n} +(S^k_{mn}L^l_{kp}+S^k_{np}L^l_{km}+S^s_{pm}L^l_{kn})\right]\omega^p\omega^m\omega^n,$$

$$\left(\frac{\partial T^l_{mn}}{\omega^p}+T^k_{mn}L^l_{kp}\right)\omega^p\wedge\omega^m\wedge\omega^n =R^l_{.pmn}\omega^p\wedge\omega^m\wedge\omega^n,$$

$$R^k_{.lmn}+R^k_{.mnl}+R^k_{.nlm} =\sum_{p<m<n}\left[\frac{\partial T^l_{mn}}{\omega^p}+\frac{\partial T^l_{np}}{\omega^m}+\frac{\partial T^l_{pm}}{\omega^n} +(T^k_{mn}L^l_{kp}+T^k_{np}L^l_{km}+T^k_{pm}L^l_{kn})\right]\omega^p\wedge\omega^m\wedge\omega^n,$$

whence (11.9) follows by the arbitrariness of $\omega^p\wedge\omega^m\wedge\omega^n$, $(p<m<n)$.

Further,

$$d\mathfrak{S}^l_k-\mathfrak{S}^t_k\theta^l_t-\theta^t_k\mathfrak{S}^l_t=-d\,(L^l_{hk}d\omega^h) +(\theta^j_kL^l_{hj}+\theta^l_jL^j_{hk})d\omega^h+d^2\theta^l_k-2\theta^j_k\theta^l_h\theta^h_j,$$

$$d\mathfrak{S}^l_k=\tfrac12\frac{\partial Q^l_{.kmn}}{\omega^p}\omega^p\omega^m\omega^n+\tfrac12 Q^l_{.kmn}d(\omega^m\omega^n),$$

$$\mathfrak{S}^t_k\theta^l_t=\tfrac12 Q^t_{.kmn}L^l_{tp}\omega^p\omega^m\omega^n,$$

$$\theta^t_k\mathfrak{S}^l_t=\tfrac12 Q^l_{.tmn}L^t_{kp}\omega^p\omega^m\omega^n.$$

$$d\mathfrak{T}^l_k+\mathfrak{T}^t_k\wedge\theta^l_t-\theta^t_k\wedge\mathfrak{T}^l_t=0,$$

$$d\mathfrak{T}^l_k=\tfrac12\frac{\partial R^l_{.kmn}}{\omega^p}\omega^p\wedge\omega^m\wedge\omega^n,$$

$$\mathfrak{T}^t_k\wedge\theta^l_t=\tfrac12 R^t_{.kmn}L^l_{tp}\omega^p\wedge\omega^m\wedge\omega^n,$$

$$\theta^t_k\wedge\mathfrak{T}^l_t=\tfrac12 R^l_{.tmn}L^t_{kp}\omega^p\wedge\omega^m\wedge\omega^n.$$

Hence

$$\left(\frac{\partial Q^l_{.kmn}}{\omega^p}-Q^l_{.kmn}L^l_{tp}-Q^l_{.tmn}L^t_{kp}\right)\omega^p\omega^m\omega^n + \tfrac{1}{2}Q^l_{.kmn}d(\omega^m\omega^n) = \left(\frac{\partial^2 L^l_{kn}}{\omega^s\omega^m}-2L^j_{ks}L^l_{tm}L^t_{jn}\right)\omega^s\omega^m\omega^n+\tfrac{1}{2}Q^l_{.kmn}d(\omega^m\omega^n),$$

$$\left(\frac{\partial R^l_{.kmn}}{\omega^p}+R^t_{.kmn}L^l_{tp} - R^l_{.tmn}L^t_{kp}\right)\omega^p\wedge\omega^m\wedge\omega^n=0,$$

since (11.8):

$$d\mathfrak{C}^l_k-\mathfrak{C}^t_k\theta^l_t-\mathfrak{C}^l_t\theta^t_k=-d\,(L^l_{hk}d\omega^h)+(\theta^j_kL^l_{hj}+\theta^l_jL^j_{hk})\,d\omega^h+d^2\theta^l_k-2\theta^j_k\theta^t_t\theta^l_j,$$

$$[d\theta^l_k=dL^l_{hk}\omega^h+L^l_{kh}d\omega^h]$$

$$=-d\,(dL^l_{hk}\omega^h)+(\theta^j_kL^l_{hj}+\theta^l_jL^j_{hk})\,d\omega^h-2\theta^j_h\theta^t_t\theta^l_j$$

