THEORY OF A BANACH SPACE WITH SEMI-NORMS

By

MASAE ORIHARA

In several functions spaces, it has two kinds of norms and it makes a Banach space by one of them. *Alexiewicz, Semadeni* and *Wiweger* named it *two norm space* and obtain fundamental and important results of it, moreover have introduced in it "*mixed topology*" by two kinds topologies.

But, some examples of the space of functions have various semi-norms in addition to the intrinsic topology (by norm) and, there exist remarkable relations between the norm and the semi-norms.

In this paper, we denote $\|x\|$ the norm of an element x of X, semi-norms of x by $\|x\|_\alpha^*$ $(\alpha\epsilon\Lambda)$ and these two norms are combined by $\|x\| = \sup_{\alpha\epsilon\Lambda} \|x\|_\alpha^*$.

The space is of course a two-norm space as point out in *Wiweger* [4]. He determined the form of the mixed topology under some conditions by pseudo-norm;

$$[X]_{(\beta i)(\alpha i)} = \sup \frac{\|x\|_{\beta i}^*}{\alpha_i}, \quad \beta_i \epsilon B, \quad 0<\alpha_i \to \infty.$$

But, the space is distinct in topology from two-norm space and it has various important examples. Therefore, I shall be study the space as extension of two-norm space in this paper.

1. Definition of many norms space. We consider a linear space with norm $\|x\|$ of an element x of set X and complete by norm-topology, that is, a Banach space. Moreover, we difine that each element x of X has some semi-norms $\|x\|_\alpha^*$ $(\alpha\epsilon\Lambda)$ satisfying the conditions $\|x+y\|_\alpha^* \leqq \|x\|_\alpha^* + \|y\|_\alpha^*$, $\|ax\|_\alpha^* = |a|\,\|x\|_\alpha^*$. ($\|x\|_\alpha^*=0$ does not implies $x=0$) The norm $\|x\|$ and semi-norms $\|x\|_\alpha^*$ $(\alpha\epsilon\Lambda)$ have the relation

$$(M) \qquad \|x\| = \sup \|x\|_\alpha^*$$

By the norm $\|x\|$ of x in X, it is a topological linear space and locally convex, denote the topology by τ.

The neighbourhoods of zero by semi-norms $\|x\|_\alpha^*$ $(\alpha\epsilon\Lambda)$ is $U(x;\ \|x\|_{\alpha i}^* < \varepsilon,\ i=1, 2, \cdots\cdots, n;\ \alpha_i\epsilon\Lambda)$, its topology is denoted by τ^*, the system of the neighbourhoods of zero by $U(\tau^*)$. The space X with topology τ^* is a linear topologicl space;

(1) if $U\epsilon U(\tau^*)$ and $\lambda\epsilon R$, $\lambda\neq 0$, then $\lambda U\epsilon U(\tau^*)$,

(2) if $U\epsilon U(\tau^*)$ and $\lambda\epsilon R$, $|\lambda| \leqq 1$, then $\lambda U \subset U$.

(3) if $U \epsilon U(\tau^*)$, then for every $x \epsilon X$ there exists $\lambda \epsilon R$, $\lambda \neq 0$ such that $\lambda x \epsilon U$.

(4) if U, $V \epsilon U(\tau^*)$, then there exists $W \epsilon U(\tau^*)$ such that $W \subset U \cap V$.

(5) if $U \epsilon U(\tau^*)$, then there exists $V \epsilon U(\tau^*)$ such that $V+V \subset U$.

Moreover, $U(\tau^*)$ satisfies the condition by the assumption (M).

(6) for every $x \epsilon X$, $x \neq 0$, there exists $U(\tau^*)$ such that $x \bar{\epsilon} U$.

And, the linear space with topology τ^* is locally convex, but incomplete. Of course, $\tau^* \leqq \tau$ and satisfies(d) of [4], so it has the various properties in [4].

The space X with the norm $\|x\|$ and semi-norms $\|x\|^*_\alpha (\alpha \epsilon \Lambda)$ is denote by $<X, \| \|, \| \|^*>$ or $<X, \tau, \tau^*>$ and call it "*many norms space*".

