ON LOCAL IMBEDDING OF KAEHLERIAN MANIFOLDS K^{2n} IN A FLAT HERMITIAN MANIFOLD H^{2n+2}.

By

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Introduction.

This paper contains mainly two concepts. One concerns the construction of a $2n$-dinensional real analytic manifold which on one hand is to be the real representation of an n-dimensional Kaehlerian space and on the other hand is to be a submanifold of a $(2n+2)$-dimensional real analytic manifoled H^{2n+2}, which whereas is the representation of an $(n+1)$-dimensional flat Hermitian space. While the other concerns the inquiry of the conditions to be imposed on the curvature tensor so that a $2n$-dimensional real manifold K^{2n} representing an arbitrary n-dimensional Kaehlerian space can be imbedded in H^{2n+2} isometrically.

In preliminary § 1 we construct such H^{2n+2} by the aid of *B.* Eckmann and *A.* Frölichers' theorem [4] which serves to endow an almost complex manifold with a complex structure. In H^{2n+2} we set up a $2n$-dimensional almost analytic submanifold M^{2n}, whose notion was previously introduced by J. A. Schouten and K. Yano in the course of study on their so-called invariant submanifolds [19]. We can show that such M^{2n} is always the real representation of an n-dimensional sub-Kaehlerian space. Further for the completion of being a submanifold in geometrical sense we define a couple of unit normals on M^{2n} by the use of the characteristic property owned by the tensor giving an almost complex structure to M^{2n}.

In § 2 we define the usual tensors accompanied with a submanifold M^{2n} together the fundamental equations followed by. The condition of integrability of the latter is derived and adding to this a remark is given on the influence of the tensor giving the complex structure upon these conditional equations.

§ 3 is the proof of the classical theorem of Bonnet following the prototype which we have and see in the arguments of imbedding a Riemanian space in a Euclidean space, e. g, [6], [20].

§ 4 is devoted to find the conditions that are to be fulfilled by the curvature tensor so that an arbitrary K^{2n} can be imbedded in H^{2n+2} isometrically. Our main theornm obtained is that its curvature tensor R_{ijkl} should satisfy

$$R_{ijkl}R^{ijkl}=R^2,\ R<0,$$

for which. As its direct consequence we prove that there exists no K^{2n}, $n\geqq 2$, of non-vanishing constant holomorphic curvature that can be imbedded in H^{2n+2} isometrically.

§ 5 is a geometrical interpretation of non-existence theorem of a K^{2n} that is one for which the Ricci equations contained in the condition of integrability vanish identically.

A remark is given on the uniqueness of the second fundamental ten- sors in accordance with the rank of the matrix composed of the right hand side of the Ricci equations.

§ 1. Sub-Kaehlerian manifold K^{2n} in a flat Hermitian manifold H^{2n+2}

We adopt the following conventions for indices:

$$A, B, C, D = 1, 2, \cdots\cdots, n+1;\ \bar{1},\ \bar{2},\cdots\cdots\cdots\cdots,\ \overline{n+1},$$
$$\Theta, \Lambda, \Pi, \Gamma = 1, 2,\cdots\cdots\cdots, n+1,$$
$$\bar{\Theta}, \bar{\Lambda}, \bar{\Pi}, \bar{\Theta} = \bar{1},\ \bar{2},\ \cdots\cdots, \overline{n+1},$$
$$a, b, c, d, e, f, g, h, i, j, k, 1 = 1, 2,\cdots\cdots, n;\ \bar{1}, \bar{2},\cdots\cdots\cdots, \bar{n},$$
$$\alpha, \beta, \gamma, \delta, \nu, \rho, \lambda, \mu, \upsilon, \omega = 1, 2,\cdots\cdots, n$$
$$\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}, \bar{\nu}, \bar{\rho}, \bar{\lambda}, \bar{\mu}, \bar{\upsilon}, \bar{\omega} = \bar{1}, \bar{2},\cdots\cdots, \bar{n}$$
$$P, Q, R, S = \mathrm{I}, \mathrm{II}$$

and consider an $(n+1)$-dimensional flat Hermitian space, covered by the complex cordinate system $(z^1, z^2,\cdots\cdots, z^{n+1})$ where $z^{\Theta}=\xi^{\Theta}+i\xi^{\bar{\Theta}}$, $i^2=-1$, and on it we presume that its element of length takes the form

$$ds^2=dz^1d\bar{z}^1+\cdots\cdots\cdots+d^{n+1}\,d\bar{z}^{n+1}, \tag{1. 1}$$

where

$$\bar{z}^{\Theta}=z^{\bar{\Theta}}=\overline{\xi^{\Theta}+i\xi^{\bar{\Theta}}}=\xi^{\Theta}-i\xi^{\bar{\Theta}}$$

Since we are dealng with a complex space, we can define a tensor having the numerical components

$$\phi_B{}^A=\begin{Vmatrix} i\delta_\Lambda{}^\Theta & 0 \\ 0 & -i\delta_\Lambda{}^\Theta \end{Vmatrix} \tag{1. 2}$$

in our cordinate system. The tensor $\phi_B{}^A$ satisfies the tensorial equation

$$\phi_B{}^A\phi_C{}^B=-\delta_C{}^A, \tag{1.3}$$

and has the numerical components

$$\left\| \begin{matrix} 0 & \delta_{\Lambda}{}^{\theta} \\ -\delta_{\Lambda}{}^{\theta} & 0 \end{matrix} \right\|$$

in the real coordinate system (ξ^A). We have then

$$(1.4) \qquad N_{BC}^{\cdot\cdot A} = 0,$$

where $N_{BC}^{\cdot\cdot A}$ is the Nijenhuis tensor [16]:

$$(1.5) \qquad N_{BC}^{\cdot\cdot A} = 2\phi_{[B}{}^{D}(\partial_{D}\phi_{C]}{}^{A} - \partial_{C]}\phi_{D}{}^{A}),$$

and $\partial = \dfrac{\partial}{\partial \xi^A}$.

