Singularities of tangent pedal curves in S^3

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1 Introduction

Let S^3 be the unit 3-dimensional sphere in $\mathbf{R}^4$ and I be an interval. For a 3-dimensional spherical unit speed curve $\mathbf{r} : I \to S^3$ and a given point $P \in S^3 - \{\alpha\mathbf{n}(s) + \beta\mathbf{b}(s) \mid s \in I, \alpha^2 + \beta^2 = 1\}$ where $\mathbf{n}(s)$, $\mathbf{b}(s)$ are the principal normal vector and the binormal vector of $\mathbf{r}(s)$ respectively, we can define two kinds of pedal curves naturally. One is the curve obtained by mapping $s \in I$ to the nearest point from P in the tangent great circle to $\mathbf{r}$ at $\mathbf{r}(s)$ and another is the curve obtained by mapping $s \in I$ to the nearest point from P in the osculating great sphere to $\mathbf{r}$ at $\mathbf{r}(s)$. We call the former (resp. latter) the *tangent pedal curve* (resp. osculating pedal curve) relative to the *pedal point* P for a 3-dimensional spherical unit speed curve $\mathbf{r}$ and denote it $Pe_{\mathbf{r},P}$ (resp. $Pe_{\mathbf{r},\mathbf{t},P}$).

In this paper, we characterize and classify singularities of tangent pedal curves in S^3 completely. Before stating our results, we introduce several notations. A *3-dimensional spherical unit speed curve* is a C^∞ map $\mathbf{r} : I \to S^3$ such that

$$\|\frac{d\mathbf{r}}{ds}(s)\| = 1, \quad \frac{d^2\mathbf{r}}{ds^2}(s) + \mathbf{r}(s) \neq \mathbf{0} \quad \text{(for any } s \in I).$$

The above two conditions for a 3-dimensional spherical unit speed curve $\mathbf{r}$ is not an essential restriction, since by using Thom transversality theorem (for instance, see [4]), for any C^∞ immersion $\mathbf{r} : I \to S^3$ we can obtain a sufficiently near C^∞ map $\widetilde{\mathbf{r}}$ in $C^\infty(I, S^3)$ with Whitney C^∞ topology such that

$$\frac{d^2\widetilde{\mathbf{r}}}{ds^2}(s), \frac{d\widetilde{\mathbf{r}}}{ds}(s) \text{ and } \widetilde{\mathbf{r}}(s) \text{ are linearly independent (for any } s \in I);$$

and the so-called *arc length* parameter gives us a C^∞ diffeomorphism $h : I \to I$ such that

$$\|\frac{d(\widetilde{\mathbf{r}} \circ h^{-1})}{ds}(s)\| = 1, \quad \frac{d^2(\widetilde{\mathbf{r}} \circ h^{-1})}{ds^2}(s) + \widetilde{\mathbf{r}} \circ h^{-1}(s) \neq \mathbf{0} \quad (\text{for any } s \in I).$$

For a 3-dimensional spherical unit speed curve $\mathbf{r}$, we put

$$\mathbf{t}(s) = \frac{d\mathbf{r}}{ds}(s), \quad \mathbf{n}(s) = \frac{\frac{d\mathbf{t}}{ds}(s) + \mathbf{r}(s)}{\|\frac{d\mathbf{t}}{ds}(s) + \mathbf{r}(s)\|}.$$

These are called the *tangent vector* and the *principal normal vector* respectively. We see easily that the vector $\mathbf{t}(s)$ is perpendicular to $\mathbf{r}(s)$ and the vector $\mathbf{n}$(s) is perpendicular to both of $\mathbf{r}(s)$, $\mathbf{t}(s)$ (see §2). Let $\mathbf{b}(s)$ be the unique unit vector which is perpendicular to all of $\mathbf{r}(s)$, $\mathbf{t}(s)$ $\mathbf{n}(s)$ and such that $\det(\mathbf{r}(s)\,\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)) = 1$. The vector $\mathbf{b}(s)$ is called the *binormal vector.* The map $\mathbf{b} : I \to S^3$, which is called the *dual* of $\mathbf{r}$, seems to be relatively well understood (for instance, see [1], [3], [7]). Furthermore, the singular surface

$$\{\alpha\mathbf{n}(s) + \beta\mathbf{b}(s) \mid s \in I,\ \alpha^2 + \beta^2 = 1\},$$

which is called the *dual surface* of $\mathbf{r}$, seems to be started to study recently ([6]). We let $C_{\mathbf{t}(s),\mathbf{n}(s)}$ be the great circle (1-dimensional sphere) of S^3 whose elements are perpendicular to both of $\mathbf{t}(s)$ and $\mathbf{n}(s)$.