First, we shall discuss the properties of the many-norms space as linear topological space.

Proposition 1. *If x_n converges to x by τ^*-topology, then the set $\{\|x_n\|\}$ is bounded.*

In fact, for any $\varepsilon > 0$ there exist $\alpha_i \epsilon \Lambda$ $(i=1, 2, \cdots\cdots, n)$ and N such that $\|x_n - x\|^*_{\alpha i} < \varepsilon$ for $n > N$. Then $\|x_n\|^*_{\alpha i} - \|x\|^*_{\alpha i} \leqq \|x_n - x\|^*_{\alpha i} < \varepsilon$, $\|x_n\|^*_{\alpha i} \leqq \|x\|^*_{\alpha i} + \varepsilon \leqq \|x\| + \varepsilon = M$. $\|x_n\| = sup \|x_n\|^*_{\alpha i} \leqq M$.

$\|x\|^*_\alpha$ is a *Minkowski*'s function, so there exists functional in $<X, \tau^*>$, $f(x) \leqq \|x\|^*_\alpha$.

This functional is continuous in τ^*-topology and $\{f\} = \Xi^*$ is total in X.

In fact, if $x \neq 0$, we have an $\alpha \epsilon \Lambda$, $\|x\|^*_\alpha \neq 0$, set $f(tx) = t\|x\|^*_\alpha$ for every real number t, so this functional $f(x)$ is extensible on X. Therfore $f(x) \neq 0$. Ξ^* is total in X.

2. Examples. The many norms space have various examples, but main space is as follows:

(1) *the space L^p.* The norm of $x \epsilon X$ is $\|x\| = (\int_0^1 |x(t)|^p dt)^{1/p}$, the semi-norms of x,

$\|x\|^*_\alpha = \int_0^1 x(t) \overline{y_\alpha(t)} dt$ where $\left(\int_0^1 |y_\alpha(t)|^q \, dt \right)^{1/q} \leqq 1$. $(^1/_p + {}^1/_q = 1)$

(2) *The space of continuous functions on the real line* $-\infty < x < +\infty$, $\|x(t)\| = sup |x(t)|$, $\|x(t)\|^*_K = \sup_{t \epsilon K} |x(t)|$, where K is a compact subset. Obviously, $\|x(t)\| = \sup_K \|x(t)\|^*_K$

(3) *X is a Banach space with norm $\|x\|$.* f_α is a bounded linear functional on X. $\|x\|^*_\alpha = |f_\alpha(x)|$ where $\|f_\alpha\| \leqq 1$. Assumption (M) is easily.

(4) *$\bar{X}$ is the conjugate space of a Banach space X.* $f \epsilon \bar{X}$ has a norm $\|f\|$ as Banach space, and $\|f\|^*_\alpha = |f(x_\alpha)|$ where $x_\alpha \epsilon X$ and $\|x_\alpha\| \leqq 1$ in X. (M) is obvious.

(5) B be a set of bounded linear operators on a Banach space X, B is also a Banach space with norm $\|A\|$ as operator of B and $\|A\|^*_\alpha$ is $\|Ax_\alpha\|$ where $x_\alpha \epsilon X$, $\|x_\alpha\| \leqq 1$. B is a many norms space and $\|A\| = \sup_\alpha \|Ax_\alpha\|$.

(6) In (5), $\|A\|^*_{\alpha,\beta} = |f_\alpha(Ax_\beta)|$ with $x_\beta \epsilon X$, $\|x_\beta\| \leqq 1$, $f_\alpha \epsilon \bar{X}$, $\|f_\alpha\| \leqq 1$.

So, $\|A\| = \sup_{\alpha, \beta} |f_\alpha(Ax_\beta)|$.

In particular, X be a Hilbert space $\mathfrak{H}$, B an operator ring M on it, the neighbourhoods of zero $\|A\| < \varepsilon$ is uniform topology, $\|Ax_i\| < \varepsilon (i = 1, 2, \cdots\cdots, n)$ is strong topology, the neighbourhoods $|(Ax_i, y_i)| < \varepsilon (i=1, 2, \cdots\cdots, n)$ is weak topology.

3. Mixed topology in the many norms space. We can introduce the mixed topology in many norms space as in the two-norm space.