$$=\frac{\partial^2 L^l_{kh}}{\omega^s\omega^m}\omega^s\omega^m\omega^n+\frac{\partial L^l_{jkm}}{\omega^n}\,d\,(\omega^m\omega^n)+L^j_{kn}L^l_{mj}\omega^n d\omega^m+L^l_{jm}L^j_{hk}\omega^m d\omega^n -2L^j_{ks}L^l_{tm}L^t_{jn}\omega^s\omega^m\omega^n$$

$$=\left(\frac{\partial^2 L^l_{kn}}{\omega^s\omega^m}-2L^j_{ks}L^l_{tm}L^t_{jn}\right)\omega^s\omega^m\omega^n+\frac{\partial L^l_{km}}{\omega^n}\,(\omega^n d\omega^m+\omega^m d\omega^n) +(L^j_{kn}L^l_{mj}+L^l_{jn}L^j_{mk})\,\omega^n d\omega^m$$

$$=\left(\frac{\partial^2 L^l_{kn}}{\omega^s\omega^m}-2L^j_{ks}L^l_{tm}L^t_{jn}\right)\omega^s\omega^m\omega^n+\left(\frac{\partial L^l_{km}}{\omega^n}+\frac{\partial L^l_{kn}}{\omega^m}+L^j_{kn}L^l_{mj}+L^l_{jn}L^j_{mk}\right)\omega^n d\omega^m$$

$$=\left(\frac{\partial^2 L^l_{kn}}{\omega^s\omega^m}-2L^j_{ks}L^l_{tm}L^t_{jn}\right)\omega^s\omega^m\omega^n+\tfrac{1}{2}Q^l_{.kmn}d\,(\omega^m\omega^n).$$

Hence (11.13) follows as before.

§ 3. Non-Connection Method in the Linear Connections in the Large.

12. Non-Connection Method for Linear Connections in the Large.

Multiplying (7.4)″ with $\omega^l_\lambda\Omega^\mu_p\Omega^\nu_q$, we obatin

$$L^l_{pq}=\omega^l_\lambda\Omega^\mu_p\Omega^\nu_q\,(L^\lambda_{\mu\nu}-\Lambda^\lambda_{\mu\nu}). \tag{12.1}$$

Hence

$$\frac{d}{dt}\frac{\omega^l}{dt}+L^l_{pq}\frac{\omega^p}{dt}\frac{\omega^q}{dt}=\frac{d}{dt}\frac{\omega^l}{dt}+\omega^l_\lambda\,(L^\lambda_{\mu\nu}-\Lambda^\lambda_{\mu\nu})\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}. \tag{12.2}$$

Now we have (2.6):

$$\frac{d^2\xi^l}{dt^2}=\frac{d}{dt}\frac{\omega^l}{dt}\equiv\omega^l_\lambda\left(\frac{d^2x^\lambda}{dt^2}+\Lambda^\lambda_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\mu}{dt}\right). \tag{12.3}$$

Adding (12.2) and (12.3) side by side, we obtain

$$\frac{d^2\xi^l}{dt^2}+L^l_{pq}\frac{d\xi^p}{dt}\frac{d\xi^q}{dt}=\omega^l_\lambda\left(\frac{d^2x^\lambda}{dt^2}+L^\lambda_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right). \tag{12.4}$$

We have set

$$\Gamma^\lambda_{\mu\nu}\overset{\text{def}}{=}\tfrac{1}{2}(L^\lambda_{\mu\nu}+L^\lambda_{\nu\mu}),\quad \Omega^\lambda_{\mu\nu}\overset{\text{def}}{=}\tfrac{1}{2}(L^\lambda_{\mu\nu}-L^\lambda_{\nu\mu}), \tag{12.5}$$

and we have

(12.6) $$\Omega^{\lambda}_{\mu\nu} dx^{\mu} dx^{\nu} \equiv 0.$$

Similarly,

(12.7) $$\Gamma^{l}_{pq} \stackrel{\text{def}}{=} \tfrac{1}{2}(L^{l}_{pq}+L^{l}_{qp}), \qquad \Omega^{l}_{pq} \stackrel{\text{def}}{=} \tfrac{1}{2}(L^{l}_{pq}-L^{l}_{qp}).$$