The metric defined above is a Riemannian, and if we express it by ε_{AB}, it has the components

$$(1.6) \quad \varepsilon_{AB} = \left\| \begin{matrix} 0 & \delta_{\bar{\theta}\Lambda} \\ \delta_{\theta\bar{\Lambda}} & 0 \end{matrix} \right\|, \left(\delta_{\theta\bar{\Lambda}} = \overline{\delta_{\theta\Lambda}} = \begin{matrix} 1 \text{ for } \theta = \Lambda \\ 0 \text{ for } \theta \neq \Lambda \end{matrix} \right),$$

in the complex coordinate system, and it is equivalent to the tensorial expression

$$(1.7) \qquad \varepsilon_{AB}\phi_{C}{}^{A}\phi_{D}{}^{B} = \varepsilon_{CD}.$$

As the space is a flat one, we have

$$(1.8) \qquad R^{A}_{\cdot BCD} = 0,$$

whero $R^{A}_{\cdot BCD}$ is the Riemann-Christoffel curvature tensor formed with ε_{AB}.

Conversely, if in a $(2n+2)$-dimensional manifold X^{2n+2} of class C^{ω} covered by a system of coordinates (ξ^A) there is given a tensor $\phi_B{}^A$ of class C^{∞} satisfying

$$(1.9) \qquad \phi_B{}^A\phi_C{}^B = -\delta_C{}^A, \ N_{BC}^{\cdot\cdot A} = 0,$$

and also a Riemannian metric G_{AB} satisfying

$$(1.10) \qquad G_{AB}\phi_C{}^A\phi_D{}^B = G_{CD}, \ \mathring{R}_{\cdot}{}^{A}{}_{BCD} = 0,$$

where $\mathring{R}_{\cdot BCD}$ is the Riemann-Christoffel curvature tensor formed with G_{AB}, we can show that an $(n+1)$-dimensional flat Hermitian space having the element of distance (1.1) can be derived from X^{2n+2}.

For the proof we first note that an even dimensional real space of class C^{∞}

is called an almost complex manifold, if it admits a tensor ϕ_{BA} of class C^{∞} satisfying $\phi_B{}^A\phi_C{}^B=-\delta_C{}^A$. If there exists a coordinate system in which $\phi_B{}^A$ has the components (1.2), we say that the almost complex structure $\phi_B{}^A$ of class C^{ω} gives a complex structure to the manifold. As the concept connecting these structural situations the following theorem is now well-known [2], [4], [5], [8], [12];

In order that the almost complex structure $\phi_B{}^A$ of class C^{ω} in a manifold of class C^{ω} give a complex structure, it is necessay and sufficient that the Nijenhuis tensor vanishes.

It is now easy to prove our assertion. Because of $(1.9)_1$, our X^{2n+2} of class C^{ω} is an almost complex manifold and the condition $(1.9)_2$ assures that $\phi_B{}^A$ gives a complex structure to X^{2n+2} by the theorem stated above. And by $(1.10)_1$, the given Riemanian metric G_{AB} has the components

$$G_{AB}=\left\| \begin{matrix} 0 & G_{\bar{\theta}\lambda} \\ G_{\theta\bar{\lambda}} & 0 \end{matrix} \right\|$$

in the complex coodinate system. But as we have $(1.10)_2$ we can take a coordinate system in which G_{AB}=const., and the element of length takes the form

$$ds^2=2G_{\theta\bar{\lambda}}d'z^{\theta}d'\bar{z}^{\lambda}.$$

Then by a suitable unitary transformation of the coordinates, we can reduce this to (1.1), which proves our statement. On this reason we call a $2n$-dimensional real analytic manihold $X^{2n+2}(\xi)$ satisfying the conditions(1.9) and (1.10) to be *the real representation of an $(n+1)$-dimensional flat Hermitian space.* Then we may call such $X^{2n+2}(\xi)$ *a flat Hermitian manifold* substantially. We shall denote it by $H^{2n+2}(\xi)$ hereafter.

We now consider a $2n$-dimensional submanifold $M^{2n}(\xi)$ of class C^{ω} in $H^{2n+2}(\xi)$:

$$\xi^A=\xi^A(\eta^i). \tag{1.11}$$

If the transform by $\phi_B{}^A$ of any vector tangent to $M^{2n}(\eta)$ is still tangent to $M^{2n}(\eta)$, we call $M^{2n}(\eta)$ an almost analytic submanifold [19], [24]. A necessary and sufficient condition for $M^{2n}(\eta)$ to be so is

$$\phi_B{}^A B_i{}^B=\phi_i{}^j B_j{}^A, \tag{1.12}$$

where $\phi_j{}^i$ is a certain tensor of class C^{ω} in $M^{2n}(\xi)$ and $B_i{}^A$ is the function defined by

$$B_i{}^A=\partial_i\xi^A, \qquad \partial_i=\frac{\partial}{\partial\eta^i}. \tag{1.13}$$

For a moment we assume that $B_i{}^A$ is of class C^{∞}.

If we multiply (1.12) by $\phi_A{}^C$ we have

$$\phi_j{}^i\phi_k{}^j=-\delta_k{}^i, \tag{1.14}$$

and this shows that $M^{2n}(\eta)$ is an almost complex manifold. By a straightforward computation we have

$$B_i{}^B B_j{}^C N_{BC}^{\cdot\cdot A} = B_a{}^A N_i{}^{\cdot}{}_j{}^{\cdot a} \tag{1.15}$$

where $N_{ij}^{\cdot\cdot a}$ is the Nijenhuis tensor:

$$N_{ij}^{\cdot\cdot a} = 2\phi^h{}_{[i}(\partial_h \phi_{j]}{}^a - \partial_{j]}\phi_h{}^a). \tag{1.16}$$

