Re-embedding Structures of Triangulations on Closed Surfaces

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Abstract

Introducing the notions called the *panel structures* and *paneled triangulations*, we shall establish a theory to analyze the re-embeddings of a given triangulation on a closed surface and conclude that there are only finitely many panel structures, up to equivalence, for each closed surface F^2, which implies the existence of a constant upper bound for the number of re-embeddings of triangulations on F^2.

1. Introduction

Let G be a graph. We denote the vertex set and edge set of G by $V(G)$ and $E(G)$, respectively. If G is already embedded on a closed surface F^2, we call each component of $F^2 - G$ a *face* of G and denote the set of faces of G by $F(G)$. However, a graph may admit many embeddings on a fixed closed surface and hence $F(G)$ depends on the embedding in general.

When we deal with two or more embeddings of a graph, we often use a map $f : G \to F^2$ to identify an embedding of a graph G into F^2, regarding G as a topological space. That is, an *embedding* $f : G \to F^2$ is an injective continuous map which induces a homeomorphism between G and $f(G)$.

Let $f,\ f' : G \to F^2$ be two embeddings of a graph G into a closed surface F^2. They are said to be *equivalent* to each other, written by $f \approx f'$, if there is a homeomorphism $h : F^2 \to F^2$ with $hf = f'$. They are *congruent* to each other, written by $f \sim f'$, if there is a homeomorphism $h : F^2 \to F^2$ with $h(f(G)) = f'(G)$ which induces a graph isomorphism. Two congruent embeddings look like the same shape, but their labelings may not coincide through the homeomorphism in general. It is obvious that:

$$f \approx f' \quad \Longrightarrow \quad f \sim f'.$$

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It is well-known that any 3-connected planar graph is uniquely embeddable in the sphere, up to equivalence, which has been proved by Whitney [27], as the uniqueness of duals. Furthermore, all embeddings of a planar graph into the sphere can be generated by two kinds of local deformations, related to the 2-isomorphism over their duals [28].

The first author was motivated by Whitney's work and wrote a series of papers [12] to [20], focusing on the uniqueness of embeddings. In particular, he has classified the structures of projective-planar graphs which generate their re-embeddings on the projective plane. However, such structures are so complicated that we cannot describe them briefly. So we need some restriction on graphs to make theorems in simple style.

For example, it follows from his theorem that if a 5-connected nonplanar graph G, except K_6, admits two or more embeddings on the projective plane, there is an essential simple closed curve on it which meets G in two points. Kitakubo proved in his thesis [6] that any 5-connected graph has at most 12 inequivalent embeddings, using this fact. This upper bound is attained by only K_6 and it will be only 1 if we count embeddings up to congruence since any two embeddings of K_6 on the projective plane are congruent. Furthermore, he has classified the re-embedding structures of 5-connected projective-planar graphs and concluded that they admits precisely 1, 2, 3, 4, 6, 9 or 12 inequivalent embeddings. So many pages in his thesis is devoted to proving these results, but he has shown a simple proof for the existence of a finite bound for the number of embeddings of 5-connected projective-planar graphs in [5].

A graph G embedded on a closed surface F^2 is said to be *n-representative* if any essential simple closed curve on F^2 meets G in at least n points. In [25] and [26], Vitray has classified the 3-representative graphs on the projective plane that are critical with respect to contraction and deletion of edges, and identified how the re-embeddings of those graphs can be generated. His classification implies that any 3-representative 3-connected graph on the projective plane has precisely 1, 2, 3, 4, 6 or 12 embeddings, up to equivalence. This is similar to Kitakubo's result, but there is no 3-representative 5-connected projective-planar graph which has precisely 9 embeddings. (It has been known that any n-representative graph on F^2 is uniquely embeddable in F^2 for a sufficiently large $n > 0$, in general. See [23].)

