ESTIMATIONS IN SOME MODIFIED POISSON DISTRIBUTIONS.

By

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Introduction.

Sometimes we observe samples that can be seen as a sample from a Poisson distribution except some classes. In the cases of this type, after a more precise observation we often find that a Poisson distribution was modified by some causes, for instances, missing or mixing of some observations.

We assume that population distribution density is $f(i)$ in (2) which is the modification of the Poisson distribution (1). Poisson parameter λ and modification parameters $\theta_1, \theta_2, \cdots\cdots, \theta_k$ are unknown and we wish to estimate them by the maximum likelihood method, where k is a known fixed number.

This corresponds to the extension of A. Clifford Cohen's study [1] to which the technique employed and results obtained here being closely related.

For instances, the case of $\theta_i=1$ $(i=0,1,2,\cdots\cdots,k)$ corresponds to the case of the truncation of the classes $i\leqq k$, and the case when $\theta_0=1$, $0\leqq\theta\leqq 1$ $(k=1)$ corresponds to the Cohen's study [2]. As we intended to cover not only the truncated or missing cases, but also the cases when the frequencies of some lower classes may significantly exceed the Poisson's, the values of θ's may be negative not only zero or positive.

(1) $$p(i)=\frac{e^{-\lambda}\lambda^i}{i!}\ (i=0,1,2,3,\cdots\cdots)$$

(2) $$f(i;\lambda,\theta_1,\theta_2,\cdots\cdots,\theta_k) = \begin{cases} \dfrac{(1-\theta_i)\,p(i)}{1-\sum\limits_{x=0}^{k}\theta_x p(x)}, & (0\leqq i\leqq k) \\[2ex] \dfrac{p(i)}{1-\sum\limits_{x=0}^{k}\theta_x p(x)}, & (k<i) \end{cases}$$

1. Maximum Likelihood Estimation

Consider a sample consisting of N observations of random variable i.

The likelihood function is

$$(3)\qquad L=\prod_{x=0}^{k}\left\{\frac{(1-\theta_x)p(x)}{1-\sum_{x=0}^{k}\theta_x p(x)}\right\}^{n_x}\cdot\prod_{y=k+1}^{\infty}\left\{\frac{p(y)}{1-\sum_{x=0}^{k}\theta_x p(x)}\right\}^{n_y}$$

where n_x is the sample frequency of $x$class $(x\leqq k)$ and n_y is the sample frequency of y class $(y>k)$. And we put

$$(4)\qquad N_x=\sum_{x=0}^{k}n_x,\qquad N_y=\sum_{y=k+1}^{\infty}n_y,\quad\text{and}$$
$$N=N_x+N_y=\sum_{x=0}^{k}n_x+\sum_{y=k+1}^{\infty}n_y$$

Then we have the following logarithms and their partial differentiations.

$$(5)\qquad \log L=\sum_{x=0}^{k}n_x\{\log(1-\theta_x)+\log p(x)\}+\sum_{y=k+1}^{\infty}n_y\log\{p(y)\}-N\log\left\{1-\sum_{x=0}^{k}\theta_x p(x)\right\}.$$

$$(6)\qquad \frac{\partial}{\partial\theta_i}\log L=\frac{-n_i}{1-\theta_i}+\frac{Np(i)}{\left\{1-\sum_{x=0}^{k}\theta_x p(x)\right\}},$$

$$(i=0,1,2,\cdots\cdots\cdots,k)$$

$$(7)\qquad \frac{\partial}{\partial\lambda}\log L=\sum_{x=0}^{k}n_x\left\{-1+\frac{x}{\lambda}\right\}+\sum_{y=k+1}^{\infty}n_y\left\{-1+\frac{y}{\lambda}\right\}$$
$$+N\sum_{x=0}^{k}\theta_x p(x)\left\{-1+\frac{x}{\lambda}\right\}\Big/\left\{1-\sum_{x=0}^{k}\theta_x p(x)\right\}$$

$$=N\left(\frac{\bar{u}}{\lambda}+\frac{-1+\sum_{x=0}^{k-1}\theta_{x+1}p(x)}{1-\sum_{x=0}^{k}\theta_x p(x)}\right)$$

where $\bar{u}$ is the total sample mean

$$(8)\qquad \bar{u}=\sum_{u=0}^{\infty}n_u u/N$$

Putting (6) to be zero, we have the following (9).

$$(9)\qquad \theta_i=1-\frac{n_i\left\{1-\sum_{x=0}^{k}\theta_x p(x)\right\}}{Np(i)}$$

$$(i=0,1,2,\cdots\cdots,k)$$

Multiplying $p(i)$ to the both sides of this equation and summing up from zero to k, we have the following relations:

$$\sum_{i=0}^{k} \theta_i p(i) = \sum_{i=0}^{k} p(i) - \frac{1}{N} \sum_{i=0}^{k} n_i \left\{1 - \sum_{i=0}^{k} \theta_x p(x)\right\}$$

Then
$$1 - \sum_{i=0}^{k} \theta_i p(i) = \sum_{y=k+1} p(y) + \frac{N_x}{N} \left\{1 - \sum_{x=0}^{k} \theta_x p(x)\right\}$$

Then we have

$$1 - \sum_{x=0}^{k} \theta_x p(x) = \frac{N}{N_y} \sum_{y=k+1}^{\infty} p(y)$$

In the right hand side we put

$$P(y>k) \equiv \sum_{y=k+1}^{\infty} p(y) \tag{10}$$

and we have the useful relations:

$$1 - \sum_{x=0}^{k} \theta_x p(x) = \frac{N}{N_y} P(y>k) \tag{11}$$

and the following (12) the estimator of θ_i by substituting (11) into (9).

$$\hat{\theta}_i = 1 - \frac{n_i\, P(y>k)}{p(i)\, N_y}, \quad (i=0,1,2,\cdots\cdots,k) \tag{12}$$

From this, we derive the folloings:

$$\sum_{x=0}^{k-1} \theta_{x+1} p(x) = \sum_{x=0}^{k-1} \left\{ p(x) - \frac{n_{x+1} p(x)}{p(x+1)} \cdot \frac{P(y>k)}{N_y} \right\} \tag{13}$$

$$= \sum_{x=0}^{k-1} p(x) - \frac{P(y>k)}{N_y \lambda} \sum_{x=0}^{k-1} n_{x+1}(x+1) = \sum_{x=0}^{k-1} p(x) - \frac{P(y>k)}{N_y \lambda} N_x \bar{x}$$

where $\bar{x}$ is the sample mean of $x \leqq k$.

$$\begin{cases} \bar{x} = \dfrac{1}{N_x} \displaystyle\sum_{x=0}^{k} n_x x, & \bar{y} = \dfrac{1}{N_y} \displaystyle\sum_{y=k+1}^{\infty} n_y y, \\ N\bar{u} = N_x \bar{x} + N_y \bar{y} \end{cases} \tag{14}$$

Similarly we define $\bar{y}$ as above.

Substituting (11), (12) and (13) into (7), put this equal to zero, and we have

$$\frac{\bar{u}}{\lambda} = \left\{1 - \sum_{x=0}^{k-1} p(x) + \frac{N_x \bar{x}}{\lambda} \cdot \frac{P(y>k)}{N_y}\right\} \Big/ \left\{\frac{NP(y>k)}{N_y}\right\} = \frac{P(y \geqq k) N_y}{P(y>k) N} + \frac{N_x \bar{x}}{\lambda N}$$

By this relation and the last relation in (14), we have the *following estimator of λ, where parameter λ in $P(y \geqq k)$ and $P(y>k)$ should be substituted by $\hat{\lambda}$.*

$$\bar{y} = \frac{P(y \geqq k)}{P(y>k)} \hat{\lambda} \tag{15}$$

Table I shows the values of the function of $F(\lambda)=\frac{\lambda P(y\geqq k)}{P(y>k)}$ for given values of λ and k. *To estimate the value of λ numerically, we take the λ whose $F(\lambda)$ is equal to a sample value $\bar{y}$ in the Table I for a given value of k. Also we have the estimated value of θ_i numerically by substituting this value of λ into* (12).

2. Variances of the estimators.

To obtain the asymptotic variances, we calculate the second order derivatives of *log L* where the estimators are assumed to coincide with the population values in the last stage.