Since we have had $(1.9)_2$, it holds from (1.15) that

$$N_{ij}^{\cdot\cdot a} = 0,$$

and hence the function $\phi_j{}^i$ gives a complex structure to $M^{2n}(\eta)$ by virtue of the theorem stated before. $\phi_j{}^i$ has the components

$$\phi_j{}^i = \begin{Vmatrix} i\delta^\lambda_\mu & 0 \\ 0 & -i\delta^{\bar\lambda}_{\bar\mu} \end{Vmatrix} \tag{1.17}$$

in the complex coordinate system (y) induced from $\phi_j{}^i$, where

$$y^\lambda = \eta^\lambda + i\eta^{\bar\lambda}, \; y^{\bar\lambda} = \eta^\lambda - i\eta^{\bar\lambda}.$$

Theh if we take Θ for A and $\bar\lambda$ for i in (1.12), we have $B_{\bar\lambda}{}^\Theta = 0$ by virtue of (1.17) and also of (1.2). Similarly if we take $\bar\Theta$ for A and λ for i, we have $B_\lambda{}^{\bar\Theta} = 0$. Thus $B_i{}^A$ has the components

$$B_i{}^A = \begin{Vmatrix} B_\lambda{}^\Theta & 0 \\ 0 & B_{\bar\lambda}{}^{\bar\Theta} \end{Vmatrix} \tag{1.18}$$

in the complex coordinate system. While we have assumed that $B_i{}^A$ is of class C^∞. Hence by the elementary theorem of function theory we see that $B_i{}^A$ is of class C^ω.

The induced metric of $M^{2n}(\eta)$ is defind by

$$g_{ij} = B_i{}^A B_j{}^B \varepsilon_{AB}, \tag{1.19}$$

and is obviously analytic and self-adjoint in the complex coordinate system. By the words "*self-adjoint quantity*" is meant a quantity which is equal to its adjoint. A self-adjoint quantity represents always a real quantity in the real coordinate system and vice versa, for which example we have seen in $\phi_B{}^A$ before. $M^{2n}(\eta)$ has the element of length

$$ds^2 = g_{ij}\, d\eta^i\, d\eta^j.$$

Our intention is now to verify that this metric submanifold $M^{2n}(\eta)$ is the real representation of an n-dimensional Kaehlerian space.

Substitution of (1.7) into (1.19) yields

$$g_{ij}=(\phi_A{}^C B_i{}^A)(\phi_B{}^D B_j{}^B)\varepsilon_{CD}$$

and as we have (1.12) and (1.19), this can be reformed to

$$g_{ij}=g_{ab}\phi_i{}^a\phi_j{}^b \tag{1.10}$$

Such a metric is called to be Hermitian [27] and is hybrid in i and j, *i. e.* $g_{\lambda\mu}=g_{\bar{\lambda}\bar{\mu}}=0$ and $g_{\lambda\bar{\mu}}=\overline{g_{\bar{\lambda}\mu}}$, where $g_{\lambda\bar{\mu}}$ satisfies

$$g_{\lambda\bar{\mu}}=B_\lambda{}^\Theta B_{\bar{\mu}}{}^\Lambda\varepsilon_{\Theta\Lambda};\ \text{conj.},$$

as we have (1.6), (1.8) and (1.19). Differentiating $g_{\lambda\bar{\mu}}$ by y^β and taking account of the facts that $\varepsilon_{AB}=$const. and $B_i{}^A$ has the numerical components (1.18), we have

$$\frac{\partial g_{\lambda\bar{\mu}}}{\partial y^\beta}=B_\lambda{}^\Theta\frac{\partial^2 z^\Lambda}{\partial y^{\bar{\mu}}\partial y^\beta}\varepsilon_{\Theta\Lambda},$$

from which we find that

$$\frac{\partial g_{\lambda\bar{\mu}}}{\partial y^\beta}=\frac{\partial g_{\lambda\bar{\beta}}}{\partial y^{\mu}};\ \text{conj.}, \tag{1.21}$$

which insures the existense of an analytic function Ψ such that

$$g_{\lambda\bar{\mu}}=\frac{\partial\Psi}{\partial y^\lambda\partial y^{\bar{\mu}}}, \tag{1.22}$$

and this shows that our n-dimensional complex manifold constitutes a Kaehlerian space [10], whose element of length is

$$\begin{aligned} ds^2&=g_{ij}dy^i dy^j\\ &=2g_{\lambda\bar{\mu}}dy^\lambda dy^{\bar{\mu}}\\ &=2\varepsilon_{\Theta\Lambda}B_\lambda{}^\Theta B_{\bar{\mu}}{}^\Lambda dy^\lambda dy^{\bar{\mu}}. \end{aligned}$$

Hence we have the

Theorem 1. *An almost analytic submanifold $M^{2n}(\eta)$ in a* $(2n+2)$*-dimensional real analytic manifold $H^{2n+2}(\xi)$* representing an $(n+1)$*-dimensional flat Hermitian space is always the real representation of an n-dimensional sub-Kaehlerian space.*

We will denote a $2n$-dimensional real analytic manifold representing an n-dimensional Kaehlerian space by $K^{2n}(\eta)$ hereafter regardless whether it is a submanifold or not, and will call $K^{2n}(\eta)$ *a Kaehlerian manifold.*

put

$$\varepsilon_{AC}\phi_B{}^C=\phi_{AB},$$

and we have for (1.7)

$$\phi_{CA}\phi_B{}^C = \varepsilon_{AB}.$$

Multiplying (1.3) by ε_{AD}, we have

$$\phi_{AC}\phi_B{}^C = -\varepsilon_{AB},$$

from which we have

$$\phi_{(AB)} = 0.$$

Similarly if we put

$$g_{ij}\phi_a{}^i = \phi_{ja}, \tag{1.22}$$

we have for (1.20)

$$\phi_{ja}\phi_b{}^j = g_{ab}, \tag{1.23}$$

and in the simlar way as above we get

$$\phi_{(ij)} = 0. \tag{1.24}$$

Put

$$\phi^{ij} = g^{aj}\phi_j{}^i, \tag{1.25}$$

and we have

$$\phi^{(ij)} = 0, \tag{1.26}$$

$$\phi^{ij}\phi_{ik} = \delta_k{}^j. \tag{1.27}$$

Let $'v^i$ be the transform by $\phi_j{}^i$ of an arbitrary contravariant vector v^i. Then we find that

$$g_{ij}v^i{}'v^j = g_{ij}\phi_h{}^j v^h v^i = 0,$$
$$g_{ij}{}'v^i{}'v^j = g_{ab}v^a v^b,$$

by virtue of (1.20) (1.22) and (1.24). These show that the transformed vector $'v^i$ is orthogonal to the original one and its length is unchanged by this transform. Using these relations we now define the two unit normals to our $2n$-dimensional submanifold.