The main result of this paper is the following.

Theorem 1 *Let $\mathbf{r} : I \to S^3$ be a 3-dimensional spherical unit speed curve. Let P be a point of $S^3 - \{\alpha\mathbf{n}(s) + \beta\mathbf{b}(s) \mid s \in I, \alpha^2 + \beta^2 = 1\}$. Then the following hold.*

1. *If $P \in S^3 - C_{\mathbf{t}(s_0),\mathbf{n}(s_0)} - \{\alpha\mathbf{n}(s) + \beta\mathbf{b}(s) \mid s \in I, \alpha^2 + \beta^2 = 1\}$, then the map-germ $Pe_{\mathbf{r},P} : (I, s_0) \to S^3$ is C^∞ right-left equivalent to the map-germ given by $s \mapsto (s, 0, 0)$.*

2. *If $P \in C_{\mathbf{t}(s_0),\mathbf{n}(s_0)} - \{\pm\mathbf{b}(s_0)\}$, then the map-germ $Pe_{\mathbf{r},P} : (I, s_0) \to S^3$ is C^∞ right-left equivalent to the map-germ given by $s \mapsto (s^2, s^3, 0)$.*

Here, two map-germs $f, g : (\mathbf{R}, 0) \to (\mathbf{R}^3, 0)$ are said to be C^∞ *right-left equivalent* if there exist germs of C^∞ diffeomorphisms $h_1 : (\mathbf{R}, 0) \to (\mathbf{R}, 0)$ and $h_2 : (\mathbf{R}^3, 0) \to (\mathbf{R}^3, 0)$ such that the identity $g = h_2 \circ f \circ h_1^{-1}$ satisfies.

By theorem 1, we see that singularities of the tangent pedal curve for a 3-dimensional spherical unit speed curve $\mathbf{r}$ are strongly restricted and no influences of the geodesic torsion of $\mathbf{r}$ occur (for the definition of the geodesic torsion, see §2).

The Serret-Frenet type formula for a 3-dimensional spherical unit speed curve, an explicit formula for $Pe_{\mathbf{r},P}$ and a lemma for the proof of theorem 3 are given in §2. In §3, theorem 1 will be proved.

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2 Serret-Frenet type formula and an application of it

For two 4-dimensional vectors $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4)$, let $\mathbf{x} \cdot \mathbf{y}$ be the standard scalar product.

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4.$$

For any C^∞ map $f : I \to \mathbf{R}^n$, $f' : I \to \mathbf{R}^n$ means the first derivative of f.

Since $\mathbf{r}(s)\cdot\mathbf{r}(s) = 1$, we see that $\mathbf{r}(s)\cdot\mathbf{t}(s) = 0$. Thus, $\mathbf{t}(s)$ is perpendicular to $\mathbf{r}(s)$. Since $\mathbf{r}(s)\cdot\mathbf{t}(s) = 0$, we see that $\mathbf{r}(s)\cdot\mathbf{t}'(s)+1 = 0$. Thus, $\mathbf{n}(s)$, which is the normalized vector of $\mathbf{t}'(s)+\mathbf{r}(s)$, is perpendicular to $\mathbf{r}(s)$. Furthermore, since $\mathbf{t}(s)\cdot\mathbf{t}(s) = 1$, we have that $\mathbf{t}(s)\cdot\mathbf{t}'(s) = 0$. Thus, $\mathbf{t}(s)\cdot(\mathbf{t}'(s)+\mathbf{r}(s)) = 0$, which implies that $\mathbf{n}(s)$ is perpendicular to $\mathbf{t}(s)$.

By the above argument, we see that $\{\mathbf{r}(s),\ \mathbf{t}(s),\ \mathbf{n}(s),\ \mathbf{b}(s)\}$ is an orthogonal moving frame, which is called *Saban frame* of $\mathbf{r}$.

Next, we put

$$\begin{aligned} \kappa_g(s) &= \|\mathbf{t}'(s) + \mathbf{r}(s)\|, \\ \tau_g(s) &= \frac{1}{\kappa_g(s)^2}\det(\mathbf{r}(s), \mathbf{r}'(s), \mathbf{r}''(s), \mathbf{r}'''(s)). \end{aligned}$$

These are called *geodesic curvature*, *geodesic torsion* of $\mathbf{r}$ at s respectively. Then, we have the following Serret-Frenet type formula .