(12,8) $$\Omega^{l}_{pq} d\xi^{p} d\xi^{q} \equiv 0.$$

Thus (12.4) may be rewritten as follows:

(12.9) $$\frac{d^2\xi^l}{dt^2}+\Gamma^{l}_{pq}\frac{d\xi^p}{dt}\frac{d\xi^q}{dt}=\omega^l_\lambda\left(\frac{d^2x^\lambda}{dt^2}+\Gamma^{\lambda}_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right).$$

(12.1) gives (cf. (7.6))

(12.10) $$\Gamma^{l}_{pq}=\omega^l_\lambda \Omega^{\mu}_{p}\Omega^{\nu}_{q}(\Gamma^{\lambda}_{\mu\nu}-\Lambda^{\lambda}_{\mu\nu}), \quad (\Lambda^{\lambda}_{\mu\nu} \stackrel{\text{def}}{=} \tfrac{1}{2}(\Lambda^{\lambda}_{\mu\nu}+\Lambda^{\lambda}_{\nu\mu})).$$

From (12.9), we obtain the

Theorem 1°. *The necessary and sufficient condition for that the* II*-geodesic curves in the large*

(12.11) $$\frac{d^2\xi^l}{dt^2}=0$$

may consist of piece-wise pasting and curvature changing continuation of the local paths

(12.12) $$\frac{d^2x^\lambda}{dt^2}+\Gamma^{\lambda}_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\kappa}{dt}=0$$

is that

(12.13) $$\Gamma^{l}_{pq} d\xi^p d\xi^q=0.$$

From (12.13) and (12.10), we obtain the

Theorem 2°. *The necessary and sufficient condition for that the* II*-geodesic curves in the large* (12.11) *may consist of the piece-wise pasting continuation of the local paths* (12.12), *is that* (cf. (7.16))

(12.14) $$\Gamma^{\lambda}_{\mu\nu} dx^\mu dx^\nu=\Lambda^{\lambda}_{\mu\nu} dx^\mu dx^\nu.$$

Theoren 3°. *The* II*-geodesic curves in the large* (12.11) *consist actually of the piece-wise pasting and curvature changing continuation of the local*

II*-geodesic curves* (2.7), (ii) | *paths* (12.12)

by the development by the developing factor $\omega^l_\lambda(x^\nu)$.

Proof. We have (12.14). Now, by Theorem 3° of Art. 8, the existence of an arbitrary vector ψ_ν (cf. (8.14)) such that

(12.15) $$\Gamma^{\lambda}_{\mu\nu}=\bar{\Lambda}^{\lambda}_{\mu\nu}+\delta^{\lambda}_{\mu}\psi_\nu+\delta^{\lambda}_{\nu}\psi_\mu$$

suffices for the commonness of paths:

(12.16) $$\frac{d^2x^\lambda}{dt^2}+\Gamma^{\lambda}_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\kappa}{dt}=\frac{d^2x^\lambda}{dt^2}+\Lambda^{\lambda}_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}=0.$$

We obtain

$$\Gamma^{\lambda}_{\mu\nu}=\bar{\Lambda}^{\lambda}_{\mu\nu}+\Omega^{\lambda}_{l}\omega^{p}_{\mu}\omega^{q}_{\nu}\Gamma^{l}_{pq} \tag{12.17}$$

from (12.10). Combining (12.15) with (12.17), we see that

$$\delta^{\lambda}_{\mu}\psi_{\nu}+\delta^{\lambda}_{\nu}\psi_{\mu}=\Omega^{\lambda}_{l}\omega^{p}_{\mu}\omega^{q}_{\nu}\Gamma^{l}_{qq} \tag{12.18}$$

and by contraction $\mu\to\lambda$, we see that the vector ψ_{ν} exists actually as follows:

$$(n+1)\,\psi_{\nu}=\omega^{q}_{\nu}\Gamma^{p}_{pq}\equiv\Gamma^{\lambda}_{\lambda\nu}-\bar{\Lambda}^{\lambda}_{\lambda\nu} \tag{12.19}$$

by (12.1). Thus from Theorem 3⁰ of Art. 8, our present Theorem 3⁰ follows.

Theorem 4⁰. *The t is an affine parameter for the paths* (12.12).