Let $B_{\mathrm{I}}{}^A$ be a unit vector orthogonal to all of the vector tangent to $M^{2n}(\gamma)$, *i. e.*

$$\varepsilon_{AB}B_i{}^A B_{\mathrm{I}}{}^B = 0, \quad \varepsilon_{AB}B_{\mathrm{I}}{}^A B_{\mathrm{I}}{}^B = 1. \tag{1.28}$$

We transform $B_{\mathrm{I}}{}^A$ by $\phi_A{}^B$ and denote it by $B_{\mathrm{II}}{}^B$, that is,

$$B_{\mathrm{II}}{}^A = \phi_B{}^A B_{\mathrm{I}}{}^B. \tag{1.29}$$

Then $B_{\mathrm{II}}{}^A$ is a unit vector orthogonal to $B_{\mathrm{I}}{}^A$. Further

$$\begin{aligned}\varepsilon_{AB}B_{\mathrm{II}}{}^A B_i{}^B &= \varepsilon_{AB}\phi_C{}^A B_{\mathrm{I}}{}^C B_i{}^B \\ &= \phi_{BC}B_{\mathrm{I}}{}^C B_i{}^B \\ &= -(\varepsilon_{DC}\phi_B{}^D)B_{\mathrm{I}}{}^C B_i{}^B \\ &= -\phi_i{}^j(\varepsilon_{CD}B_{\mathrm{I}}{}^C B_j{}^D) \\ &= 0,\end{aligned}$$

because of the definition of ϕ_{BC} and of (1.12) and (1.28), which shows that the vector $B_{\text{II}}{}^A$ is orthogonal to $B_i{}^A$ too. Therefore we can adopt these two vectors $B_P{}^A$, $(P=\text{I}, \text{II})$, as a couple of unit normals to $M^{2n}(\eta)$.

We have had

$$\varepsilon_{AB}B_i{}^A B_P{}^B=0, \quad \varepsilon_{AB}B_P{}^A B_Q{}^B=\delta_{QP}, \tag{1.30}$$

and together with the introduction of the notations

$$B_{PA}=\varepsilon_{AB}B_P{}^B, \qquad B_{iA}=\varepsilon_{AB}g^{ij}B_j{}^B, \tag{1.31}$$

where $g^{ij}g_{jk}=\delta_k{}^i$, we have

$$B^i{}_A B_j{}^A=\delta_j{}^i,\ B_{QA}B_P{}^A=\delta_{QP},\ B^i{}_A B_P{}^A=0,\ B_{PA}B_i{}^A=0, \tag{1.32}$$

$$B_i{}^A B^i{}_B+B_P{}^A B_{PB}=\delta_B{}^A. \tag{1.33}$$

Multiplication of (1.12) by $B^k{}_A$ yields

$$\phi_j{}^i=B^i{}_A B_j{}^B \phi_B{}^A. \tag{1.34}$$

§ 2. The second and third fundamental tensors. Associated equations.

If we construct the Christoffel symbol by g_{ab}:

$$\left\{{a \atop bc}\right\}=\frac{1}{2}g^{ad}\left(\frac{\partial g_{bd}}{\partial y^c}+\frac{\partial g_{dc}}{\partial y^b}-\frac{\partial g_{bc}}{\partial y^d}\right), \tag{2.1}$$

it contains such components

$$\left\{{\nu \atop \bar{\mu}\bar{\lambda}}\right\}=0;\ \text{conj.} \tag{2.2}$$

$$\left\{{\nu \atop \bar{\mu}\lambda}\right\}=\frac{1}{2}g^{\nu\bar{\rho}}\left(\frac{\partial g_{\mu\bar{\rho}}}{\partial y^{\lambda}}-\frac{\partial g_{\mu\lambda}}{\partial y^{\bar{\rho}}}\right)=0;\ \text{conj.} \tag{2.3}$$

because of (1.20) and (1.21).

Then for

$$\phi^i_{j;k}=\frac{\partial\phi_j{}^i}{\partial y^k}+\phi_j{}^a\left\{{i \atop ak}\right\}-\phi_a{}^i\left\{{a \atop jk}\right\},$$

we have

$$\phi_{\mu}{}^{\nu}{}_{;k}=0, \quad \phi_{\bar{\mu}}{}^{\bar{\nu}}{}_{;k}=0,$$

by (1.17), and also

$$\phi_{\bar{\mu}}{}^{\nu}{}_{;\lambda}=-2i\left\{{\nu \atop \bar{\mu}\lambda}\right\}=0;\quad \text{conj.}, \tag{2.4}$$

$$\phi_{\bar{\mu}}{}^{\nu}{}_{;\bar{\lambda}}=-2i\left\{{\nu \atop \bar{\mu}\bar{\lambda}}\right\}=0;\quad \text{conj.}, \tag{2.5}$$

by (2.2) and (2.3). Hence we have

$$\phi^{i}_{j;k} = 0,$$

and consequently

$$\phi_{ij;k} = 0, \tag{2.6}$$

as we have

$$g_{ij;k} = 0. \tag{2.7}$$

Conversely, if in a $2n$-dimensional real analytic manifold of complex structure we are given the conditions (1.20) and $\phi_{ij;k}=0$, we can have (2.2) in the complex coordinate system by virtue of the hybridity of g_{ij} being assured by (1.20), and can also have (2.3) by (2.4) and (2.5) which are the results of the tensorial condition $\phi_{ij;k}=0$. Then from (2.3) we derive (1.21). Hence Kaehlerian manifold is defined as one for which $\phi_{ij;k}=0$ [27].