For each $U \epsilon U(\tau)$ and for each sequence $U_1^*, U_2^*, \cdots\cdots \epsilon U(\tau^*)$, we shall denote by $\gamma(U_1^*, U_2^*, \cdots\cdots; U)$ the set $\bigcup_{n=1}^{\infty} (U_1^* \cap U + U_2^* \cap 2U + \cdots\cdots + U_n^* \cap nU)$. Wiweger [4] has called this topology the *mixed topology* determined by the topologies τ and τ^*.

Mixed topology is weaker than τ and stronger than τ^* as the topology τ^* is weaker than the topology τ $(\tau^* \leqq \tau)$.

Both of $U(\tau)$, $U(\tau^*)$ are locally convex, so also the mixed topology is locally convex. We shall denote by γ-topology the mixed topology in order to *Wiweger.*

We define that B is bounded set in $<X, \tau>$ if for each $U \epsilon U(\tau)$ there exists $\lambda \epsilon R$ such that $B \subset \lambda U$.

In general, τ^* doesn't satisfies the first countability axiom, so it has not also in mixed topology.

But, if τ^* satisfies the first countability axiom, by *Wiweger* [3] and proposition 1 the convergence in τ^* and γ-convergence are equivalent.

4. Conjugate space. Let be Ξ, Ξ_γ, Ξ^* the set of linear continuous functional concerning τ, γ or τ^* on X, so $\Xi \supset \Xi_\gamma \supset \Xi^*$. For τ^*-continuous functional $\xi(x)$, $\|\xi\|_\alpha^* = sup|\xi(x)|$, where $\|x\|_\alpha^* \leqq 1$. Then, $|\xi(x)| \leqq \|\xi\|_\alpha^* \cdot \|x\|_\alpha^*$ and $\|\xi_1 + \xi_2\|_\alpha^* \leqq \|\xi_1\|_\alpha^* + \|\xi_2\|_\alpha^*$, $\|a\xi\|_\alpha^* = |a| \, \|\xi\|_\alpha^*$ for every $\alpha \epsilon \Lambda$. $\|\xi\|_\alpha^* = 0$ doesn't implies $\xi = 0$. $\Xi^* \subset \Xi$, so $\xi \epsilon \Xi^*$ has the norm $\|\xi\| = \sup_{\|x\| \leqq 1} |\xi(x)|$.

Proposition 2. *$<\Xi^*, \| \|^*>$ is closed in $<\Xi, \| \|>$.*

Proof. y is an element in the closure of Ξ^*, so there exists $y_0 \epsilon \Xi^*$ such that $\|y - y_0\| < \varepsilon$. y_0 is a functional of Ξ^*. So for x of $\|x - x_0\|_\alpha^* < \delta$, $|y_0(x) - y_0(x_0)| < \varepsilon$. We have $\|x\|_\alpha^* \leqq \delta + \|x_0\|_\alpha^* \leqq 2 \|x\|_\alpha^* \leqq 2\|x\|$. y, y_0 are τ-continuous, so we have $|y(x) - y(x_0)| \leqq |y(x) - y_0(x)| + |y_0(x) - y_0(x_0)| + |y_0(x_0) - y(x_0)| \leqq \|y - y_0\| \|x\| + \|y - y_0\| \|x_0\| + \varepsilon$
$= \|y - y_0\| (\|x\| + \|x_0\| + \varepsilon \leqq 3\varepsilon \|x_0\| + \varepsilon = (3\|x_0\| + 1)$

Therefore y_0 is continuous in τ^*. So, $<\Xi^*, \| \|^*>$ is closed in $<\Xi, \| \|>$.

Theorem. *$<\Xi^*, \| \|, \| \|^*>$ is many norms space.*

Form the proposition 2, we have following theorem.

Theorem. *$<X, \tau, \tau^*>$ be a many norms space, then the following conditions are equivalent;*

(i) *$<X, \| \|, \| \|^*>$ is $*$-reflexive and $\Xi^* = \Xi$.*

(ii) *$<X, \| \|>$ is reflexive.*

We can prove as same as 3. 7, theorem in [2].

Bibliography

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