A simple graph G is called a *triangulation* on a closed surface F^2 if it is embedded on F^2 so that each face is bounded by a 3-cycle and that any two faces share at most one edge. (We call a cycle of length n an *n-cycle*.) The second condition necessarily holds under the first one unless G is isomorphic to K_3. It is easy to see that any triangulation on a closed surface is 3-representative. Lawrencenko [8] has discussed the re-embeddings of triangulations on the projective plane and proved the same fact on the number of embeddings of triangulations as Vitray showed.

For the present, we do not have a general theory enough to analyze the re-embeddings of graphs on closed surfaces. So we shall confine ourself to discussing triangulations on closed surfaces in this paper. Our purpose is to establish a theory to classify the re-embedding structures of triangulations. The fundamental notions in our theory are the *panel structure* and a *paneled triangulation*. The former describes the flexibility and partial rigidity of triangulations while the latter is a formal object to control the panel structures.

In Section 2, we shall show general observations about triangulations on closed surfaces, some of which are well-know, but a basis for our theory developed later. We shall define and discuss the panel structures of triangulations with related notions in Section 3 and paneled triangulations in Section 4. The irreducibility of paneled triangulations defined in Section 5 is the most important notion in our theory, which suggests a method to classify the equivalence classes of panel structures. As its application, we shall prove Lawrencenko's result as mentioned above. In Section 6, we shall prove the finiteness of panel structures in number, which implies that there is a constant $N = N(F^2)$ for any closed surface F^2 such that every triangulation on F^2 has at most N embeddings, up to equivalence.

2. General observations

Let G be a triangulation on a closed surface F^2 and $C_3(G)$ the set of 3-cycles in G. Since the boundary cycle ∂A of a face $A \in F(G)$ is a 3-cycle, it is convenient to identify a face $A \in F(G)$ with its boundary cycle ∂A and to denote it by uvw with its three corners u, v, $w \in V(G)$. So we shall regard $F(G)$ as a subset of $C_3(G)$.

Let $N(v)$ denote the set of neighbors of a vertex $v \in V(G)$, called the *neighborhood* of v while $\bar{N}(v) = N(v) \cup \{v\}$ is called the *closed neighborhood* of v. (The neighborhood of a subset X in $V(G)$ will be denoted by $N(X) = \bigcup_{v \in X} N(v)$.) The neighbors of any vertex $v \in V(G)$ lie around v and form a cycle. This cycle around v is called the *link* of v and is denoted by $\mathrm{lk}(v)$. The subgraph obtained as $\mathrm{lk}(v) \cup \{v\}$ with edges incident to v is often called the *wheel neighborhood* of v and is denoted by $W(v)$ here. Note that $\mathrm{lk}(v)$ and $W(v)$ depend on the embedding of G.

The link of v is one of hamilton cycles of the subgraph $\langle N(v) \rangle$ in G induced by $N(v)$. If $\langle N(v) \rangle$ has two or more hamilton cycles, the vertex v is said to be *skew*. We can find a theory on skew vertices in [12] which is closely related to the uniqueness of embeddings of triangulations. For, if v is not skew, then it will have a unique rotation over its neighborhood, up to reversion, which the unique hamilton cycle induces.

LEMMA 1. *The closed neighborhood $\bar{N}(v)$ of a skew vertex v induces a nonplanar subgraph.*

Proof. Let C be a hamilton cycle of $\langle N(v) \rangle$ other than $\mathrm{lk}(v)$ and choose an edge $xy \in E(C) - E(\mathrm{lk}(v))$. Then the two segments along $\mathrm{lk}(v)$ bounded by $\{x, y\}$ has length at least 2 and there is another edge $st \in E(C) - E(\mathrm{lk}(v))$ joining these segments. It is clear that $W(v) + \{xy, st\}$ contains a subdivision of K_5 and is nonplanar. This implies that $\langle \bar{N}(v) \rangle$ is nonplanar. ■