For later use we shall derive some fomulas that concern the curvature tensor [1], [25], [27];

$$R^{i}_{.jkl}=\partial_l\{{}_j{}^i{}_k\}-\partial_k\{{}_j{}^i{}_l\}+\{{}_j{}^h{}_k\}\{{}_h{}^i{}_l\}-\{{}_j{}^h{}_l\}\{{}_h{}^i{}_k\},$$

It satisfies

$$R^{i}_{.j(kl)}=0,\ R_{ijkl}=R_{klij},\ \text{where } R_{ijkl}=g_{ih}R^{h}_{.jkl}$$

$$R^{i}_{.(jkl)}=0,\ \text{(Bianchi iden tity of the first kind)},$$

$$R^{i}_{.j(kl;h)}=0,\ \text{(Bianchi identity of the second kind)}.$$

From the Ricci identity

$$\phi^{i}_{j;k;l}-\phi^{i}_{j;l;k}=\phi_j{}^a R^{i}_{.akl}-\phi_a{}^i R^{a}_{.jkl}, \tag{2.8}$$

and (2.6) we have

$$\phi_a{}^i R^{a}_{.jkl}=\phi_j{}^a R^{i}_{.akl},$$

Contracting by g_{ih} we have

$$\phi_{(i}{}^{a}R_{|a|j)kl}=0, \tag{2.9}$$

or, what amounts to the same

$$R_{ijkl}-\phi_i{}^a\phi_j{}^b R_{abkl}=0 \tag{2.10}$$

Contracting (2.8) by g^{jk} and using the relation (1.25) we have

$$\phi_{ia}R^{a}_{.l}=-\frac{1}{2}\phi^{ab}R_{ilab}, \tag{2.11}$$

from which we have

(2.12) $$\phi_{(i|a|}R^{a}_{\cdot\,|\,l)}=0,$$

or, what amount to the same

(2.13) $$R^{a}_{\cdot\,l}+\phi_i{}^a\phi_l{}^bR^{i}_{\cdot b}=0.$$

If we differentiate (2.1) covariantly with respect to η^k and sum up those three equations obtained by the cyclic interchange of the indices i, j and k, we have

(2.14) $$\phi_{(i|a|}R^{a}_{\cdot\,|j;k)}=0,$$

in consequence of the Bianchi identity of the second kind and of (2.6).

These formulas are, of course, valid for any $2n$-dimensional real analytic manifold as long as it is the real representation of an n-dimensional Kaehlerian space.

Differentiating $B_j{}^A$ covariantly, we have the equations of Gauss

(2. 15) $$B^{A}_{i;k}\overset{\text{def}}{=}H_{jk}^{\cdot\cdot A}$$
$$=\partial_kB_j{}^A-\{{}_j{}^i{}_k\}B_i{}^A.$$

$H_{jk}^{\cdot\cdot A}$ is symmetric in j and k and is a vector orthogonal to $B_i{}^A$.

$$H_{jk}^{\cdot\cdot A}=H_{jkP}B_P{}^A.$$

H_{jkP} is called the second fundamental tensor of the submanifold. If we differentiate (1.29) and (2.15), we have

(2.16) $$H_{jk\,\mathrm{I}}=\phi_j{}^aH_{ak\,\mathrm{II}},\quad H_{jk\,\mathrm{II}}=-\phi_j{}^aH_{ak\,\mathrm{I}},$$

or, what amounts to the same

(2.17) $$H_{ijP}+\phi_i{}^a\phi_j{}^bH_{abP}=0,$$

which shows that H_{ijP} is a pure tensor. Hence by the hybridity of g_{ij} we have

(2.18) $$g^{ij}H_{ijP}=0.$$

If we put

(2.19) $$L_{PQk}=\varepsilon_{AB}B_P{}^A{}_{;k}\,B_Q{}^B,$$

we find that L_{PQk} is anti-symetric in P and Q. It is called the third fundamental tensor and by the use of which we have the equations of Weingarten

(2.20) $$B_P{}^A{}_{;k}=-B_i{}^AH^{i}_{\cdot kP}+L_{PQk}B_Q{}^A,$$

where $H^{i}_{\cdot kP}=g^{ih}H_{hkP}$ But if we take, for example, I for P in (2.20) and transform them by $\phi_A{}^B$, we have the equations for $P=II$, as we have (1.29), (2.16) and $L_{(PQ)k}=0$. Therefore the equations of Weingarten to either I or II for P will suffice for all.

As for the condition of integrability for (2.15) and (2.16) we have [18]

(2.21) $$R_{ijkl}=2H_{j(k|P|}H_{|i|l)}{}^P, \qquad \text{(Gauss equations),}$$

(2.22) $$H_{a(k|P|;h)}=L_{PQ(h}H_{|a|k)}{}^Q, \qquad \text{(Coddazzi equations),}$$

(2.23) $$L_{PQ(k;h)}=g^{ab}H_{(a|Q|}H_{|b|kP)}. \qquad \text{(Ricci equations).}$$

If we transform H_{ijP}'s contained in (2.21) by $\phi_a{}^i$, we obtain

$$\phi_a{}^i\phi_b{}^jR_{ijkl}=2\phi_b{}^jH_{j(k|P|}\phi^{\,i}_{|a|}H_{|i|l)}{}^P.$$

But as we have (2.11) and (2.17), these transformed equations reduce to the original epuations (2.21) themselves, and thus $\phi_j{}^i$ does not change the Gauss equations. Also the Codacci equations are only transferred within the indices I and II by this transform. As for the Ricci equations we have but the secondary ones wich are trivial, and thus the condition of integrability is given by the above three kinds of equations solely.

§ 3 Existence theorem.