Proof. From (12.14) and (12.15), we have

$$(\delta^{\lambda}_{\mu}\psi_{\nu}+\delta^{\lambda}_{\nu}\psi_{\mu})\,dx^{\mu}dx^{\nu}=2\,(\psi_{\nu}dx^{\nu})\,dx^{\mu}=0$$

for all values of dx^{μ}. Hence

$$\psi_{\nu}dx^{\nu}\equiv\frac{1}{n+1}\,\omega^{q}\Gamma^{p}_{pq}\equiv\frac{1}{n+1}\,\theta^{p}_{p}\equiv\frac{1}{n+1}(\theta^{\nu}_{\nu}-\Theta^{\nu}_{\nu})=0 \tag{12.20}$$

(by (7.6) and (12.19)) at all points of the path. Thus, by Theorem 4⁰ of Art. 8, we see that t is an affine parameter for the paths (12.12).

Conclusion. *The* II-*geodesic curves* (12.11):

$$\frac{d^2\xi^l}{dt^2}=0, \tag{12.21}$$

whose finite equations are of the forms

$$\xi^l=a^l t+c^l,\qquad (a^l,\,c^l=\text{const.}), \tag{12.22}$$

behave as for meet and join like straight lines and yield us a non-connection method for linear connections L^l_{pq} in the large.

13. Non-Connection Method for Extended Affine Connections in the Large. If we restrict ourselves to the case of the symmetric part of the general linear connections $L^{\lambda}_{\mu\nu}$, the results of the present § 3 yield us a ***non-connection method for extended affine connections in the large.*** The identities (12.3):

$$\frac{d^2\xi^l}{dt^2}=\omega^l_{\lambda}\left(\frac{d^2x^{\lambda}}{dt^2}+\Lambda^{\lambda}_{\mu\nu}\frac{dx^{\mu}}{dt}\frac{dx^{\nu}}{dt}\right) \tag{13.1}$$

and (12.9):

$$\frac{d^2\xi^l}{dt^2}=\omega^l_{\lambda}\left(\frac{d^2x^{\lambda}}{dt^2}+\Gamma^{\lambda}_{\mu\nu}\frac{dx^{\mu}}{dt}\frac{dx^{\nu}}{dt}\right) \tag{13.2}$$

may be converted into their inverses

$$\Omega^{\lambda}_{l}\frac{d^2\xi^l}{dt^2}=\frac{d^2x^{\lambda}}{dt^2}+\Lambda^{\lambda}_{\mu\nu}\frac{dx^{\mu}}{dt}\frac{dx^{\nu}}{dt} \tag{13.3}$$

and

(13.4) $$\Omega^{\lambda}_{l}\frac{d^2\xi^l}{dt^2}=\frac{d^2x^{\lambda}}{dt^2}+\Gamma^{\lambda}_{\mu\nu}\frac{dx^{\mu}}{dt}\frac{dx^{\nu}}{dt}$$

respectively. *The factor* Ω^{λ}_{l} *projects piece-wise the global path* $\frac{d^2\xi^l}{dt^2}=0$ *piece-wise onto the local path* (13.3)=(13.4)=0, *which is a* II*-geodesic curve as well as a geodesic curve.*

Thus the inverse transformation

(13·5) $$x^{\lambda}=\Omega^{\lambda}_{l}\xi^l+\Omega^{\lambda}_{o},\qquad (\,|\,\Omega^{\lambda}_{l}\,|\neq 0,\ \Omega^{\lambda}_{o}=\text{const.})$$

of the transformation

(13.6) $$\xi^l=\omega^l_{\mu}(x^{\nu})\,x^{\mu}+a^l_o,\ (\,|\,\omega^l_{\mu}\,|\neq 0,\ a^l_o=\text{const.})$$

maps the local affine connection space $\{x^{\lambda}\}$ ***continuingly onto the global extended affine space, where no connection is necessary.***

§ 4. Non-Connection Method for the Extended Euclidean Connections in the Large.

14. Local Riemannian Connection. The non-connection method for linear connections in the large stated in Art. 12 has been made substantially from the view point of *principal fibre bundles with extended structure groups, the group parameters being appropriate functions of local coordinates. It is valid for the extended affine connections in the large, for which the local connection parameters are* $\Gamma^{\lambda}_{\mu\nu}$, *and the structure group is the author's extended affine group* (cf. (3.23)). *The extended affine connections in the large imply the extended Euclidean connections in the large, for which the local connection parameters are* $\{^{\lambda}_{\mu\nu}\}$, *special* $\Gamma^{\lambda}_{\mu\nu}$, *and the structure group is the author's extended Euclidean transformations*:

(14.1) $$\bar{\xi}^l=a^l_m(\xi^p)\,\xi^m+a^l_o,\qquad (a^l_o=\text{const.}),$$

where

(14.2) $$|\,a^l_m(\xi^p)\,|=1 \quad\text{or}\quad -1$$

is an orthogonal determinant.