Lemma 2.1

$$\begin{pmatrix} \mathbf{r}'(s) \\ \mathbf{t}'(s) \\ \mathbf{n}'(s) \\ \mathbf{b}'(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \kappa_g(s) & 0 \\ 0 & -\kappa_g(s) & 0 & \tau_g(s) \\ 0 & 0 & -\tau_g(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{r}(s) \\ \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix}.$$

By lemma 2.1 we see that the dual $\mathbf{b}$ is non-singular at s if and only if $\tau(s) \neq 0$.

Proof of lemma 2.1 We put

$$\mathbf{n}'(s) = a_1\mathbf{r}(s) + b_1\mathbf{t}(s) + c_1\mathbf{n}(s) + d_1\mathbf{b}(s).$$

and we show that $a_1 = 0,\ b_1 = -\kappa_g(s),\ c_1 = 0,\ d_1 = \tau_g(s)$.

Since $\mathbf{r}(s) \cdot \mathbf{n}(s) = 0$, we have that $\mathbf{r}(s) \cdot \mathbf{n}'(s) = 0$. Thus, $a_1 = 0$. Since $\mathbf{n}(s) \cdot \mathbf{n}(s) = 1$, we have that $\mathbf{n}(s) \cdot \mathbf{n}'(s) = 0$. Thus, $c_1 = 0$. Since $\mathbf{t}(s) \cdot \mathbf{n}(s) = 0$, we have that $\mathbf{t}'(s) \cdot \mathbf{n}(s) + \mathbf{t}(s) \cdot \mathbf{n}'(s) = 0$. Thus, $\kappa_g(s) + b_1 = 0$. Finally,

$$\begin{aligned}
\tau_g(s) &= \frac{1}{\kappa_g(s)^2}\det(\mathbf{r}(s), \mathbf{r}'(s), \mathbf{r}''(s), \mathbf{r}'''(s)) \\
&= \frac{1}{\kappa_g(s)^2}\det(\mathbf{r}(s), \mathbf{t}(s), \kappa_g(s)\mathbf{n}(s) - \mathbf{r}(s), \kappa_g'(s)\mathbf{n}(s) + \kappa_g\mathbf{n}'(s) - \mathbf{t}(s)) \\
&= \frac{1}{\kappa_g(s)^2}\det(\mathbf{r}(s), \mathbf{t}(s), \kappa_g(s)\mathbf{n}(s), \kappa_g(s)\mathbf{n}'(s)) \\
&= \det(\mathbf{r}(s), \mathbf{t}(s), \mathbf{n}(s), d_1\mathbf{b}(s)) \\
&= d_1.
\end{aligned}$$

Next, we put

$$\mathbf{b}'(s) = a_2\mathbf{r}(s) + b_2\mathbf{t}(s) + c_2\mathbf{n}(s) + d_2\mathbf{b}(s).$$

and we show that $a_2 = 0,\ b_2 = 0,\ c_2 = -\tau_g(s),\ d_2 = 0$.

Since $\mathbf{r}(s) \cdot \mathbf{b}(s) = 0$, we have that $\mathbf{r}(s) \cdot \mathbf{b}'(s) = 0$. Thus, $a_2 = 0$. Since $\mathbf{b}(s) \cdot \mathbf{b}(s) = 1$, we have that $\mathbf{b}(s) \cdot \mathbf{n}'(s) = 0$. Thus, $d_2 = 0$. Since $\mathbf{t}(s) \cdot \mathbf{b}(s) = 0$, we have that

$$\begin{aligned}
0 &= \mathbf{t}'(s) \cdot \mathbf{b}(s) + \mathbf{t}(s) \cdot \mathbf{b}'(s) \\
&= \mathbf{t}(s) \cdot \mathbf{b}'(s) = b_2.
\end{aligned}$$

Finally, since $\mathbf{n}(s) \cdot \mathbf{b}(s) = 0$, we have that

$$\begin{aligned}
0 &= \mathbf{n}'(s) \cdot \mathbf{b}(s) + \mathbf{n}(s) \cdot \mathbf{b}'(s) \\
&= \tau_g(s) + c_2.
\end{aligned}$$

q.e.d

Lemma 2.2

$$Pe_{\mathbf{r},P}(s) = \frac{1}{\sqrt{(P \cdot \mathbf{r}(s))^2 + (P \cdot \mathbf{t}(s))^2}}((P \cdot \mathbf{r}(s))\mathbf{r}(s) + (P \cdot \mathbf{t}(s))\mathbf{t}(s))$$

Proof of lemma 2.2 For any $s \in I$, by subtracting $(P \cdot \mathbf{n}(s))\mathbf{n}(s) + (P \cdot \mathbf{b}(s))\mathbf{b}(s)$ from P we obtain the vector

$$P - (P \cdot \mathbf{n}(s))\mathbf{n}(s) - (P \cdot \mathbf{b}(s))\mathbf{b}(s) = (P \cdot \mathbf{r}(s))\mathbf{r}(s) + (P \cdot \mathbf{t}(s))\mathbf{t}(s)$$

in $\mathbf{R}^4$ which is positive scalar multiple of $Pe_{\mathbf{r},P}(s)$. Normalizing this vector gives the right hand side of the formula in lemma 2.2, which must be the vector $Pe_{\mathbf{r},P}(s)$. q.e.d

By this formula, we can characterize singularities of the tangent pedal curve relative to P as follows.