Partially differentiating (6) by λ and applying (10)$\sim$(15) we have

$$\frac{\partial^2}{\partial\lambda\partial\theta_i}\log L=\Big(Np(i)\Big(-1+\frac{i}{\lambda}\Big)\Big\{1-\sum_{x=0}^{k}\theta_x p(x)\Big\}$$

$$+Np(i)\sum_{x=0}^{k}\theta_x p(x)\Big(-1+\frac{x}{\lambda}\Big)\Big)\Big/\Big\{1-\sum_{x=0}^{k}\theta_x p(x)\Big\}^2$$

$$=\frac{N_y}{P(y>k)}\Big\{-p(i)+p(i)\frac{i}{\lambda}\Big\}+Np(i)\Big\{\frac{N_y}{NP(y>k)}\Big\}^2\cdot\Big\{-\sum_{x=0}^{k}\theta_x p(x)+\sum_{x=0}^{k}\theta_x p(x)\frac{x}{\lambda}\Big\}$$

$$=\frac{N_y}{P(y>k)}\Big\{-p(i)+p(i-1)\Big\}+Np(i)\Big\{\frac{N_y}{NP(y>k)}\Big\}^2\Big\{\frac{NP(y>k)}{N_y}-P(y\geqq k)-\frac{P(y>k)}{N_y\lambda}N_x\bar{x}\Big\}$$

$$=\frac{N_y}{P(y>k)}\Big(-p(i)+p(i-1)+p(i)\Big\{1-\frac{N_yP(y\geqq k)}{NP(y>k)}-\frac{N_x\bar{x}}{N\lambda}\Big\}\Big)$$

$$=\frac{N_y}{P(y>k)}\Big(\frac{i}{\lambda}p(i)-p(i)\frac{\bar{u}}{\lambda}\Big)=\frac{N_yp(i)(i-\bar{u})}{P(y>k)\lambda}$$

When $i=0$, we can assum $p(i-1)=0$ in the above relation, the reason may be clear in the above process. Then we have the following relations for all values of $i=0, 1, 2,\cdots\cdots, k$

$$\text{(16)}\qquad \frac{\partial^2}{\partial\lambda\partial\theta_i}\log L=\frac{N_yp(i)(i-\bar{u})}{P(y>k)\lambda}$$

Similarly we have the following relation by partially differentiating (7) by λ.

$$\frac{\partial^2}{\partial\lambda^2}\log L=\frac{-N\bar{u}}{\lambda^2}+\frac{N}{\{1-\sum_{x=0}^{k}\theta_x p(x)\}^2}\Big(\sum_{x=0}^{k-1}\theta_{x+1}p(x)\Big\{-1+\frac{x}{\lambda}\Big\}\Big\{1-\sum_{x=0}^{k}\theta_x p(x)\Big\}$$

$$-\Big\{-1+\sum_{x=0}^{k-1}\theta_{x+1}p(x)\Big\}\Big\{-\sum_{x=0}^{k}\theta_x p(x)\Big(-1+\frac{x}{\lambda}\Big)\Big\}\Big)$$

$$=\frac{-N\bar{u}}{\lambda^2}+N\Big\{\frac{NP(y>k)}{N_y}\Big\}^{-2}\Big(\Big\{-P(x<k)+\frac{N_x\bar{x}\,P(y>k)}{\lambda N_y}+\sum_{x=0}^{k-2}\theta_{x+2}p(x)\Big\}\frac{NP(y>k)}{N_y}$$

$$+\left\{P(y\geqq k)+\frac{N_x\bar{x}P(y>k)}{\lambda N_y}\right\}\left\{1-\frac{NP(y>k)}{N_y}-\sum_{x=0}^{k-1}\theta_{x+1}\,p(x)\right\}\Big)$$

From (12) we calculate as follows:

$$\sum_{x=0}^{k-2}\theta_{x+2}\,p(x)=\sum_{x=0}^{k-2}\left\{p(x)-n_{x+2}\frac{P(y>k)}{N_y}\frac{p(x)}{p(x+2)}\right\}$$

$$=\sum_{x=0}^{k-2}p(x)-\frac{P(y>k)}{N_y}\sum_{x=0}^{k-2}\frac{n_{x+2}(x+2)(x+1)}{\lambda^2}.$$

Putting S_x^2 to be equal to the variance of $x\leqq k$, as

$$\sum_{x=0}^{k}x^2n_x=N_x(S_x^2+\bar{x}^2), \tag{17}$$

we have

$$\sum_{x=0}^{k-2}n_{x+2}(x+2)(x+1)=\sum_{x=0}^{k}n_x x(x-1)=N_x(S_x^2+\bar{x}^2-\bar{x}).$$

Applying these relations we have

$$\frac{\partial^2}{\partial\lambda^2}\log L=\frac{-N\bar{u}}{\lambda^2}+\left(\frac{N_y}{NP(y>k)}\right)^2N\left(\left\{-p(k-1)+\frac{P(y>k)}{N_y\lambda}\left(N_x\bar{x}-\frac{N_x}{\lambda}(S_x^2+\bar{x}^2-\bar{x})\right)\right\}\cdot\right.$$

$$\frac{NP(y>k)}{N_y}+P(y>k)\left\{\frac{\bar{y}}{\lambda}+\frac{N_x\bar{x}}{\lambda N_y}\right\}\left\{1-\frac{NP(y>k)}{N_y}-\sum_{x=0}^{k-1}p(x)\right.$$

$$\left.\left.+\frac{N_x\bar{x}P(y>k)}{\lambda N_y}\right\}\right)=\frac{-N\bar{u}}{\lambda^2}-\frac{N_yp(k-1)}{P(y>k)}+\frac{1}{\lambda}\left\{N_x\bar{x}-\frac{N_x}{\lambda}(S_x^2+\bar{x}^2-\bar{x})\right\}$$

$$+\frac{N_y\bar{u}}{P(y>k)\lambda}\cdot\left\{P(y\geqq k)-\frac{NP(y>k)}{N_y}+\frac{N_x\bar{x}P(y>k)}{\lambda N_y}\right\}\Big)$$

$$=\frac{-N\bar{u}}{\lambda^2}-\frac{N_yp(k-1)}{P(y>k)}+\frac{N_x\bar{x}}{\lambda}-\frac{N_x(S_x^2+\bar{x}^2-\bar{x})}{\lambda^2}+\frac{\bar{u}^2N}{\lambda^2}-\frac{N\bar{u}}{\lambda}\Big)$$

$$=\frac{-N\bar{u}}{\lambda^2}-\frac{N_yp(k-1)}{P(y>k)}+\frac{N_x\bar{x}-N\bar{u}}{\lambda}-\frac{N_x(S_x^2+\bar{x}^2-\bar{x})}{\lambda^2}+\frac{N\bar{u}^2}{\lambda^2}\Big)$$

$$=\frac{-N\bar{u}+N\bar{u}^2-N_x(S_x^2+\bar{x}^2-\bar{x})}{\lambda^2}-\frac{N_y\,p(k-1)}{P(y>k)}-N_y\frac{P(y\geqq k)}{P(y>k)}$$

$$=\frac{-N_y\bar{y}+N\bar{u}^2-N_x(S_x^2+\bar{x}^2)}{\lambda^2}-\frac{N_yP(y\geqq k-1)}{P(y>k)}$$

On the other hand, we have the following relation by (1), (2) and (9).

$$P(y\geqq k-1)=e^{-\lambda}\sum_{y=k-1}^{\infty}\frac{\lambda^y}{y!}=e^{-\lambda}\sum_{y=k+1}^{\infty}\frac{\lambda^y}{y!}\cdot\frac{y(y-1)}{\lambda^2}=\frac{1}{\lambda^2}\sum_{y=k+1}^{\infty}(y^2-y)\,p(y)$$

$$=\frac{1}{\lambda^2}\sum_{y=k+1}^{\infty}(y^2-y)\,f(y|y>k)\sum_{y=k+1}^{\infty}p(y),$$

where the conditional probability

$$f(y|y>k)=\frac{p(y)}{1-\sum_{x=0}^{k}{}'\,\theta_x p(x)}\Big/\sum_{y=k+1}^{\infty}\left\{\frac{p(y)}{1-\sum_{x=0}^{k}{}'\,\theta_x p(x)}\right\}.$$

Let μ_y and σ_y^2 represent the mean and variance of $y>k$, we have

$$\frac{P(y\geqq k-1)}{P(y>k)}=\frac{\sigma_y^2+\mu_y^2-\mu_y}{\lambda^2}$$

where μ_y and σ_y^2 can be substituted by the corresponding sample values $\bar{y}$ and S_y^2 in our asymptotic case.