We now consider the inverse problem. Given three kinds of functions of class C^ω

$$g_{ij}(=g_{ji}),\quad H_{jkP}(=H_{kjP}),\quad L_{PQk}(=-L_{QPk}),$$

satisfying

(3.1) $$g_{ij}=g_i{}^a\phi_j{}^bg_{ab},$$

(3.2) $$H_{ijP}=-\phi_i{}^a\phi_j{}^bH_{abP},$$

if it is posible to determine a $2n$-dimensional Kaehlerian manifold which is a submanifold of $H^{2n+n}(\xi)$ and has precisely these three kinds of functions as its fundamental tensors. For which we should solve two systems of the following partial differential equations for $B_i{}^A$, $B_P{}^A$ and $\xi^A=\xi^A(\eta^i)$ simultaneously:

(3.3) $$\left\{\begin{aligned} &\phi_B{}^AB_i{}^B=\phi_i{}^jB_j{}^A,\\ &\varepsilon_{AB}B_i{}^AB_j{}^B=g_{ij},\\ &\varepsilon_{AB}B_i{}^AB_{\mathrm{I}}{}^B=0,\\ &\varepsilon_{AB}B_{\mathrm{I}}{}^AB_{\mathrm{I}}{}^B=1.\end{aligned}\right.$$

$$(3.4)\quad \begin{cases} \partial_i \xi^A = B_i{}^A, \\ \partial_k B_j{}^A = B_i{}^A \{{}_j{}^i{}_k\} + H_{jkP} B_P{}^A, \\ \partial_k B_{\mathrm{I}}{}^A = -B_i{}^A H^i_{\cdot kP} + L_{\mathrm{I}\,\mathrm{II}k} B_{\mathrm{II}}{}^A, \end{cases}$$

where

$$B_{\mathrm{II}}{}^A = \phi_B{}^A B_{\mathrm{I}}{}^B$$

by definition.

Differentiating the equations (3.3) partially with respect to η^h by turn and using whole of (3.1), (3.2), (3.3) and (3.4) we have

$$\partial_h(\phi_B{}^A B_j{}^B - \phi_i{}^i B_i{}^A) = (\phi_B{}^A B_i{}^B - \phi_i{}^k B_k{}^A)\{{}_j{}^i{}_h\},$$

$$\begin{aligned} \partial_h(\varepsilon_{AB} B_j{}^A B_k{}^B - g_{jk}) = {} & 2(\varepsilon_{AB} B_i{}^A B_{(j}{}^B - g_{(j|i|})\{{}_{k)}{}^i{}_h\} \\ & + 2H_{(j|h\,\mathrm{I}|}\varepsilon_{|AB|} B_{|\mathrm{I}|}{}^A B_{k)}{}^B \\ & - 2\phi_{(j}{}^a H_{|h|k)\,\mathrm{II}}\,\varepsilon_{AB} B_a{}^A B_{\mathrm{I}}{}^B, \end{aligned}$$

$$\begin{aligned} \partial_h(\varepsilon_{AB} B_j{}^A B_{\mathrm{I}}{}^B) = {} & \varepsilon_{AB} B_i{}^A B_{\mathrm{I}}{}^B \{{}_j{}^i{}_h\} \\ & - (\varepsilon_{AB} B_i{}^A B_j{}^B - g_{ij}) H^i_{\cdot h\,\mathrm{I}} \\ & + (\varepsilon_{AB} B_{\mathrm{I}}{}^A B_{\mathrm{I}}{}^B - 1) H_{jh\,\mathrm{I}} \\ & + \varepsilon_{AB} B_i{}^A B_{\mathrm{I}}{}^B \phi_j{}^i L_{\mathrm{II}\,\mathrm{I}\,h}, \end{aligned}$$

$$\partial_h(\varepsilon_{AB} B_{\mathrm{I}}{}^A B_{\mathrm{I}}{}^B - 1) = -2\varepsilon_{AB} B_i{}^A B_{\mathrm{I}}{}^B H^i_{\cdot h\,\mathrm{II}},$$

in this order. Now, when the functions g_{ij}, H_{jkP} and L_{PQk} are given so that they may satisfy the Gauss, Codazzi and Ricci equations, we can solve $(3.4)_3$ for $B_i{}^A$ and $B_{\mathrm{I}}{}^A$, and consequently $B_{\mathrm{II}}{}^A$ too. Then, as has been seen above, the conditions (3.3) to be a submanifold are fulfilled by these solutions throughout the manifold if we give a set of equations (3.3) at a point (ξ^A) under the consideration as the initial condition, and thus is obtainable our required Kaehlerian submanifold by an integration of $(3.4)_1$. Hence we have the

Theorem. 2. *If the functions g_{ij}, H_{ikP} and L_{PQk} satisfying (3.1) and (3.2) are given so that they satisfy the Gauss, Codazzi, and Ricci equations, then it is always possible to determine a $2n$-dimensional Kaehlerian submanifold $K^{2n}(\eta)$ in a given $(2n+2)$-dimensional flat*

Hermitian manifold $H^{2n+2}(\xi)$ in such a way that it has those given functions as the fundamental tensors.

On comparison of the number of the unknown quantities with that of the arbitrary constants appearing at the integration of those differential equations (3.4), we can simply say that the Kaehlerian submanifold whose existence is insured by the above theorem is one that can be determined to within a translation and unitary transformation.

§ 4. Curvatura conditions for isometric imbedding.

The problem we have set up in the preceding paragraph is identical to the inquiry that given an arbitrary $K^{2n}(\eta)$ if it is possible to imbed a manifold in $H^{2n+2}(\xi)$ isometrically, and for which it is necessary and sufficiant that H_{jkP} and L_{PQk} be chosen intrinsically in such a way that the Gauss, Codazzi and Ricci equations hold good together with (3.2).