Let G be a triangulation on a closed surface F^2 and e an edge of G. The *contraction* of e or *contracting* e is to shrink e and to replace each of the resulting multiple edges with one edge. (The inverse operation is called a *vertex splitting*.) Let G/e denote the graph obtained from G by contracting e and $[e]$ the vertex into which e shrinks. If G/e also is a triangulation on F^2, then e is said to be *contractible*. Thus, an edge e in a triangulation G, except K_4, is contractible if and only if G/e is simple. This criterion can be rephrased into:

LEMMA 2. *Let G be a triangulation of a closed surface, except K_4 on the sphere. An edge $e \in E(G)$ is not contractible if and only if e lies on at least three 3-cycles.* ■

Since this lemma presents a combinatorial property, the contractibility of an edges does not depend on the embedding of a triangulation. Thus, if e is contractible in G, then so is $f(e)$ in $f(G)$ for any embedding $f : G \to F^2$ and $f(G)/f(e)$ is isomorphic to G/e as graphs.

LEMMA 3. *Let v be a vertex of a triangulation G on a closed surface F^2, except K_4 on the sphere. If no contractible edge is incident to v, then $\langle \bar{N}(v) \rangle$ is nonplanar.*

Proof. Let $\mathrm{lk}(v) = v_0 \cdots v_{n-1}$ be the link around v. By Lemma 2, for any vertex v_i, there is an edge v_iv_j with $j \not\equiv i \pm 1 \pmod{n}$. Choose v_i to minimize $|i-j| \geq 2$ and suppose that $i < j$. Then there is an edge v_kv_h with $i < k < j < h$ and $W(v) + \{v_iv_j, v_kv_h\}$ contains a subdivision of K_5. This implies that $\langle \bar{N}(v) \rangle$ is nonplanar. ■

Let $\mathrm{Emb}(G, F^2)$ denote the set of all embeddings of G into F^2. It is clear that both $\mathrm{Emb}(G, F^2)/\sim$ and $\mathrm{Emb}(G, F^2)/\approx$ are finite sets and that

$$|\mathrm{Emb}(G, F^2)/\sim| \leq |\mathrm{Emb}(G, F^2)/\approx|$$

in general. In particular, there is a good relationship between $|\mathrm{Emb}(G, F^2)/\approx|$ and edge contraction, as follows.

LEMMA 4. *Let G be a triangulation on a closed surface F^2 and e a contractible edge in G. Then we have:*

$$|\mathrm{Emb}(G, F^2)/\approx| \leq |\mathrm{Emb}(G/e, F^2)/\approx|$$

Proof. Let $f : G \to F^2$ be any embedding of G into F^2 and contract the edge $f(e)$ in $f(G)$ on F^2. By Lemma 2, $f(G)/f(e)$ is isomorphic to G/e via the natural isomorphism $f' : G/e \to f(G)/f(e)$ induced by f and this isomorphism can be regarded as an embedding map $f' : G/e \to F^2$. So we define a correspondence $\Phi : \mathrm{Emb}(G, F^2) \to \mathrm{Emb}(G/e, F^2)$ by $\Phi([f]) = [f']$, where $[f]$ stands for the equivalence class including f. It is clear that Φ is well-defined.

Let $g : G \to F^2$ be another embedding of G into F^2 and suppose that g' is equivalent to f', that is, there is a homeomorphism $h' : F^2 \to F^2$ with $h'f' = g'$. Then G/e contains a cycle C such that $f'(C)$ is the link around $f([e])$ and necessarily $g'(C) = h'f'(C)$ also is the link around $g([e])$. This cycle C can be regarded as one in G and $f(C)$ and $g(C)$ bound 2-cells including $f(e)$ and $g(e)$ inside. Deforming h' suitably within these 2-cells, we can obtain a homeomorphism $h : F^2 \to F^2$ with $hf = g$. Thus, f and g are equivalent to each other. This implies that Φ is injective and the lemma follows. ■

Note that the same statement as above does not hold for the congruence in general. For example, Figure 1 presents such a counter example. The labels over vertices indicate a graph isomorphism between the two triangulations on the torus, and hence they are two embeddings of one graph, say G. It is easy to see that there is no other isomorphism