Substituting these into the former relation, we have the following result in a large sample case.

$$\frac{\partial^2}{\partial\lambda^2}\log L=\frac{\{N\bar{u}^2-N_x(S_x^2+\bar{x}^2)-N_y(S_y^2+\bar{y}^2)\}}{\lambda^2} \tag{18}$$

Partially differentiating (6) by θ_j $(j=1, 2,\cdots\cdots\cdots, k)$ we have the following (19) and (20).

$$\frac{\partial^2}{\partial\theta_i\partial\theta_j}\log\ L=\frac{N_y^2 p(i)\,p(j)}{N\{P(y>k)\}^2}\qquad(i\neq j) \tag{19}$$

$$\frac{\partial^2}{\partial\theta_i^2}\log\ L=\frac{-1}{n_i}\left\{\frac{p(i)N_y}{P(y>k)}\right\}^2+\frac{1}{N}\left\{\frac{p(i)N_y}{P(y>k)}\right\}^2 \tag{20}$$

From (16), (18), (19) and (20), we have the following set of formulas.

$$\left\{\begin{aligned}
&-\frac{1}{N}\frac{\partial^2}{\partial\theta_i\partial\theta_j}\log L=\frac{-N_y^2\,p(i)\,p(j)}{N^2P(y>k)^2},\qquad(i\neq j)\\
&-\frac{1}{N}\frac{\partial^2}{\partial\theta_i^2}\log L=\left(-1+\frac{N}{n_i}\right)\left\{\frac{N_y\,p(i)}{NP(y>k)}\right\}^2\\
&-\frac{1}{N}\frac{\partial^2}{\partial\lambda\partial\theta_i}\log L=\frac{-N_y\,p(i)}{NP(y>k)}\cdot\frac{(i-\bar{u})}{\lambda}\\
&-\frac{1}{N}\frac{\partial^2}{\partial\lambda^2}\log L=\frac{-1}{N\lambda^2}\{N\bar{u}^2-N_x(S_x^2+\bar{x}^2)-N_y(S_y^2+\bar{y}^2)\equiv A
\end{aligned}\right. \tag{21}$$

Now we shall set up the inverse dispersion matrix whose determinant will be equal to (22) by the formulas of (21):

(22)

$$
\varDelta \equiv \begin{vmatrix}
\frac{-N_y{}^2 p(0)^2}{N^2 P(y>k)^2}\left(1-\frac{N}{n_0}\right) & \frac{-N_y{}^2 p(1) p(0)}{N^2 P(y>k)^2} & \frac{-N_y{}^2 p(2) p(0)}{N^2 P(y>k)^2} & \cdots\cdots & \frac{-N_y{}^2 p(k) p(0)}{N^2 P(y>k)^2} & \frac{-N_y p(0)(0-\bar{u})}{N P(y>k)\lambda} \\
\frac{-N_y{}^2 p(0) p(1)}{N^2 P(y>k)^2} & \frac{-N_y{}^2 p(1)^2}{N^2 P(y>k)^2}\left(1-\frac{N}{n_1}\right) & \frac{-N_y{}^2 p(2) p(1)}{N^2 P(y>k)^2} & \cdots\cdots & \frac{-N_y{}^2 p(k) p(1)}{N^2 P(y>k)^2} & \frac{-N_y p(1)(1-\bar{u})}{N P(y>k)\lambda} \\
\frac{-N_y{}^2 p(0) p(2)}{N^2 P(y>k)^2} & \frac{-N_y{}^2 p(1) p(2)}{N^2 P(y>k)^2} & \frac{-N_y{}^2 p(2)^2}{N^2 P(y>k)^2}\left(1-\frac{N}{n_2}\right) & \cdots\cdots & \frac{-N_y{}^2 p(k) p(2)}{N^2 P(y>k)^2} & \frac{-N_y p(2)(2-\bar{u})}{N P(y>k)\lambda} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{-N_y{}^2 p(0) p(k)}{N^2 P(y>k)^2} & \frac{-N_y{}^2 p(1) p(k)}{N^2 P(y>k)^2} & \frac{-N_y{}^2 p(2) p(k)}{N^2 P(y>k)^2} & \cdots\cdots & \frac{-N_y{}^2 p(k)^2}{N^2 P(y>k)^2}\left(1-\frac{N}{n_k}\right) & \frac{-N_y p(k)(k-\bar{u})}{N P(y>k)\lambda} \\
\frac{-N_y p(0)(0-\bar{u})}{N P(y>k)\lambda} & \frac{-N_y p(1)(1-\bar{u})}{N P(y>k)\lambda} & \frac{-N_y p(2)(2-\bar{u})}{N P(y>k)\lambda} & \cdots\cdots & \frac{-N_y p(k)(k-\bar{u})}{N P(y>k)\lambda} & A
\end{vmatrix}
$$

$$=\frac{(-1)^k}{\lambda^2}\left(\prod_{i=0}^{k}\left(\frac{N_y p(i)}{NP(y>k)}\right)\right)^2 \begin{vmatrix} 1-\frac{N}{n_0} & 1 & 1 & \cdots\cdots & 1 & 0-\bar{u} \\ 1 & 1-\frac{N}{n_1} & 1 & \cdots\cdots & 1 & 1-\bar{u} \\ 1 & 1 & 1-\frac{N}{n_2} & \cdots\cdots & 1 & 2-\bar{u} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots\cdots & 1-\frac{N}{n_k} & k-\bar{u} \\ 0-\bar{u} & 1-\bar{u} & 2-\bar{u} & \cdots\cdots & k-\bar{u} & -A\lambda^2 \end{vmatrix}$$

We put this as following.

$$\Delta=\frac{(-1)^k}{\lambda^2}\left\{\frac{N_y}{NP(y>k)}\right\}^{2k+2}\left\{\prod_{i=0}^{k}p(i)\right\}^2\Delta' \tag{23}$$

where we defined as follow:

$$\Delta'\equiv\begin{vmatrix} 1-\frac{N}{n_0} & 1 & 1 & \cdots\cdots & 1 & 0-\bar{u} \\ 1 & 1-\frac{N}{n_1} & 1 & \cdots\cdots & 1 & 1-\bar{u} \\ 1 & 1 & 1-\frac{N}{n_2} & \cdots\cdots & 1 & 2-\bar{u} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots\cdots & 1-\frac{N}{n_k} & k-\bar{u} \\ 0-\bar{u} & 1-\bar{u} & 2-\bar{u} & \cdots\cdots & k-\bar{u} & -A\lambda^2 \end{vmatrix}$$

$$=\left\{\prod_{i=0}^{k}n_i\right\}^{-1}\begin{vmatrix} n_0-N & n_0 & n_0 & \cdots\cdots & n_0 & n_0(0-\bar{u}) \\ n_1 & n_1-N & n_1 & \cdots\cdots & n_1 & n_1(1-\bar{u}) \\ n_2 & n_2 & n_2-N & \cdots\cdots & n_2 & n_2(2-\bar{u}) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n_k & n_k & n_k & \cdots\cdots & n_k-N & n_k(k-\bar{u}) \\ 0-\bar{u} & 1-\bar{u} & 2-\bar{u} & \cdots\cdots & k-\bar{u} & -A\lambda^2 \end{vmatrix}$$

$$
=\left\{\prod_{i=0}^{k} n_i\right\}^{-1}
\begin{vmatrix}
n_0-N & N & 0 & \cdots & 0 & n_0(0-\bar{u}) \\
n_1 & -N & N & \ddots & \vdots & n_1(1-\bar{u}) \\
n_2 & 0 & -N & \ddots & 0 & n_2(2-\bar{u}) \\
\vdots & \vdots & \ddots & \ddots & N & \vdots \\
n_k & 0 & \cdots & 0 & -N & n_k(k-\bar{u}) \\
-\bar{u} & 1 & \cdots & \cdots & 1 & -A\lambda^2
\end{vmatrix}
$$

$$
=\left\{\prod_{i=0}^{k} n_i\right\}^{-1}
\begin{vmatrix}
N_x-N & 0 & 0 & \cdots & 0 & N_x(\bar{x}-\bar{u}) \\
n_1 & -N & N & \ddots & \vdots & n_1(1-\bar{u}) \\
n_2 & 0 & -N & \ddots & 0 & n_2(2-\bar{u}) \\
\vdots & \vdots & \ddots & \ddots & N & \vdots \\
n_k & 0 & \cdots & 0 & -N & n_k(k-\bar{u}) \\
-\bar{u} & 1 & 1 & \cdots & 1 & -A\lambda^2
\end{vmatrix}
$$

where the triangles with zeros on their vertices are to be filled with zeros.