There are some peculiar points to be espacially noticed. So I will expose in the following lines such aspects.

Let the fundamental quadratic differential form for the classical *n*-dimensional local Riemannian geometry be

(14.3) $$ds^2=g_{\mu\nu}dx^{\mu}dx^{\nu}>0,\qquad (g_{\mu\nu}=g_{\nu\mu},\ |\,g_{\mu\nu}\,|\neq 0).$$

It is known that (14.3) *is always expressible in the following form*:

(14.4) $$ds^2=\omega^l\omega^l,\qquad (l=1,2,\cdots,n)$$

but for undergoing orthogonal transformations, where

(14.5) $$\omega^l=\omega^l_{\mu}(x^{\nu})\,dx^{\mu},\qquad (\,|\,\omega^l_{\mu}\,|\neq 0).$$

But the orthogonal transformations hitherto known have exclusively been of constant coefficients. Now the present author has discovered [2] *orthogonal*

tarnsformations with appropriate functions of coordinates as coefficients, the concerning invariants being retained. We will refer to such transformations as ***extended orthogonal transformations.*** They constitute a group, an ***extended orthogonal transformation group.***

We utilize the ω^l appearing in (14.4) as the ω^l in (2.1).

It is readily seen that

$$\omega^l\omega^l=\omega^l_\mu\omega^l_\nu dx^\mu dx^\nu, \tag{14.6}$$

so that

$$g_{\mu\nu}=\omega^l_\mu\omega^l_\nu, \tag{14.7}$$

$$|g_{\mu\nu}|=|\omega^l_\mu|\cdot|\omega^l_\nu|=|\omega^l_\mu|^2\neq 0. \tag{14.8}$$

$$g^{\mu\nu}=[\text{cofactor of } g_{\mu\nu} \text{ in } |g_{\mu\nu}|]\,/\,|g_{\mu\nu}|$$

$$=\frac{\text{cofactor of } \omega^l_\mu \text{ in } |\omega^l_\mu|}{|\omega^l_\mu|}\cdot\frac{\text{cofactor of } \omega^l_\nu \text{ in } |\omega^l_\nu|}{|\omega^l_\nu|},$$

$$g^{\mu\nu}=\Omega^\mu_l\Omega^\nu_l. \tag{14.9}$$

Hence

$$\left\{{}^{\lambda}_{\mu\nu}\right\}=\tfrac{1}{2}g^{\lambda\sigma}\left(\frac{\partial g_{\mu\sigma}}{\partial x^\nu}+\frac{\partial g_{\sigma\nu}}{dx^\mu}-\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right)$$

$$=\tfrac{1}{2}\Omega^\lambda_h\Omega^\sigma_h\left(\frac{\partial\omega^l_\mu}{\partial x^\nu}\omega^l_\sigma+\omega^l_\mu\frac{\partial\omega^l_\sigma}{\partial x^\nu}+\frac{\partial\omega^l_\sigma}{\partial x^\mu}\omega^l_\nu+\omega^l_\sigma\frac{\partial\omega^l_\nu}{\partial x^\mu}-\frac{\partial\omega^l_\mu}{\partial x^\sigma}\omega^l_\nu-\omega^l_\mu\frac{\partial\omega^l_\nu}{\partial x^\sigma}\right),$$

$$\left\{{}^{\lambda}_{\mu\nu}\right\}=\tfrac{1}{2}(\Lambda^\lambda_{\mu\nu}+\Lambda^\lambda_{\nu\mu})+\tfrac{1}{2}\Omega^\sigma_h\left[\omega^l_\mu\left(\frac{\partial\omega^l_\sigma}{\partial x^\nu}-\frac{\partial\omega^l_\nu}{\partial x_\sigma}\right)+\omega^l_\nu\left(\frac{\partial\omega^l_\sigma}{\partial x^\mu}-\frac{\partial\omega^l_\mu}{\partial x^\sigma}\right)\right]. \tag{14.10}$$