If we contract the Gauss equations (2.21):

$$R_{ijkl}=2H_{j(k|P|}H_{|i|l)P}, \tag{4.1}$$

by g^{jk} and take account of (2.13), we have

$$R_{il}=-g^{ab}H_{aiP}H_{blP}, \tag{4.2}$$

Take $2n$-orthogonal ennuple $\underset{a}{\lambda^i}$ and we have

$$g^{il}=\sum_{a=1}^{2n}\underset{a}{\lambda^i}\,\underset{a}{\lambda^l}.$$

If we multiply the left and right members of (4.2) by the respective members of the last equality, we have

$$R=-\sum_{a=1}^{2n}g^{ab}\,(H_{aiP}\underset{a}{\lambda^i})(H_{bjP}\underset{a}{\lambda^j}), \tag{4.3}$$

from which we find that $R\leqq 0$ because of the positive definiteness of $g^{ab}X_aX_b$. But when $R=0$, we should have

$$H_{aiP}\underset{a}{\lambda^i}=0,$$

and as $\underset{a}{\lambda^i}$ are $2n$ independent vectors, we have then

$$H_{jkP}=0,$$

from which we deduce

$$R_{ijkl}=0,$$

by the Gauss equations (4.1) Thus we have first of all the

Theorem 3. *In order that a non-flat Kaehlenrian manifold $K^{2n}(\eta)$ may be imbedded isometrically in a flat Hermitian manifold $H^{2n+n}(\xi)$, its scalar curvature R should be negative.*

Put [14]

$$L_{ahbk}=2\{H_{a(h|\mathrm{I}|}H_{|b|k)\mathrm{II}}-H_{a(h|\mathrm{II}|}H_{|b|k)\mathrm{I}}\}. \tag{4.4}$$

Then L_{ahbk} is symmetric in a and b, and anti-symmetric in h and k. Construct the quantity

$$K_{hk}=-\frac{1}{2}g^{ab}L_{ahbk},$$

and we have

$$K_{ij}=2g^{ab}H_{a(i|\mathrm{II}|}H_{|b|j)\mathrm{I}}, \tag{4.5}$$

which shows that K_{ij} is an anti-symmetric tensor and coincides with the right member of Ricci equations (2.23). By the use of the Codazzi equations (2.22) we can derive

$$K_{(ij;k)}=0, \tag{4.6}$$

and hence *it is a necessary condition for $K^{2n}(\eta)$ to be isometrically imbedded in $H^{2n+2}(\xi)$.*

If we write K_{hk} as

$$K_{PQhk}=2g^{ab}H_{a(h|P|}H_{|b|k)Q}, \tag{4.7}$$

it holds

$$K_{(PQ)hk}=0,\quad K_{PQ(hk)}=0.$$

Multiplying (4.7) by H_{diQ} and summing up for Q, and summing those three equations obtained by the cyclic interchange of the letters i, j and k, we have

$$H_{d(i|Q|}K_{|PQ|jk)}=H_{a(i|P|}R_{|d|}{}^{a}{}_{\cdot jk)}, \tag{4.8}$$

in consequence of the Gauss equations (4.1)

Again multiplying (4.8) by H_{biP} and summing up for P, and subtracting from it the equations obtained by the interchange of the letter i and j, we get

$$L_{aib(j}K_{kl)}=-R_{cbi(j}R_{|a|}{}^{c}{}_{\cdot kl)}. \tag{4.9}$$

On the other hand (4.8) can be written as

$$(H^{b}{}_{i\mathrm{II}}H_{c(j|\mathrm{I}|}-H^{b}{}_{i\mathrm{I}}H_{c(j|\mathrm{II}|})K_{|\mathrm{II}\,\mathrm{I}|kl)}$$
$$=H_{biP}H_{a(j|P|}R_{|c|}{}^{a}{}_{\cdot kl)}.$$

Contracting by g^{bc} and summing up those two epuations obtained by the interchange of the letters i and j, we get

$$K_{i(j}L_{|a|k|b|l)} = -R_{a}{}^{c}{}_{\cdot i(j}R_{|cb|kl)}. \tag{4.10}$$

Adding (4.10) to (4.9) and multiplying the resulting equations by $-\frac{1}{2}g^{ab}$ we have

$$K_{i(j}K_{kl)} = \frac{1}{2}R_{b}{}^{a}{}_{\cdot i(j}R_{|a|}{}^{b}{}_{\cdot kl)}. \tag{4.11}$$

Thus, *in order that a $K^{2n}(\eta)$ may be imbedded isometrically in $H^{2n+2}(\xi)$, it is necessary that there exists an anti-symmetric tensor K_{ij} satisfying* (4.11) [14].

If we multiply the Gauss equations (4.1) by $\phi_a{}^i$ and take account of (3.2), or equivalently, of (2.16), we have

$$\phi_a{}^i R_{ijkl} = 2(H_{j(k|\mathrm{II}|}H_{|a|l)\mathrm{I}} - H_{j(k|\mathrm{I}|}H_{|a|l)\mathrm{II}}),$$

or contrbcting by g^{ja}

$$g^{ja}\phi_a{}^i R_{ijkl} = 2K_{kl}.$$

But as we have $g^{ja}\phi_a{}^i = \phi^{ij}$ and also obtain

$$\phi^{ij}R_{ijkl} = -2g_{hk}\phi_a{}^h R^{a}_{\cdot l}$$

by virtue of (2.11), the above equation becomes

$$K_{hl} = \phi_{ak}R^{a}_{\cdot l} \tag{4.12}$$

Subst:tution of this relation into the conditional equations (4.11) gives

$$R_{b}{}^{a}{}_{\cdot i(j}R_{|a|}{}^{b}{}_{\cdot kl)} = 2\phi_{ir}R^{r}_{\cdot (j}\phi_{k|s|}R^{s}_{\cdot l)}. \tag{4.13}$$

For the simplification we multiply this by $\phi^{ij}\phi^{kl}$ and using the relations (1.14), (1.22), (1.23), (1.27) and (2.12), we finally have

$$R_{ijkl}R^{ijkl} = R^2,$$

as one of the necessary conditions.