The notations N_x, $\bar{x}$ and $\bar{u}$ are defined in (4), (14) and (8) respectively, and here after we may use the notations defined in the previous sections without any interpretations.

Expanding the last determinant with respect to the first row, we have

(24) $$\Delta' = \frac{(-1)^k N^{k-1} N_y^2 S_y^2}{\prod_{i=0}^{k} n_i},$$

then

(25) $$\Delta = \frac{(-1)^k}{\lambda^2}\left\{\frac{N_y}{NP(y>k)}\right\}^{2k+2}\left\{\prod_{i=0}^{k} p(i)\right\}^2 \Delta' = \frac{N_y^{2k+1} S_y^2}{\lambda^2 N^{k+3} P(y>k)^{2k+2}} \prod_{i=0}^{k}\left\{\frac{p(i)^2}{n_i}\right\}$$

In the determinant of (22), we note the cofactor of A by D_A and calculate as follow.

(26)

$$D_A=\begin{vmatrix} -\dfrac{N_y{}^2p(0)^2}{N^2P(y>k)^2}\left(1-\dfrac{N}{n_0}\right) & \cdots\cdots & -\dfrac{N_y{}^2p(k)p(0)}{N^2P(y>k)^2} \\ \vdots & \ddots & \vdots \\ -\dfrac{N_y{}^2p(0)p(k)}{N^2P(y>k)^2} & \cdots\cdots & -\dfrac{N_y{}^2p(k)^2}{N^2P(y>k)^2}\left(1-\dfrac{N}{n_k}\right) \end{vmatrix}$$

$$=\frac{(-1)^{k+1}\prod\limits_{i=0}^{k}p(i)^2\,N_y{}^{2k+2}}{N^{2k+2}P(y>k)^{2k+2}}\begin{vmatrix} 1-\dfrac{N}{n_0} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 1-\dfrac{N}{n_k} \end{vmatrix}$$

$$=(-1)^{k+1}\left\{\frac{N_y}{NP(y>k)}\right\}^{2k+2}\prod_{i=0}^{k}\left\{\frac{p(i)^2}{n_i}\right\}\begin{vmatrix} n_0-N & n_0 & \cdots & n_0 \\ n_1 & n_1-N & \cdots & n_1 \\ \vdots & \vdots & \ddots & \vdots \\ n_k & n_k & \cdots & n_k-N \end{vmatrix}$$

$$=(-1)^{k+1}\left\{\frac{N_y}{NP(y>k)}\right\}^{2k+2}\prod_{i=0}^{k}\left\{\frac{p(i)^2}{n_i}\right\}\begin{vmatrix} n_0-N & N & 0 & \cdots & 0 \\ n_1 & -N & N & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & N \\ n_k & 0 & \cdots & 0 & -N \end{vmatrix}$$

$$=(-1)^{k+1}\left\{\frac{N_y}{NP(y>k)}\right\}^{2k+2}\prod_{i=0}^{k}\left\{\frac{p(i)^2}{n_i}\right\}\begin{vmatrix} N_x-N & 0 & 0 & \cdots & 0 \\ n_1 & -N & N & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & N \\ n_k & 0 & \cdots & 0 & -N \end{vmatrix}$$

$$=\frac{N_y{}^{2k+3}}{N^{k+2}P(y>k)^{2k+2}}\prod_{i=0}^{k}\left\{\frac{p(i)^2}{n_i}\right\}.$$

Deviding D_A by ΔN, we have the variance of the estimator of λ, noted in (27).

$$(27) \qquad Var(\hat{\lambda}) = \frac{\lambda^2}{N_y S_y^2}$$

To investigate the relative accuracy of the estimator, we compute the square of the coefficient of variation as follow:

$$(28) \qquad \{C.\ V.(\hat{\lambda})\}^2 = \frac{1}{N_y S_y^2}$$

$$(29) \qquad \frac{N_y S_y^2}{N} = \frac{\sum_{y=k+1}^{\infty} p(y)y^2 - \bar{y}^2 P(y>k)}{\left\{1 - \sum_{x=0}^{k} \theta_x p(x)\right\}} = \frac{\sum_{y=k+1}^{\infty} \{p(y)y(y-1) + P(y)y\} - \bar{y}^2 P(y>k)}{\left\{1 - \sum_{x=0}^{k} \theta_x p(x)\right\}}$$

$$= \frac{\lambda^2 P(y \geqq k-1) + \lambda P(y \geqq k) - \dfrac{\lambda^2 P(y \geqq k)^2}{P(y>k)}}{\left\{1 - \sum_{x=0}^{k} \theta_x p(x)\right\}}$$

$$= \frac{P(y \geqq k)}{\{1 - \sum_{x=0}^{k} \theta_x p(x)\}} \left(\frac{\lambda P(y \geqq k-1)}{P(y>k-1)} + 1 - \frac{\lambda P(y \geqq k)}{P(y>k)}\right)$$

The contents of the square bracket of the last side is a decreasing function of k, and of course, $P(y \geqq k)$ also. The contents of the parenthis is complex, but clearly the large values of $\theta_i (i = 1, 2, \cdots\cdots, k)$ are the factors of the large value of $N_y S_y^2$.

In conclusion, if k is relatively small and θ_i are large the coefficient of variation of $\hat{\lambda}$ will be small for any given value of λ. *Whether the estimator $\hat{\lambda}$ is usefull or not may be investigated in the right hand side of* (28) *by a sample value.*

Before sampling, the rough estimation can be done by the calculation of the last side of (29) for a given k and rough estimate of λ and θ_i. *In general, the large θ's and small k is desirable and the k greater than λ is not desirable.*

Now we shall seek for the cofactor corresponding to the element of r-th row r-th column in the determinant of (22). We denote this cofactor D^{rr} and compute as follow:

$$(30) \qquad D^{rr} = \frac{(-1)^{k-1}}{\lambda^2} \left\{\frac{N_y}{NP(y>k)}\right\}^{2k} \prod_{i=0}^{k} \left\{\frac{p(i)^2}{n_i}\right\} \frac{n_r}{p(r)^2}.$$

$$\left|\begin{array}{cccccccccc}
n_0-N & N & 0 & \cdots & & & & & 0 & n_0(0-\bar{u}) \\
n_1 & -N & N & \ddots & & & & & \vdots & \vdots \\
n_2 & 0 & \ddots & \ddots & \ddots & & & & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & & & \vdots & \vdots \\
n_{r-1} & \vdots & & \ddots & -N & N & \ddots & & \vdots & n_{r-1}(r-1-\bar{u}) \\
n_{r+1} & \vdots & & & \ddots & -N & N & \ddots & \vdots & n_{r+1}(r+1-\bar{u}) \\
\vdots & \vdots & & & & \ddots & \ddots & \ddots & 0 & \vdots \\
\vdots & \vdots & & & & & \ddots & \ddots & N & \vdots \\
n_k & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & -N & n_k(k-\bar{u}) \\
-\bar{u} & 1 & & 1 & 2 & 1 & \cdots & & 1 & -A\lambda^2
\end{array}\right|$$

$$\equiv\frac{(-1)^{k-1}}{\lambda^2}\left\{\frac{N_y}{NP(y>k)}\right\}^{2k}\prod_{i=0}^{k}\left\{\frac{p(i)^2}{n_i}\right\}\frac{n_r}{p(r)^2}\cdot D'^{rr}$$

where D'^{rr} is defined to be equal to the last determinant.