Lemma 2.3

$$Pe'_{\mathbf{r},P}(s) = 0 \quad \Longleftrightarrow P \in C_{\mathbf{t}(s),\mathbf{n}(s)}.$$

Proof of lemma 2.3 By using lemma 2.1, we have

$$((P \cdot \mathbf{r}(s))^2 + (P \cdot \mathbf{t}(s))^2)' = 2\kappa_g(s)(P \cdot \mathbf{t}(s))(P \cdot \mathbf{n}(s))$$

and

$$((P \cdot \mathbf{r}(s))\mathbf{r}(s) + (P \cdot \mathbf{t}(s))\mathbf{t}(s))' = \kappa_g(s)(P \cdot \mathbf{n}(s))\mathbf{t}(s) + \kappa_g(s)(P \cdot \mathbf{t}(s))\mathbf{n}(s).$$

Thus, simple calculations show

$$Pe'_{\mathbf{r},P}(s) = \frac{\kappa_g(s)}{((P \cdot \mathbf{r}(s))^2 + (P \cdot \mathbf{t}(s))^2)^{\frac{3}{2}}} \Big(\xi_{\mathbf{r}}(s)\mathbf{r}(s) + \xi_{\mathbf{t}}(s)\mathbf{t}(s) + \xi_{\mathbf{n}}(s)\mathbf{n}(s) \Big),$$

where $\xi_{\mathbf{r}}(s) = -(P \cdot \mathbf{r}(s))(P \cdot \mathbf{t}(s))(P \cdot \mathbf{n}(s))$, $\xi_{\mathbf{t}}(s) = (P \cdot \mathbf{r}(s))^2(P \cdot \mathbf{n}(s))$ and $\xi_{\mathbf{n}}(s) = ((P \cdot \mathbf{r}(s))^2 + (P \cdot \mathbf{t}(s))^2)(P \cdot \mathbf{t}(s))$. Since $P \in S^3 - \{\alpha\mathbf{n}(s) + \beta\mathbf{b}(s) \mid s \in I,\ \alpha^2 + \beta^2 = 1\}$, we see that $(P \cdot \mathbf{r}(s))^2 + (P \cdot \mathbf{t}(s))^2 \neq 0$. Thus, by the above calculations we see that $Pe'_{\mathbf{r},P}(s) = 0$ if and only if $P \in C_{\mathbf{t}(s),\mathbf{n}(s)}$. q.e.d

Let s_0 be an element of I. We put

$$\varphi(s) = (P \cdot \mathbf{t}(s + s_0), P \cdot \mathbf{n}(s + s_0)),$$

for any $s \in I$ such that $s + s_0 \in I$. Let $\mathcal{E}_1$, $\mathcal{E}_2$ be the set of all C^∞ function-germs $(\mathbf{R}, s_0) \to \mathbf{R}$, $(\mathbf{R}^2, \varphi(s_0)) \to \mathbf{R}$ respectively. We furthermore let m_2 be the subset of $\mathcal{E}_2$ consisting of all function-germs with zero constant terms. Then, $\varphi^* m_2 \mathcal{E}_1$ is an $\mathcal{E}_1$-submodule of $\mathcal{E}_1$ and we would like to consider the following quotient $\mathcal{E}_1$ module:

$$\frac{\mathcal{E}_1}{\varphi^* m_2 \mathcal{E}_1}.$$

Lemma 2.4 *For $Pe_{\mathbf{r},P}(s+s_0)$, the following hold.*

1. $Pe'_{\mathbf{r},P}(s+s_0)\cdot\mathbf{r}(s+s_0)=0.$

2. $Pe'_{\mathbf{r},P}(s+s_0)\cdot\mathbf{t}(s+s_0)\in\varphi^*m_2\mathcal{E}_1.$