According to (8.14), we set

$$\left\{{}^{\lambda}_{\mu\nu}\right\}=\tfrac{1}{2}(\Lambda^\lambda_{\mu\nu}+\Lambda^\lambda_{\nu\mu})+\delta^\lambda_\mu\psi_\nu+\delta^\lambda_\nu\psi_\mu, \tag{14.11}$$

so that

$$\delta^\lambda_\mu\psi_\nu+\delta^\lambda_\nu\psi_\mu=\Omega^\lambda_h\Omega^\sigma_h\left[\omega^l_\mu\left(\frac{\partial\omega^l_\sigma}{\partial x^\nu}-\frac{\partial\omega^l_\nu}{\partial x^\sigma}\right)+\omega^l_\nu\left(\frac{\partial\omega^l_\sigma}{\partial x^\mu}-\frac{\partial\omega^l_\mu}{\partial x^\sigma}\right)\right]. \tag{14.12}$$

Contracting $\mu\rightarrow\lambda$:

$$(n+1)\,\psi_\nu=\Omega^\sigma_h\Omega^\lambda_h\left[\omega^l_\lambda\left(\frac{\partial\omega^l_\sigma}{\partial x^\nu}-\frac{\partial\omega^l_\nu}{\partial x^\sigma}\right)+\omega^l_\nu\left(\frac{\partial\omega^l_\sigma}{\partial x^\lambda}-\frac{\partial\omega^l_\lambda}{\partial x^\sigma}\right)\right]$$

$$=\Omega^\sigma_l\left(\frac{\partial\omega^l_\sigma}{\partial x^\nu}-\frac{\partial\omega^l_\nu}{\partial x^\sigma}\right)=\Lambda^\sigma_{\sigma\nu}-\Lambda^\sigma_{\nu\sigma},$$

$$(n+1)\,\psi_\nu=\Lambda^\sigma_{\sigma\nu}-\Lambda^\sigma_{\nu\sigma}, \tag{14.13}$$

what proves the unexpected result:

$$\frac{d^2x^\lambda}{ds^2}+\left\{{}^{\lambda}_{\mu\nu}\right\}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}=\frac{d^2x^\lambda}{ds^2}+\Lambda^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}. \tag{14.14}$$

Second proof for (14.14). It suffices to prove that

$$(14.15) \qquad \Omega^\sigma_h\left[\omega^l_\mu\left(\frac{\partial\omega^l_\sigma}{\partial x^\nu}-\frac{\partial\omega^l_\nu}{\partial x^\sigma}\right)+\omega^l_\nu\left(\frac{\partial\omega^l_\sigma}{\partial x^\mu}-\frac{\partial\omega^l_\mu}{\partial x^\sigma}\right)\right]dx^\mu dx^\nu\equiv 0.$$

For it, we proceed as follows.

$$\begin{aligned}(14.15) &= 2\Omega^\sigma_h\omega^l_\mu\left(\frac{\partial\omega^l_\sigma}{\partial x^\nu}-\frac{\partial\omega^l_\nu}{\partial x^\sigma}\right)dx^\mu dx^\nu = 2\omega^l\left(\Omega^\sigma_h d\omega^l_\sigma-\frac{\partial\omega^l_\nu}{\omega^h}\Omega^\lambda_p\omega^p\right)\\
&= 2\omega^l\left(\Omega^\sigma_h d\omega^l_\sigma-\Omega^\nu_p\frac{\partial\omega^l_\nu}{\omega^q}\frac{\omega^q}{\omega^h}\omega^p\right)\\
&= 2\omega^l\left(\Omega^\sigma_h d\omega^l_\sigma-\Omega^\nu_p d\omega^l_\nu\delta^p_h\right)\\
&= 2\omega^l\left(\Omega^\sigma_h d\omega^l_\sigma-\Omega^\nu_h d\omega^l_\nu\right)\equiv 0.\end{aligned}$$

Third proof for (14.14).