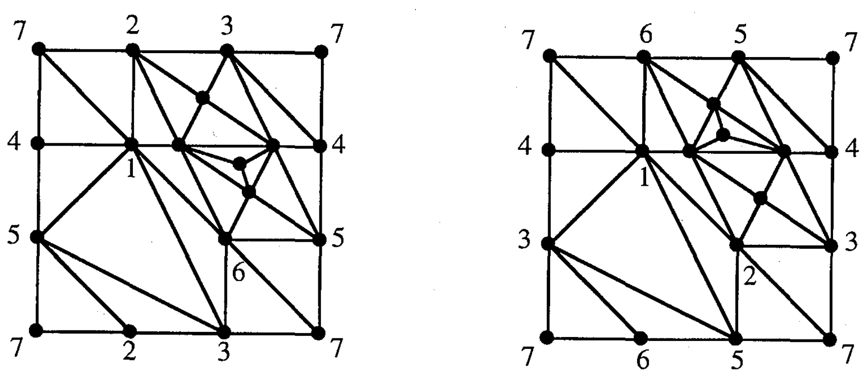

Figure 1: Two incongruent embeddings of a triangulation on the torus

between them and the unique isomorphism does not extend to any homeomorphism over the torus. Thus, these triangulations are not congruent, but contracting an edge e yields two congruent embeddings of G/e if e is incident to the unique vertex of degree 3.

A triangulation G is said to be *contractible* to another triangulation T if T can be obtained from G by a sequence of contraction of edges, and is *irreducible* if G has no contractible edges. Thus, any triangulation is contractible to one of irreducible triangulations. The tetrahedron K_4 is a unique irreducible triangulation of the sphere [24] while there are precisely two irreducible triangulations of the projective plane, which are isomorphic to K_6 and $K_4 + \overline{K}_3$ as graphs [1]. Those for the torus and for the Klein bottle have been classified in [7] and [9].

There are only finitely many irreducible triangulations of a closed surface in general, which has been shown in [2] to [4] and [11]. In particular, Nakamoto and Ota [11] have given a linear upper bound for the number of their vertices with respect to the genus of closed surfaces. (See Section 6 for more details.) The finiteness of irreducible triangulations in number plays an important role in many contexts as well as in our proof of the following theorem.

THEOREM 5. *Given a closed surface F^2, there is a natural number $N = N(F^2)$ such that any triangulation G on F^2 has at most N embeddings into F^2, up to equivalence.*

Proof. Define N as the maximum value of $|\mathrm{Emb}(T, F^2)/\approx|$ taken over all irreducible triangulations T of F^2, which actually exists by the finiteness of irreducible triangulations. Since any triangulation G is contractible to one of irreducible triangulations, say T, we have $|\mathrm{Emb}(G, F^2)/\approx| \leq |\mathrm{Emb}(T, F^2)/\approx| \leq N$ by Lemma 4. ■

The value of $N = N(F^2)$ can be determined by estimating the number of embeddings of irreducible triangulations of F^2. For example,

$$N(S^2) = 1, \quad N(P^2) = 12, \quad N(T^2) = 120 \quad \text{and} \quad N(K^2) = 36$$

for the sphere S^2, the projective plane P^2, the torus T^2 and the Klein bottle K^2. The first three are attained by the complete graphs K_4, K_6 and K_7 on these surfaces in order while the last one is attained by the triangulation obtained from two copies of K_6 on the projective plane by joining them along one pair of faces.

3. Panel structures

As is shown in the previous section, we can decide the exact upper bound for the number of inequivalent embeddings of triangulations on a closed surface F^2 if the complete list of irreducible triangulations of F^2 is given. However, we need more accurate arguments to the set of those natural numbers that can be realized as $|\mathrm{Emb}(G,F^2)|/\approx|$ for a triangulation G on F^2. In this section, we shall define and discuss some notions to do it.