(31)

$$D'^{rr}=\left|\begin{array}{cccccccccc}
\sum_{i=0}^{k} n_i-n_r-N & 0 & \cdots & & & & & 0 & & \sum_{i=0}^{k} n_i(i-\bar{u})-n_r(r-\bar{u}) \\
\sum_{i=1}^{k} n_i-n_r & -N & \ddots & & & & & \vdots & & \sum_{i=1}^{k} n_i(i-\bar{u})-n_r(r-\bar{u}) \\
\sum_{i=2}^{k} n_i-n_r & 0 & -N & \ddots & & & & \vdots & & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & & & \vdots & & \vdots \\
\sum_{i=r-1}^{k} n_i-n_r & \vdots & & \ddots & -N & \ddots & & \vdots & & \sum_{i=r-1}^{k} n_i(i-\bar{u})-n_r(r-\bar{u}) \\
\sum_{i=r+1}^{k} n_i & \vdots & & & \ddots & -N & \ddots & \vdots & & \sum_{i=r+1}^{k} n_i(i-\bar{u}) \\
\sum_{i=r+2}^{k} n_i & \vdots & & & & \ddots & \ddots & \vdots & & \sum_{i=r+2}^{k} n_i(i-\bar{u}) \\
\vdots & \vdots & & & & & \ddots & 0 & & \vdots \\
n_k & 0 & \cdots & & & & 0 & & N & n_k(k-\bar{u}) \\
-\bar{u} & \underbrace{1 \cdots\cdots 1}_{r\ \text{columns}} & & & & & & & \underbrace{2 \quad 1 \cdots\cdots 1}_{k-r+1\ \text{columns}} & -A\lambda^2
\end{array}\right|$$

Letting Δ'_1 and Δ'_2 be the cofactors corresponding to the nonzero elements in the last but one column, we proceed as follows:

(32) $$D'^{rr} = -\Delta'_1 - N\Delta'_2$$

where

(33) $$\Delta'_1 = (-1)^{k-1} N^{k-2} \begin{vmatrix} \sum_{i=0}^{k} n_i(i-\bar{u}) - n_r(r-\bar{u}) & \sum_{i=0}^{k} n_i - n_r - N \\ n_k(k-\bar{u}) & n_k \end{vmatrix}$$

and (34)

$$\Delta'_2 = (-1)^{k-1} \begin{vmatrix} \sum_{i=0}^{k} n_i(i-\bar{u}) - n_r(r-\bar{u}) & \sum_{i=0}^{k} n_i - n_r - N & 0 & \cdots & \cdots & \cdots & 0 \\ \sum_{i=1}^{k} n_i(i-\bar{u}) - n_r(r-\bar{u}) & \sum_{i=1}^{k} n_i - n_r & -N & & & & \vdots \\ \vdots & \vdots & 0 & \ddots & & & \vdots \\ \sum_{i=r-1}^{k} n_i(i-\bar{u}) - n_r(r-\bar{u}) & \sum_{i=r-1}^{k} n_i - n_r & \vdots & -N & \ddots & & \vdots \\ \sum_{i=r+1}^{k} n_i(i-\bar{u}) & \sum_{i=r-1}^{k} n_i & \vdots & & -N & \ddots & 0 \\ \vdots & \vdots & \vdots & & & \ddots & \vdots \\ \sum_{i=k-1}^{k} n_i(i-\bar{u}) & \sum_{i=k-1}^{k} n_i & 0 & \cdots & \cdots & 0 & -N \\ -Ak^2 & -\bar{u} & 1 \cdots 1 & 2 & 1 \cdots 1 & & 1 \end{vmatrix}$$

$$\underbrace{\qquad\qquad}_{r+1 \text{ columns}} \qquad \underbrace{\qquad\qquad}_{k-r-1 \text{ columns}}$$

We have the value of cofactor corresponding to the element of last row last column in Δ'_2 as follow:

$$N^{k-3} \begin{vmatrix} \sum_{i=0}^{k} n_i(i-\bar{u}) - n_r(r-\bar{u}) & \sum_{i=0}^{k} n_i - N - n_r \\ \sum_{i=k-1}^{k} n_i(i-\bar{u}) & \sum_{i=k-1}^{k} n_i \end{vmatrix}$$

The cofactor corresponding to $-N$ in the last column of Δ'_2 is the similar type to Δ'_2 itself. So, we can repeat this process and obtain the following result:

$$D'^{rr} = (-1)^k N^{k-2} \left(\begin{vmatrix} \sum_{i=0}^{k} n_i(i-\bar{u}) - n_r(r-\bar{u}) & \sum_{i=0}^{k} n_i - n_r - N \\ n_k(k-\bar{u}) & n_k \end{vmatrix} \right.$$

$$+\begin{vmatrix} \sum_{i=0}^{k} n_i(i-\bar{u})-n_r(r-\bar{u}) & \sum_{i=0}^{k} n_i-n_r-N \\ \sum_{i=k-1}^{k} n_i(i-\bar{u}) & \sum_{i=k-1}^{k} n_i \end{vmatrix} + \cdots + \begin{vmatrix} \sum_{i=0}^{k} n_i(i-\bar{u})-n_r(r-\bar{u}) & \sum_{i=0}^{k} n_i-n_r-N \\ \sum_{i=r+2}^{k} n_i(i-\bar{u}) & \sum_{i=r+2}^{k} n_i \end{vmatrix}$$

$$+2\begin{vmatrix} \sum_{i=0}^{k} n_i(i-\bar{u})-n_r(r-\bar{u}) & \sum_{i=0}^{k} n_i-n_r-N \\ \sum_{i=r+1}^{k} n_i(i-\bar{u}) & \sum_{i=r+1}^{k} n_i \end{vmatrix} + \begin{vmatrix} \sum_{i=0}^{k} n_i(i-\bar{u})-n_r(r-\bar{u}) & \sum_{i=0}^{k} n_i-n_r-N \\ \sum_{i=r-1}^{k} n_i(i-\bar{u})-n_r(r-\bar{u}) & \sum_{i=r-1}^{k} n_i-n_r \end{vmatrix}$$

$$+\cdots\cdots+\begin{vmatrix} \sum_{i=0}^{k} n_i(i-\bar{u})-n_r(r-\bar{u}) & \sum_{i=0}^{k} n_i-n_r-N \\ \sum_{i=1}^{k} n_i(i-\bar{u})-n_r(r-\bar{u}) & \sum_{i=1}^{k} n_i-n_r \end{vmatrix} + \left.\begin{vmatrix} \sum_{i=0}^{k} n_i(i-\bar{u})-n_r(r-\bar{u}) & \sum_{i=0}^{k} n_i-n_r-N \\ -A\lambda^2 N & -\bar{u}N \end{vmatrix}\right\}$$

Noting that the corresponding elements of the first rows of all above determinants are common, we put as follows.

$$D'^{rr}=(-1)^k N^{k-2}\begin{vmatrix} \sum_{i=0}^{k} n_i(i-\bar{u})-n_r(r-\bar{u}) & \sum_{i=0}^{k} n_i-N-n_r \\ B & C \end{vmatrix} \tag{35}$$

where

$$\begin{aligned} B &= \sum_{j=r+1}^{k}\sum_{i=j}^{k} n_i(i-\bar{u})+\sum_{i=r+1}^{k} n_i(i-\bar{u})+\sum_{j=1}^{r-1}\{\sum_{i=j}^{k} n_i(i-\bar{u})-n_r(r-\bar{u})\}-A\lambda^2 N \\ &= \sum_{i=1}^{k} in_i(i-\bar{u})-n_r r(r-\bar{u})-A\lambda^2 N \\ &= \bar{u}(N_y\bar{y}+n_r r)-N_y(S_y^2+\bar{y}^2)-n_r r^2 \end{aligned}$$

$$\begin{aligned} C &= -N\bar{u}+\sum_{j=r+1}^{k}\sum_{i=j}^{k} n_i+\sum_{i=r+1}^{k} n_i+\sum_{j=1}^{r-1}\sum_{i=j}^{k} n_i-n_r(r-1) \\ &= -N\bar{u}+\sum_{i=1}^{k} n_i i-\sum_{i=r}^{k} n_i+\sum_{i=r+1}^{k} n_i-n_r(r-1)=-N\bar{u}+N_x\bar{x}-n_r r \\ &= -N_y\bar{y}-n_r r \end{aligned} \tag{36}$$

Applying this to the last side of B, we have

$$B=-\bar{u}C-r^2 n_r-N_y(S_y^2+\bar{y}^2) \tag{37}$$

Substituting (36) and (37) into (35), we have the following result.

$$\begin{aligned} D'^{rr} &= (-1)^k N^{k-2}\begin{vmatrix} \sum_{i=0}^{k} n_i(i-\bar{u})-n_r(r-\bar{u}) & \sum_{i=0}^{k} n_i-N-n_r \\ -\bar{u}C-r^2 n_r-N_y(S_y^2+\bar{y}^2) & C \end{vmatrix} \\ &= (-1)^k N^{k-2} N_y \begin{vmatrix} r-\bar{y} & -N_y-n_r \\ -S_y^2 & n_r(\bar{y}-r) \end{vmatrix} \\ &= (-1)^{k-1} N^{k-2} N_y\{n_r(\bar{y}-r)^2+(N_y+n_r)S_y^2\} \end{aligned} \tag{38}$$

Then we have the following variances and square of coefficients of variation of the estimators of θ_r $(r=1,2,\cdots\cdots\cdots,k)$ by (25), (30) and (38).