3. $Pe'_{\mathbf{r},P}(s+s_0)\cdot\mathbf{n}(s+s_0)\in\varphi^*m_2\mathcal{E}_1$

4. $Pe'_{\mathbf{r},P}(s+s_0)\cdot\mathbf{b}(s+s_0)\in\varphi^*m_2\mathcal{E}_1$

5. $Pe''_{\mathbf{r},P}(s+s_0)\cdot\mathbf{r}(s+s_0)\in\varphi^*m_2\mathcal{E}_1$

6. $Pe''_{\mathbf{r},P}(s+s_0)\cdot\mathbf{t}(s+s_0)+\varphi^*m_2\mathcal{E}_1=\tau_g(s+s_0)(P\cdot\mathbf{r}(s+s_0))^2(P\cdot\mathbf{b}(s+s_0))+\varphi^*m_2\mathcal{E}_1$

7. $Pe''_{\mathbf{r},P}(s+s_0)\cdot\mathbf{n}(s+s_0)+\varphi^*m_2\mathcal{E}_1=-P\cdot\mathbf{r}(s+s_0)^3+\varphi^*m_2\mathcal{E}_1$

8. $Pe''_{\mathbf{r},P}(s+s_0)\cdot\mathbf{b}(s+s_0)\in\varphi^*m_2\mathcal{E}_1$

9. $Pe'''_{\mathbf{r},P}(s+s_0)\cdot\mathbf{t}(s+s_0)+\varphi^*m_2\mathcal{E}_1=\kappa_g(s+s_0)(P\cdot\mathbf{r}(s+s_0))^3+\varphi^*m_2\mathcal{E}_1$

Proof of lemma 2.4 Since $\{\mathbf{r},\mathbf{t},\mathbf{n},\mathbf{b}\}$ is an orthogonal moving frame, we see that the proof of lemma 2.3 shows that 1-4 of lemma 2.4 hold. Further calculations by using lemma 2.1 show that 5-9 of lemma 2.4 hold. q.e.d

3 Proof of theorem 1

[*Proof of 1*] By lemma 2.3, $Pe'_{\mathbf{r},P}(s_0)\neq 0$ in this case. Thus, the map-germ $Pe_{\mathbf{r},P}(s_0)$ is non-singular and by the rank theorem ([2]) the result holds. q.e.d

[*Proof of 2*] By a suitable rotation of S^3 we may assume that $\mathbf{r}(s_0)=(1,0,0,0)$, $\mathbf{t}(s_0)=(0,1,0,0)$, $\mathbf{n}(s_0)=(0,0,1,0)$ and $\mathbf{b}(s_0)=(0,0,0,1)$.

Then, by lemma 2.4 we may put

$$Pe_{\mathbf{r},P}(s+s_0) = \begin{pmatrix} a+\alpha_3(s) \\ \frac{1}{2}\tau_g(s+s_0)a^2bs^2 + \frac{1}{3!}\kappa_g(s+s_0)a^3s^3 + \alpha_4(s) \\ -\frac{1}{2}a^3s^2 + \beta_3(s) \\ \gamma_3(s) \end{pmatrix},$$

where $a = (P \cdot \mathbf{r}(s_0))$, $b = (P \cdot \mathbf{b}(s_0))$ and α_i, β_i, $\gamma_i : (\mathbf{R}, 0) \to (\mathbf{R}, 0)$ are certain C^∞ function-germs which satisfy $\frac{d^k\alpha_i}{ds^k}(0) = \frac{d^k\beta_i}{ds^k}(0) = \frac{d^k\gamma_i}{ds^k}(0) = 0$ for $k \leq i-1$. Note that $a \neq 0$ in the case of 2 of theorem 1.

Let $\mathcal{E}_1$ be the set of all C^∞ function germs with one variable $(\mathbf{R}, 0) \to \mathbf{R}$, m_1 be its subset consisting of all function-germs with zero constant terms. Then, $m_1^3\mathcal{E}_1$ is a finitely generated $\mathcal{E}_1$-module. We put $f(s) = s^2$ and apply the Malgrange preparation theorem (for instance, see [2], [4], [8]) to $m_1^3\mathcal{E}_1$ and f. Then we see that for any function-germ $g \in m_1^3\mathcal{E}_1$ there exists a certain C^∞ function-germ ψ such that

$$g(s) = \psi(s^2, s^3).$$

Thus, for the map-germ $Pe_{\mathbf{r},P} : (I, s_0) \to S^3$ there exists a germ of C^∞ diffeomorphism $h_t : (S^3, Pe_{\mathbf{r},P}(s_0)) \to (\mathbf{R}^3, 0)$ such that

$$h_t \circ Pe_{\mathbf{r},P}(s+s_0) = (s^2, s^3, 0).$$

q.e.d

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