Let G be a triangulation on a closed surface F^2 and regard $F(G)$ as a subset in $C_3(G)$. A 3-cycle C of G is called a *panel* of G if $f(C)$ bounds a face of $f(G)$ for any embedding $f: G \to F^2$. A face also is called a *panel* of G if it is bounded by such a cycle. However, a panel will be a face, rather than an abstract 3-cycle, in the below and will be indicated as a shaded region in figures. A face is called a *hole* of G if it is not a panel.

We shall denote the set of panels of G by $\wp(G)$. That is,

$$\wp(G) = \bigcap \{F(f(G)) : f \in \mathrm{Emb}(G,F^2)\}.$$

The composite structure $(G,\wp(G))$ is called the *panel structure* of G. This notion is closely related to embeddings, but $(G,\wp(G))$ is uniquely determined, not depending on the embedding of G.

LEMMA 6. *A face incident to a contractible edge is a panel.*

Proof. Let e be a contractible edge in a triangulation G. By Lemma 2, e lies on precisely two 3-cycles in G and they must bound faces incident to e in any embedding of G. Thus, the lemma follows. ∎

A vertex v of G is said to be *flat* if every face incident to v is a panel of G, while v is *twistable* otherwise. It is clear that a unique cycle in $\langle N(v)\rangle$ becomes the link of a flat vertex v, not depending on the embedding, although v might be skew.

LEMMA 7. *A twistable vertex is skew.*

Proof. Let G be a triangulation on a closed surface F^2 and v a twistable vertex. Then there is another embedding $f: G \to F^2$ of G, not equivalent to the original, in which some of facial cycles incident to v does not bound a face, and hence a cycle in $\langle N(v)\rangle$ other than $\mathrm{lk}(v)$ becomes the link of $f(v)$ in $f(G)$. Thus, $\langle N(v)\rangle$ contains at least two hamilton cycles and v must be skew. ∎

LEMMA 8. *Any vertex of degree* 3 *is flat. Two adjacent vertices are flat if both of them have degree at most* 4.

Proof. It is easy to see that those vertices are not skew, and hence they are flat by Lemma 7. ∎

Let G be a triangulation on a closed surface F^2. The graph obtained from G by removing all the flat vertices is called the *frame* of G and is denoted by $\mathrm{Fr}(G)$. On the other hand, the 2-complex induced by the panels $\wp(G)$ is called the *panel complex* of G and is denoted by $\mathrm{P}(G)$. That is, each edge in $\mathrm{P}(G)$ is incident to a panel.

The frame $\mathrm{Fr}(G)$ is embedded on F^2 as a subembedding of G, but is not a triangulation in general. The faces of $\mathrm{Fr}(G)$ can be classified into two classes; the first kind is triangular and comes from a face of G, which is a hole of G. The other faces, called *plates*, contain only panels of G and their union is homeomorphic to the panel complex $\mathrm{P}(G)$. It is clear that $G = \mathrm{Fr}(G) \cup (\mathrm{P}(G) \cap G)$ and that the restriction $f|_{\mathrm{P}(G)\cap G}$ extends to an embedding of $\mathrm{P}(G)$ for any embedding $f : G \to F^2$.

Let $(G_1, \wp(G_1))$ and $(G_2, \wp(G_2))$ be the panel structures of two triangulations G_1 and G_2 on a closed surface F^2. They are said to be *equivalent* to each other if there is an graph isomorphism $\phi : \mathrm{Fr}(G_1) \to \mathrm{Fr}(G_2)$ which induces a homeomorphism $\tilde{\phi} : \mathrm{P}(G_1) \to \mathrm{P}(G_2)$.