As for the condition (4.6) the substitution of (4.12) in (4.6) gives the formula coinciding with (2.14), which we have obtained for a $2n$-dimensional real analytic manifold representing an n-dimensional Kaehlerian space. Therefore the condition (4.6) is already fulfilled from the beginning when we deal with a Kaehlerian manifold. Summarizing the results hitherto we have obtained, we have the

Theorem 4. *In order that a 2n-dimensional real analytic manifold $K^{2n}(\eta)$ which is the real representation of an n-dimensional non-flat Kaehlerian space can be imbedded isometrically in a $(2n+2)$-dimensional real analytical manifold $H^{2n+2}(\xi)$ which is the representation of an $(n+1)$-dimensional flat Hermitian space, its curvature tensor $R^{i}_{\cdot jkl}$ should satisfy*

$$R_{ijkl}R^{ijkl} = R^2, \quad R < 0. \tag{4.14}$$

A Kaehlerian manifold $K^{2n}(\eta)$ is called a manifold of constant holomorphic curvature, if the sectional curvature spanned by two vectors v^i and $\phi_j^i v^j$ is everywhere constant [1], [23], [25], [26], [27]. Its curvature tensor is given by

$$R^i_{\cdot jkl}=\frac{k}{2}(\delta_{(}{}^i{}_l g_{|j|k)}+\phi^i{}_{(l}\phi_{|j|k)}-\phi_j{}^i\phi_{kl}), \tag{4.15}$$

where k is an absolute constant. We have then

$$R_{ij}=\frac{n+1}{2}kg_{ij},\quad R=kn(n+1) \tag{4.16}$$

If we substitute (4.15) into (4.14) and make use of (4.16) we get

$$R\,(n-1)(n+2)=0.$$

Hence R should vanish for $n\geqq 2$, and k should do so. Thus we have the

Theorm 5. *These exists no Kaehlerian manifold $K^{2n}(\eta)$, $n\geqq 2$, of non-vanishing constant holomorphic curvature which can be imbedded isometrically in a $(2n+2)$-dimensional flat Hermitian manifold $H^{2n+2}(\xi)$.*

It is interesting to compare this fact with the problem to imbed a Riemannian space of constant curvature in a Euclidean space as its hypersurface. It is well-known as the theorem of H. Levy [11], L. P. Eisenhart [6] or of the present author [17] that there exists no Riemannian space V^n, $n>2$, of negative constant curvature which can be regarded as a hypersurface of a Euclidean space, that is, of class one in the sense of T. Y. Thomas [20], [21].

§ 5. Non-ezistance of parallel submanifolds.

Let us consider the case that K_{ij} vanishes. Then by (4.12) we should have $\phi_{ai}R^a_{\cdot j}=0$, or equivalently

$$\phi_i{}^a R_{aj}=0,$$

and as $|\phi_i{}^a|\neq 0$, we have

$$R_{aj}=0,$$

and consequently $R=0$. Then by the argument made in the proof of theorm 3, the space is flat. Thus we have the

Theorem 6. *These exists no Kaehlerian manifold $K^{2n}(\eta)$ other than flat one that can be imbedded isometrically in a flat Hermitian manifold $H^{2n+n}(\xi)$ in such a way that the fnnctions K_{ij} vanish identically, or in an equivalent sence, both of the members of the Ricci equatious (2.23) vanish.*

We shall interpret this fact geometrically. Given a submanifold $M^{2n}(\eta)$ representing a given Kaehlerian space, if these exists an another $'M^{2n}(\eta)$ having its two normals in common with $M^{2n}(\eta)$, we say that $'M^{2n}(\eta)$ is a parallel submanifold [15]. If we denote by $'\xi^A$ the position vector of the point on $'M^{2n}(\eta)$ which corresponds to a point on $M^{2n}(\eta)$ with the position vector ξ^A, $'\xi^A$ has the form

$$'\xi^A = \xi^A + C^P B_P{}^A, \tag{5.1}$$

Then on using the equations of Weingarten the vectors tangent to $'M^{2n}(\eta)$ can be written as

$$\begin{aligned} 'B_i{}^A = \partial_i{}'\xi^A = (\delta_i{}^j - C^P H^j_{.iP}) B_j{}^A \\ + (\partial_i C^P + C^Q L_{QPi}) B_P{}^A. \end{aligned} \tag{5.2}$$

Since $B_P{}^A$ is common to both $M^{2n}(\eta)$ and $'M^{2n}(\eta)$, $'B_i{}^A$ should be orthogonal to $B_P{}^A$, and accordingly the second term in the right member of (5.2) vanishes and we have

$$'B_i{}^A = (\delta_i{}^j - C^P H^j_{.iP}) B_j{}^A, \tag{5.3}$$

$$\partial_j C^P + C^Q L_{QPj} = 0. \tag{5.4}$$

The condition of integrability for C^P, i. e.

$$\partial_{(i}\partial_{j)} C^P = 0$$

gives

$$C^Q(\partial_{(i} L_{|PQ|j)} + L_{RR|(i} L_{|PR|j)}) = 0.$$

or

$$C^Q L_{QP(i;j)} = 0.$$

Since C^Q, ($Q =$ I, II), are independent, we have

$$L_{PQ(i;j)} = 0, \tag{5.5}$$

from which we find that

$$K_{ij} = 0.$$

For (5.5) is nothing but the left member of the Ricci equations (2.28). Hence a necessary and sufficeant condition that a submanifold $M^{2n}(\eta)$ be accompanied by a parallel submanihold is that K_{ij} vanishes. But as we have proved the theorem 6, we can state the

Theorem 7. *These exists no Kaehlerian manifold $K^{2n}(\eta)$ other than flat one that can be imbedded isometrically in $H^{2n+2}(\xi)$ in such a way that it is always accompanied by a parallel submanifold.*

To the last, as we have dealt with our local imbedding problem on manifolds of real representation, whose metrics are actually Riemannian. If then so, the existence of the second fundamental tensor appearing in the theory of submanifolds can be studied quite in the similar way as is done in the case to imbed a Riemannian space V^n in a Euclidean space E^{n+2}, that is, in the case of class two. The latter problem was studied in detail by M. Matsumoto [14] as the direct generalization of the computations made by T. Y. Thomas for class one [21]. Then we can repeat the arguments analogously and have the

Theorem 8. *If the rank of the matrix* $\|K_{ij}\|$, *whose elements satisfy the condition* (4.12), *is equal or greater than 4, the second fundameutal tensors* H_{jkP} *are unique to whithin a unitary transformation.*

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