$$Var(\hat{\theta}_r)=\frac{D^{rr}}{\varDelta N}=\frac{P(y>k)^2 n_r}{N_y{}^3 S_y{}^2 p(r)^2}\{(N_y+n_r)S_y{}^2+n_r(\bar{y}-r)^2\} \tag{39}$$

or
$$=\frac{(1-\theta_r)^2}{n_r}\left(1+\frac{n_r}{N_y}+\frac{n_r}{N_y}\cdot\frac{(\bar{y}-r)^2}{S_y{}^2}\right)$$

$$=(1-\theta_r)^2\left(\frac{1}{n_r}+\frac{1}{N_y}+\frac{(y-r)^2}{N_y S_y{}^2}\right)$$

$$\{C.V.(\hat{\theta}_r)\}^2=\left(\frac{1-\theta_r}{\theta_r}\right)^2\left(\frac{1}{n_r}+\frac{1}{N_y}+\frac{(y-r)^2}{N_y S_y{}^2}\right) \tag{40}$$

Then the right hand side of (39) and (40) show that *the estimator* $\hat{\theta}_r$ *in* (12) *is the more useful*

(i) *the larger (the nearer to* 1) θ_r *and other* θ*'s are,*

(ii) *the larger (the nearer to* k) r *is,*

(iii) *the larger* $p(r)$ *is,*

(iv) *the smaller* k *is,*

and especially

(v) *the larger* N *is.*

3. Estimators in the case when some θ's are known and other θ's are unknown.

The estimator (15) and (12) in the section **1** are available also in the case when the paramaters $\theta_l(l=l_1,l_2,\cdots\cdots,l_s)$ are known and other θ's $\theta_j(j=j_0,j_1,j_2,\cdots\cdots,j_{k-s})$ are unknown.

If we wish to improve the relation (15) and (12) by the informations of known values of θ's, we may use the following formula.

$$\bar{z}=\frac{\lambda[P(y\geqq k)+\sum_{i=1}^{s}(1-\theta_{l_i})p(l_i-1)]}{[P(y>k)+\sum_{i=1}^{s}(1-\theta_{l_i})p(l_i)]},\quad \hat{\theta}_j=1-\frac{n_j\{P(y>k)+\sum_{i=1}^{s}(1-\theta_{l_i})p(l_i)\}}{p(j)(N-\sum_{i=0}^{k-s}n_{j_i})}, \tag{41}$$

where $\bar{z}$ is the mean in the total classes except the classes with unknown θ's

Proof. Putting (6) and (7) to be zero, we have follows:

$$\theta_j=1-\frac{n_j\{1-\sum_{x=0}^{k}\theta_x p(x)\}}{p(j)N}\quad (j=j_1,j_2,j_3,\cdots\cdots\cdots,j_{k-s}). \tag{42}$$

$$\frac{\bar{u}}{\lambda}=\frac{1-\sum_{x=0}^{k-1}\theta_{x+1}p(x)}{1-\sum_{x=0}^{k}\theta_x p(x)} \tag{43}$$

Putting $1-\sum_{x=0}^{k}\theta_x p(x)=G$, we have the following relations from (42).

$$G=1-\sum_{i=0}^{k-s}p(j_i)+\frac{G}{N}\sum_{i=0}^{k-s}n_{ji}-\sum_{i=1}^{s}\theta_{li}p(l_i)=P(y>k)+\sum_{i=1}^{s}p(l_i)(1-\theta_{li})+\frac{G}{N}N'_x$$

where N'_x is the total sum of frequencies in the classes whose θ's are unknown. Then we have

$$G=\frac{N}{N_y+N''_x}\left(P(y>k)+\sum_{i=1}^{s}p(l_i)(1-\theta_{li})\right),\ \text{where}\ N''_x=N_x-N'_x,$$

$$N_x\bar{x}=N'_x\bar{x}'+N''_x\bar{x}''$$

Substituting this into (42), we have the improved $\hat{\theta}_j$ in (41).

The numerator of the right hand side of (43) is as follow.

$$\begin{aligned}1-\sum_{x=0}^{k-1}\theta_{x+1}p(x)&=1-\sum_{x=1}^{k}\theta_x p(x-1)=1-\sum_{i=1}^{s}\theta_{li}p(l_i-1)-\sum_{i=0}^{k-s}\theta_{ji}p(j_i-1)\\&=1-\sum_{i=1}^{s}\theta_{li}p(l_i-1)-\sum_{i=0}^{k-s}p(j_i-1)+\sum_{i=0}^{k-s}\frac{n_{ji}j_iG}{N\lambda}\\&=P(y\geqq k)+\sum_{i=1}^{s}(1-\theta_{li})p(l_i-1)+\frac{G}{N\lambda}\sum_{i=0}^{k-s}n_{ji}j_i\end{aligned}$$

Substituting them into (43), we have

$$\frac{\bar{u}}{\lambda}=\frac{P(y\geqq k)+\sum_{i=1}^{s}(1-\theta_{li})p(l_i-1)}{G}+\frac{N'\bar{x}'}{\lambda N}$$

Then $$\frac{N_y\bar{y}+N''_x\bar{x}''}{\lambda N}=\frac{(N_y+N''_x)}{N}\left\{\frac{P(y\geqq k)+\sum_{i=1}^{s}(1-\theta_{li})p(l_i-1)}{P(y>k)+\sum_{i=1}^{s}(1-\theta_{li})p(l_i)}\right\}$$

If we put $\bar{z}=\dfrac{N_y\bar{y}+N''_x\bar{x}''}{N_y+N''_x}$ the mean in the total classes except the classes with unknown θ's, (41) is derived from this formula.

But the relation (41) is so complex that it can not be considered convenient for practical use in general.

We shall show some special simple cases which may be for practical use.

(i) Case when $\theta_0=\theta_1=\cdots\cdots=\theta_s=1$ (i. e. the truncated case)

$$\bar{z}=\frac{\lambda P(y\geqq k)}{P(y>k)} \quad (N''_x=0, \quad \bar{z}=\bar{y})$$

perfectly coincides with (15)

(ii) Case when $\theta_0=\theta_1=\cdots\cdots\cdots\theta_s=0$

$$\bar{z}=\lambda\left(\frac{P(y\geqq k)+P(x\leqq s-1)}{P(y>k)+P(x\leqq s)}\right) \tag{44}$$

(iii) Case when $\theta_{k-s+1}=\theta_{k-s+2}=\cdots\cdots\cdots=\theta_k=1$
coincides with (i).

(iv) Case when $\theta_{k-s+1}=\theta_{k-s+2}=\cdots\cdots\cdots=\theta_k=0$

$$\bar{z}=\frac{\lambda P(i\geqq k-s)}{P(i>k-s)}$$

The same case as the case with (15)

4. Variances of estimators in the case when some θ's are known.

In the case when $\theta_l(l=l_1,l_2,\cdots\cdots\cdots\cdots,l_s;\ s\leqq k)$ are known, the number of estimators decreases by s, and in the determinants (22)$\sim$(26), some s rows and some s columns must vanish.

Let the unknown θ's be $\theta_j(j=j_0,j_1,j_2,\cdots\cdots\cdots,\ j_{k-s};\ s\leqq k)$.

In the case, the variances of $\hat{\lambda}$ and $\hat{\theta}_r$ are following (53) and (57).

The formulas in this case corresponding to (23)$\sim$(28) are the following (45)$\sim$(54) respectively:

$$\Delta=\frac{(-1)^{k-s}}{\lambda^2}\prod_{i=0}^{k-s}\left\{\frac{N_y p(j_i)}{NP(y>k)}\right\}^2\cdot\Delta'' \tag{45}$$

where

$$\Delta''=\begin{vmatrix} 1-\dfrac{N}{nj_0} & 1 & \cdots\cdots & 1 & j_0-\bar{n} \\ 1 & 1-\dfrac{N}{nj_1} & \ddots & \vdots & j_1-\bar{n} \\ \vdots & \ddots & \ddots & 1 & \vdots \\ 1 & \cdots\cdots & 1 & 1-\dfrac{N}{nj_{k-s}} & j_{k-s}-\bar{n} \\ j_0-\bar{n} & j_1-\bar{n} & \cdots\cdots & j_{k-s}-\bar{n} & -A\lambda^2 \end{vmatrix}$$

To simplify this, we proceed as follows.