$$\Omega^\lambda_k\delta^p_k=\Omega^\lambda_p,$$

$$\Omega^\lambda_k\frac{\omega^p}{\omega^k}=\Omega^\lambda_p,$$

$$\Omega^\lambda_k\frac{\omega^p}{\omega^k}\omega^h=\Omega^\lambda_k\omega^p\delta^h_k=\Omega^\lambda_p\omega^h,$$

$$\begin{aligned}\Omega^\lambda_h\omega^p&=\Omega^\lambda_p\omega^h,\\
\Omega^\lambda_h\omega^p\,(\Omega^\sigma_h\Omega^\nu_p)&=\Omega^\lambda_p\omega^h\,(\Omega^\sigma_h\Omega^\nu_p),\\
\Omega^\lambda_h\Omega^\sigma_h\Omega^\nu_p\omega^p&=\Omega^\lambda_p\Omega^\nu_p\Omega^\sigma_h\omega^h,\\
\Omega^\lambda_h\Omega^\sigma_h dx^\nu&=\Omega^\lambda_p\Omega^\nu_p dx^\sigma,\end{aligned}$$

$$\Omega^\lambda_k\Omega^\sigma_k\frac{\partial\omega^l_\nu}{\partial x^\sigma}dx^\nu=\Omega^\lambda_k\Omega^\nu_k\frac{\partial\omega^l_\nu}{\partial x^\sigma}dx^\sigma=\Omega^\lambda_k\Omega^\nu_k d\omega^l_\nu.$$

Hence, by multiplying with ω^h_λ,

$$\Omega^\sigma_h\left(\frac{\partial\omega^l_\nu}{\partial x^\sigma}dx^\nu-d\omega^l_\sigma\right)=0,$$

so that

$$\begin{aligned}(14.15) &= 2\Omega^\sigma_h\omega^l\left(\frac{\partial\omega^l_\sigma}{\partial x^\nu}-\frac{\partial\omega^l_\nu}{\partial x^\sigma}\right)dx^\nu\\
&= 2\omega^l\Omega^\sigma_h\left(d\omega^l_\sigma-\frac{\partial\omega^l_\nu}{\partial x^\sigma}dx^\nu\right)=0\end{aligned}$$

Fourth proof for (14.14). It is evident that

$$\left(\frac{\partial\omega^l_\sigma}{\partial x^\nu}-\frac{\partial\omega^l_\nu}{\partial x^\sigma}\right)dx^\sigma dx^\nu\equiv 0.$$

Now along any II-geodesic curve, we have

$$dx^\sigma=\Omega^\sigma_k d\xi^k=a^k\Omega^\sigma_k ds, \qquad (a^h=\text{const.})$$

by (2.15). Hence

$$\left(\frac{\partial\omega^l_\sigma}{\partial x^\nu}-\frac{\partial\omega^l_\nu}{\partial x^\sigma}\right)a^k\Omega^\sigma_k dx^\nu=0.$$

Along the II-geodesic ξ^h—axis, we have

$$a^k=\delta^k_h,$$

so that

$$\left(\frac{\partial\omega^l_\sigma}{\partial x^\nu}-\frac{\partial\omega^l_\nu}{\partial x^\sigma}\right)\Omega^\sigma_h dx^\nu=0$$

and that

$$\Omega^\sigma_h\omega^l_\mu\left(\frac{\partial\omega^l_\sigma}{\partial x^\nu}-\frac{\partial\omega^l_\nu}{\partial x^\sigma}\right)dx^\mu dx^\nu=0.$$

Fifth proof for (14.14). We obtain

$$\frac{d^2x^\lambda}{ds^2}+\{{}^{\,\lambda}_{\mu\nu}\}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}=0 \qquad \Bigg| \qquad \frac{d^2x^\lambda}{ds^2}+\Lambda^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}=0$$

as solutions of one and the same extremal problem $\delta s=0$, the variations of parameters being

$$\delta x^\sigma,\ \delta\frac{dx^\sigma}{ds}. \qquad \Bigg| \qquad \delta\frac{d\xi^l}{ds}=\frac{\delta\omega^l_\mu}{\partial x^\nu}\delta\frac{dx^\nu}{ds}\frac{dx^\mu}{ds}+\omega^l_\mu\delta\frac{dx^\mu}{ds}.$$

(The cyclic case!)

Sixth proof for (14.14). In the theory of anholonomic system [33, 34], the following formulas are known:

(14.16) $$ds^2=g_{\mu\nu}dx^\mu dx^\nu=g_{hk}\omega^h\omega^k=g_{hk}\omega^h_\mu\omega^k_\nu dx^\mu dx^\nu,$$

(14.17) $$\frac{d}{ds}\frac{\omega^l}{ds}+\{{}^{\,l}_{hk}\}\frac{\omega^h}{ds}\frac{\omega^k}{ds}=\omega^l_\lambda\left(\frac{d^2x^\lambda}{ds^2}+\{{}^{\,\lambda}_{\mu\nu}\}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right),$$

where $\{{}^{\,l}_{hk}\}$ is constructed in terms of g_{hk}. In case (14.4), we have

(14.18) $$g_{hk}=\delta_{hk},$$

so that

(14.19) $$\{{}^{\,l}_{hk}\}=0$$

and thus (14.17) becomes

(14.20) $$\frac{d^2\xi^l}{ds^2}=\omega^l_\lambda\left(\frac{d^2x^\lambda}{ds^2}+\{{}^{\,\lambda}_{\mu\nu}\}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right),$$

which, taken together with the author's identity (13.1), shows our relation (14.14).

15. Equations of Structure. In our case $\Gamma^\lambda_{\mu\nu}=\{{}^{\,\lambda}_{\mu\nu}\}$, owing to the peculiar relations (14.13), (14.17) and (14.11), (14.13), (14.19) and (14.15), the equations of structure (10.3) and (10.4) become extremely simple: —

In order that n linearly independent linear differential forms

(15.1) $$\omega^l=\omega^l_\mu(x^\nu)\,dx^\mu$$

defined on the n^2+n dimensional principal fibre bundle $\mathfrak{B}$ consisting of $\{x^\lambda\}$ and $\{\omega^l_\mu\}$ may define an extended linear connection $\{{}^{\lambda}_{\mu\nu}\}$ over a differentiable manifold M, it is necessary and sufficient that the condition

$$(15.2)\quad d\omega^l \equiv \omega^l_\lambda(d^2x^\lambda + \Lambda^\mu_{\mu\nu}dx^\mu dx^\nu) \equiv \omega^l_\lambda(d^2x^\lambda + \{{}^{\lambda}_{\mu\nu}\}\, dx^\mu dx^\nu)$$

$$d\omega^l = \omega^l_\lambda\,[ddx^\lambda + \{{}^{\lambda}_{\mu\nu}\}dx^\nu \wedge dx^\mu] = \omega^l_\lambda\,[ddx^\lambda + \tfrac{1}{2}T^{\lambda}_{\mu\nu}dx^\mu \wedge dx^\nu] \equiv 0, \quad (ddx^\lambda \equiv 0)$$

is satisfied, what is now the case.

Proof. In the Theorem of Art. 10, we have

$$\theta^l_h = 0,\ T^l_{mn} = 0,\ T^{\lambda}_{\mu\nu} = 0,\ R^k_{.lmn} = 0,\ L^k_{hl} = 0,\ Q^k_{.lmn} = 0.$$

Thus the

Theorem. *n linearly independent linear differential forms*

$$(15.3)\qquad \omega^l = \omega^l_\mu(x^\nu)\, dx^\mu$$

defined on the n^2+n dimensional principal fibre bundle $\mathfrak{B}$ consisting of $\{x^\lambda\}$ and $\{\omega^l_\mu\}$ define surely an extended linear connection $\{{}^{\lambda}_{\mu\nu}\}$ over a differentiable manifold M.

Cor. *In case of* (15.3), *we have*

$$(15.4)\quad d\theta^\lambda_\mu + \theta^\lambda_\nu\theta^\nu_\mu = (d\Theta^\lambda_\mu + \Theta^\lambda_\nu\Theta^\nu_\mu), \qquad d\theta^\lambda_\mu + \theta^\lambda_\nu \wedge \theta^\nu_\mu = d\Theta^\lambda_\mu + \Theta^\lambda_\nu \wedge \Theta^\nu_\mu,$$

$$(15.5)\quad \tfrac{1}{2}(Q^\lambda_{.\mu\alpha\beta} - \Theta^\lambda_{.\mu\alpha\beta})\, dx^\alpha dx^\beta + (\{{}^{\lambda}_{\sigma\mu}\} - \Lambda^\lambda_{\sigma\mu})d^2x^\sigma = 2\Theta^\nu_\mu\theta^\lambda_\nu. \qquad R^\lambda_{.\mu\alpha\beta}dx^\alpha \wedge dx^\beta = R^l_{.\mu\alpha\beta}dx^\alpha \wedge dx^\beta.$$

16. Bianchi Identities and Some Formulas. By the reason mentioned in Art. 14, the formulas (11.1), (11.2), (11.3), (11.4) and the Bianchi identities (11.7), (11.8) as well as (11.9) and (11.10) become all

$$0 = 0.$$

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(Received January 10, 1964)