(46) $$\Delta''=\left\{\sum_{i=0}^{k-s}n_{ji}\right\}^{-1}\begin{vmatrix} n_{j_0}-N & n_{j_0} & \cdots\cdots & n_{j_0} & n_{j_0}(j_0-\bar{u}) \\ n_{j_1} & n_{i_1}-N & \cdots\cdots & n_{j_1} & n_{j_1}(j_1-\bar{u}) \\ \vdots & \vdots & & \vdots & \vdots \\ n_{j_{k-s}} & n_{j_{k-s}} & \cdots\cdots & n_{j_{k-s}}-N & n_{j_{k-s}}(j_{k-s}-\bar{u}) \\ j_0-\bar{u} & j_1-\bar{u} & \cdots\cdots & j_{k-s}-\bar{u} & -A\lambda^2 \end{vmatrix}$$

$$=\left\{\sum_{i=0}^{k-s}n_{ji}\right\}^{-1}\begin{vmatrix} N_x'-N & 0 & 0 & \cdots\cdots & 0 & N_x'(\bar{x}'-\bar{u}) \\ n_{j_1} & -N & N & \ddots & \vdots & n_{j_1}(j_1-\bar{u}) \\ \vdots & 0 & -N & \ddots & 0 & \vdots \\ & \vdots & \ddots & \ddots & N & \\ n_{j_{k-s}} & 0 & \cdots\cdots & 0 & -N & n_{j_{k-s}}(j_{k-s}-\bar{u}) \\ j_0-\bar{u} & j_1-j_0 & \cdots\cdots & & j_{k-s}-j_{k-s-1} & -A\lambda^2 \end{vmatrix}$$

where

(47) $$N'_x=\sum_{i=0}^{k-s}n_{ji},\qquad \bar{x}'=\frac{1}{N'_x}\sum_{i=0}^{k-s}n_{ji}j_i,\qquad S'^2_x=\frac{1}{N'_x}\sum_{i=0}^{k-s}n_{ji}j_i^2-\bar{x}'^2$$

: sum of frequencies, mean and variance of all classes of which θ's are unknown respectively. And further, we put as follows with respect to the classes with known values of θ's:

(48) $$N''_x=N_x-N'_x,\qquad N''_x\bar{x}''=N_x\bar{x}-N'_x\bar{x}'$$
$$N''_x(S''^2_x+\bar{x}''^2)=N_x(S_x^2+\bar{x}^2)-N'_x(S'^2_x+\bar{x}'^2)$$

With these notations, we have the following (49) by (46)~(48)

(49) $$\Delta''=\left\{\prod_{i=0}^{k-s}n_{ji}\right\}^{-1}(-1)^{k-s}N^{k-s-1}\Big[-(N_y\bar{y}+N''_x\bar{x}'')^2+(N_y+N''_x)\cdot\{N''_x(S''^2_x+\bar{x}''^2)+N_y(S_y^2+\bar{y}^2)\}\Big].$$

Then we have following (50) by (45) and (49)

(50) $$\Delta=\frac{N_y^{2(k-s+1)}N_z^2S_z^2}{N^{k-s+3}P(y>k)^{2(k-s+1)}\lambda^2}\prod_{i=0}^{k-s}\left\{\frac{p(j_i)^2}{n_{ji}}\right\}$$

where N_z and S_z^2 are the total sum of frequencies and variance with respect to the all classes except the classes with unknown θ's respectively, holding the relations:

(51)
$$N_z=N_y+N''_x,\qquad N_z z=N_y\bar{y}+N''_x\bar{x}'',$$
$$N_z(S_z^2+\bar{z}^2)=N''_x(S_x''^2+\bar{x}''^2)+N_y(S_y^2+\bar{y}^2).$$

By the similar process, we have the value of the cofactor of (45) corresponding to (26) as follow.

(52)
$$D_A=\left\{\frac{N_y}{NP(y>k)}\right\}^{2(k-s+1)}N^{k-s}N_z\prod_{i=0}^{k-s}\left\{\frac{p(j_i)^2}{n_{ji}}\right\}$$

Then we have the *variance of estimator in the same type as* (27) *as follow*

(53)
$$Var(\hat{\lambda})=\lambda^2/N_zS_z^2$$

where N_z and S_z^2 are defined in (51).

Then the square of coefficient of variation is shown in (54).

(54)
$$\{C.\,V.(\hat{\lambda})\}^2=1/N_zS_z^2$$

As (53) and (54) are similar type to (27) and (28) in the section **2** respectively, our conclusion in this case contains that of the section **2** as described later.

To investigate (54) precisely, we consider following relations

(55)
$$\frac{N_zS_z^2}{N}=\frac{N_yS_y^2+N''_xS''_x{}^2}{N}+\frac{N_yN''_x(\bar{y}-\bar{x}'')^2}{N(N_y+N''_x)}$$

(56)
$$\frac{N''_x}{N}=\frac{\sum_{i=1}^{s}(1-\theta_{l_i})p(i_i)}{1-\sum_{x=0}^{k}\theta_x p(x)},\qquad \frac{N_y}{N}=\frac{P(y>k)}{1-\sum_{x=0}^{k}\theta_x p(x)}$$

where l_i is the class number with known θ.

Then we have the conclusion in this case as follows from (54), (55) and (56): the estimator $\hat{\lambda}$ *is the more usefull*

(i) *the smaller* k *is,*

(ii) *the greater the unknown* θ*'s are,*

(iii) *the smaller the known* θ*'s are,*

(iv) *the smaller the mean in the classes with known* θ *is,*

(v) *the greater the variance in the classes with known* θ *is,*

and especially (vi) *the larger* N *is.*

As to the variance and C. V. of $\hat{\theta}$ in this case, we have the following results (57) and (58) in the similar type to (39) and (40) respectively after a process similar to that of (53) and (39) though more complex.

(57)
$$Var(\hat{\theta}_{jr})=\frac{(1-\theta_{jr})^2}{n_{jr}}\cdot\frac{(N_z+n_{jr})^2\,S'_z{}^2}{N_z^2\,S_z^2}$$

or
$$=(1-\theta_{jr})^2\left(\frac{1}{n_{jr}}+\frac{1}{N_z}+\frac{(j_r-z)^2}{N_z\,S_z^2}\right)$$

where S'^2_z is the variance in the total sum of classes $i>k$, those with known θ and θ_{jr}.

$$\{C.\ V.\ (\theta_{jr})\}^2=\left(\frac{1-\theta_{jr}}{\theta_{jr}}\right)^2\left(\frac{1}{n_{jr}}+\frac{1}{N_z}+\frac{(j_r-\bar{z})^2}{N_z S_z^2}\right) \tag{58}$$

Then the *estimator* $\hat{\theta}_{jr}$ *is the more useful,*

(i) *the larger (the nearer to 1)* θ_{jr} *and other* θ*'s are,*

(ii) *the larger* $p(j_r)$ *is,*

(iii) *the nearer to* $\bar{z}$ j_r *is,*

(iv) *the smaller* k *is,*

(v) *the more* θ*'s of the classes with large densities are known,*

and especially, (vi) *the larger* N *is.*

5. Illustrative examples

Example 1. Table A shows the frequencies of deaths by traffic accidents per zone per month recorded in some zones in the city Yokohama. It contains too large frequencies in the classes of 0 and 1 to regard as a Poisson distribution. The test of goodness of fit proved this fact very clearly.

We applied our method in **1** of the case when $k=0$ and $k=1$ as followings.

Table A. Distribution of number of deaths by traffic accidents

No. of deaths per zone per month i	Frequencies				
	Original data (1)	Expected data $k=0$ (2)	Expected data $k=1$ (3)	Original (4)	Expected (5)
0	204	204	204	52	41.2
1	69	59.3	69	36	44.0
2	24	32.6	20.3	18	23.5
3	5	12.0	11.5	4	8.4
4	7	3.3	4.9	7	2.3
5	2	0.7	1.7	2	0.5
6	1	0.1	0.5	1	0.1
7	0	0.0	0.1	0	0.0
Total	312	312.0	312.0	120	120.0

(i) Case $k=0$

$$\bar{y}=\frac{176}{108}=1.63,\quad \hat{\lambda}=1.1,\quad C.\ V.(\hat{\lambda})=0.0908,$$

$$\hat{\theta}_0=1-\frac{204\ /\ 0.667}{108\ /\ 0.333}=-2.7778,\quad C.\ V.(\hat{\theta}_0)=0.2584,$$

(ii) Case $k=1$

$$\bar{y}=\frac{107}{39}=2.74,\quad \hat{\lambda}=1.7,\quad C.V.(\hat{\lambda})=0.148$$

$$\hat{\theta}_0=-13.507,\qquad C.V.(\hat{\theta}_0)=0.476$$

$$\hat{\theta}_1=-1.887,\qquad C.V.(\hat{\theta}_1)=0.498$$

From the point of view of the coefficient of variation, we should give up the case $k=1$.

Agreement between (1) and (2) for the classes of $i\geqq 1$ is satisfactory. The goodness of fit $\chi^2=3.931$ is smaller than $\chi^2_{0.05}=5.9915$ with 2 degrees of freedom. From the frequencies in column (1) we excluded those in the zones with deaths less than 6 and placed the rests on the column (4). Column (5) is the expected Poisson frequencies fitted to (4) with $\lambda=1.1$. The goodness of fit is $\chi^2=6.228$ which is less than $\chi^2_{0.05}=7.8147$ with 3 degrees of freedom. The agreement is fairly good.

The fact shows that the data (1) is the mixed frequencies of the Poisson zones and nearly deathless zones.

Example 2.

We applied our method of **3** to the Cohen's illustrative example in [2] page 347 as shown in the following Table B.

Expected frequencies in column (4) are the frequencies fitted to (2) by our method of the case $s=0$, $k=1$, $\theta_0=1$. The rsults are almost coincident with Cohen's results in the column (3), and the values of $\hat{\lambda}$, $\hat{\theta}$, etc.

Table B. Distribution of number of gall-cells

No. of Gall-cells i	Frequency			
	Observed data (1)	Altered data (2)	Expected	
			by Cohen (3)	by the auther $k=1, s=0, \theta_0=1$ (4)
0	0	0	0	0
1	90	60	60	60
2	96	96	88.6	89.1
3	57	57	60.6	60.6
4	26	26	31.0	30.9
5	10	10	12.7	12.6
6	4	4	4.3	4.3
7	5	5	1.3	1.2
8	0	0	0.3	0.3
9	1	1	0.1	0.1
10	0	0	0.1	0.0
Total	289	259	259.0	259.1

$$\bar{z}=\bar{y}=2.93, \quad \hat{\lambda}=2.04, \quad C.V.(\hat{\lambda})=0.05$$
$$\hat{\theta}_1=0.32, \quad Var(\hat{\theta}_1)=0.0172$$

Example 3.

If the 30 observations were missed from the 96 observations in the class of $i=2$ in column (1) of Table B, we have the column (1) in the Table C.

We shall estimate λ and θ_2 from this data applying the theory in **3** and **4.**

Table C.

No. of Gall-cells i	Frequency	
	Altered data (1)	Expected data (2) $\theta_0=1$ $\theta_1=0$
0	0	0
1	90	85.3
2	66*	66.0
3	57	59.2
4	26	30.3
5	10	12.3
6	4	4.2
7	5	1.3
8	0	0.3
9	1	0.1
10	0	0.0
Total	259	259.0

$$N_z=193$$

$$\bar{z}=\frac{483}{193}=2.5026$$

$$N_z S_z^2=795.5$$

We have the following relation by (44).

$$\bar{z}=\frac{\lambda[P(y\geqq 2)+p(0)]}{P(y>2)+p(1)}$$

$$=\frac{\lambda[1-p(1)]}{1-p(2)-p(0)}\equiv \Phi(\lambda)$$

λ	2.0	2.1
$\Phi(\lambda)$	2.451	2.566

Then we have

$$\hat{\lambda}=2.04$$

by interpolation. Applying (53), (41) and (57), we have the following results.

$$Var(\hat{\lambda})=0.007,$$
$$\hat{\theta}_2=0.2424, \quad Var(\hat{\theta}_2)=0.012.$$

The goodness of fit between column (1) and column (2) except the classes $i=0$ and $i=2$ is $\chi^2=1.0301$ which is less than $\chi^2_{0.05}=7.8147$ with 3 degree of fredom.

Table I Numerical values of $\frac{\lambda P(y \geqq k)}{P(y > k)}$

k \ λ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	1.0508	1.1033	1.1575	1.2133	1.2708	1.3298	1.3906	1.4528	1.5166
1	2.0339	2.0689	2.1051	2.1424	2.1810	2.2208	2.2618	2.3039	2.3475
2	3.0254	3.0514	3.0784	3.1062	3.1347	3.1642	3.1943	3.2256	3.2576
3	4.0202	4.0412	4.0625	4.0844	4.1071	4.1300	4.1539	4.1783	4.2033
4		5.0341	5.0518	5.0700	5.0883	5.1075	5.1269	5.1470	5.1674
5				6.0596	6.0756	6.0916	6.1080	6.1248	6.1420
6					7.0656	7.0795	7.0938	7.1081	7.1229

Table I (continued)

k \ λ	1	2	3	4	5	6	7	8	9	10	λ / k
0	1.5820	2.3130	3.1572	4.0748	5.0340	6.0149	7.0064	8.0027	9.0011	10.0005	0
1	2.3922	2.9114	3.5595	4.3224	5.1756	6.0908	7.0451	8.0215	9.0100	10.0045	1
2	3.2906	3.6744	4.1652	4.7692	5.4811	6.2854	7.1611	8.0870	9.0453	10.0228	2
3	4.2290	4.5258	4.9053	5.3796	5.9550	6.6307	7.3974	8.2392	9.1379	10.0765	3
4	5.1883	5.4272	5.7285	6.1056	6.5680	7.1234	7.7721	8.5086	9.3212	10.1948	4
5	6.1594	6.3576	6.6042	6.9096	7.2846	7.7386	8.2785	8.9062	9.6181	10.4056	5
6	7.1380	7.3068	7.5129	7.7660	8.0742	8.4479	8.8954	9.4230	10.0337	10.7248	6
7	8.1219	8.2680	8.4441	8.6572	8.9158	9.2266	9.5991	10.0414	10.5598	11.1552	7
8	9.1086	9.2374	9.3912	9.5744	9.7931	10.0558	10.3689	10.7407	11.1783	11.6877	8
9		10.2134	10.3482	10.5080	10.6972	10.9213	11.1880	11.5026	11.8742	12.3080	9
10		11.1932	11.3136	11.4548	11.6203	11.8145	12.0432	12.3135	12.6299	13.0005	10
11		12.1768	12.2853	12.4112	12.5571	12.7277	12.9268	13.1601	13.4326	13.7511	11
12			13.2615	13.3736	13.5063	13.6564	13.8315	14.0345	14.2712	14.5471	12
13			14.2413	14.3444	14.4623	14.5969	14.7529	14.9316	15.1395	15.3735	13
14				15.3176	15.4252	15.5463	15.6855	15.8456	16.0290	16.2405	14
15				16.2956	16.3931	16.5044	16.6294	16.7727	16.9357	17.1231	15
16				17.2761	17.3658	17.4668	17.5815	17.7106	17.8574	18.0238	16
17					18.3419	18.4352	18.5396	18.6565	18.7898	18.9396	17
18					19.3205	19.4068	19.5029	19.6110	19.7314	19.8678	18
19						20.3818	20.4709	20.5699	20.6796	20.8045	19
20						21.3602	21.4428	21.5342	21.6369	21.7482	20
21							22.4167	22.5026	22.5965	22.7001	21
22							23.3952	23.4744	23.5612	23.6576	22
23								24.4478	24.5300	24.6204	23
24								25.4257	25.5020	25.5852	24
25									26.4765	26.5548	25
26										27.5265	26
27										28.5019	27

REFERENCES

[1] A. Cliford Cohen, J_R: Estimating the parameters of a modified Poisson distribution, Journal of American Statistical Society Vol. 55; No. 289. pp. 139-143 (1960).

[2] ______; Estimation in the truncated Poisson distribution when zeros and some ones are missing, Journal of American Statistical Society vol. 55, No. 290 pp. 342-348 (1960).

[3] Research Association of statistical Science: Statistical Tables.

Errata

In Vol. IX (1961), the following corrections should be made.

1. In the article "Fundamentaj Teoremoj por Elipsa Sistemo de du Laŭpartaj Diferencialaj Ekvacioj de la Unua Ordo kun du Sendependaj Variantoj" by Kanji NAKAMORI (pages 1—28), page 9 and 10 should be exchanged.
2. In the article "Some Estimations of the Parameters of Multinormal Populations from Linearly Truncated Samples. I" by Keizo YONEDA (page 149—161), on page 156, line 3—4, "less" and "greater" should be exchanged.