THEOREM 9. *If two triangulations on a closed surface F^2 have equivalent panel structures, then they admit the same number of embeddings, up to equivalence.*

Proof. Let G_1 and G_2 be two triangulations with equivalent panel structures. Given an embedding $f_2 : G_2 \to F^2$, we define an embedding $f_1 : G_1 \to F^2$, as follows. First set $f_1|_{\mathrm{Fr}(G_1)} = f_2\phi$. Let $\phi_2 : \mathrm{P}(G_2) \to F^2$ be the extension of $f_2|_{\mathrm{P}(G_2)\cap G_2}$, which is an embedding of $\mathrm{P}(G_2)$ into F^2 and set $f_1|_{\mathrm{P}(G_1)\cap G_1} = \phi_2\tilde{\phi}|_{\mathrm{P}(G_1)\cap G_1}$. This correspondence $f_1 \leftrightarrow f_2$ induces the bijection between $\mathrm{Emb}(G_1)/\approx$ and $\mathrm{Emb}(G_2)/\approx$. ■

Let G be a triangulation on a closed surface F^2. Another triangulation G' is called a *refinement* of G if G' contains a subdivision of G as its subembedding. Furthermore, the panel structure $(G', \wp(G'))$ is a *refinement* of $(G, \wp(G))$ if G' can be embedded or re-embedded on F^2 as a refinement of G where only panels of G are subdivided.

THEOREM 10. *Two panel structures are equivalent to each other if and only if they have a common refinement.*

Proof. Embed $\mathrm{P}(G_1)$ on F^2 together with $\mathrm{P}(G_2)$ by $\tilde{\phi}$ given in the definition. Then, we can make their common refinement, adding edges and vertices to $\tilde{\phi}(\mathrm{P}(G_1)) \cup \mathrm{P}(G_2)$. ■

4. Paneled triangulations

The panel structure $(G, \wp(G))$ exists a priori and can be said to generate the varieties of embeddings of a triangulation. In this section, we shall define and analyze an artificial object, called a paneled triangulation, which will be used to classify the panel structures in the next section.

Let G be a triangulation on a closed surface F^2 and let $\wp$ be a subset of $F(G)$. We call the pair $(G, \wp)$ a *paneled triangulation* over G with panel $\wp$ and denote it by $G_\wp$. A face belonging to $\wp$ is called a *panel* or is said to be *paneled*. A *flat* vertex, a *twistable* vertex, a *hole*, the *frame* $\mathrm{Fr}(G_\wp)$ and the *panel complex* $\mathrm{P}(G_\wp)$ of G are defined in the same way as in the previous section.

An *embedding* $f : G_\wp \to F^2$ is an embedding $f : G \to F^2$ such that $f(\partial A)$ bounds a face in $f(G)$ for each face $A \in \wp$. The *equivalence* and the *congruence* over embeddings of paneled triangulations are defined in the same fashion as for ordinary triangulations. Two paneled triangulations $G_\wp$ and $G'_{\wp'}$ are said to be *isomorphic* to each other if there is a homeomorphism $h : F^2 \to F^2$ with $h(G) = G'$ which induces a bijection between $\wp$ and $\wp'$.

A paneled triangulation $G_\wp$ is said to be *saturated* if there is at least one embedding $f : G_\wp \to F^2$, for each face $A \notin \wp$, such that $f(\partial A)$ does not bound a face in $f(G)$. For example, the *full-paneled triangulation* $G_{F(G)}$ is saturated and it has a unique embedding, up to equivalence. If $\wp = \wp(G)$, then $G_\wp$ is saturated and $\mathrm{Emb}(G_\wp, F^2) = \mathrm{Emb}(G, F^2)$. The *empty-paneled triangulation* $G_\emptyset$ with no panels is saturated if and only if the panel structures of the triangulation G itself has no panels.

Now let G be a triangulation of a closed surface F^2, not paneled. For a subset S in $\mathrm{Emb}(G, F^2)$, define $\wp_S$ as the set of faces whose boundary cycles bound faces in any embedding belonging to S. Then S is said to be *saturated* if there is a face $A \in \wp_S$, for any embedding $f \notin S$, such that $f(\partial A)$ does not bound a face in $f(G)$. It is clear that if $f \in S$ and $f \approx f'$, then $f' \in S$, provided that S is saturated.

LEMMA 11. *Let G be a triangulation on a closed surface F^2. Then, there is a bijection between the saturated subsets of embeddings of G into F^2 and the saturated paneled triangulations over G.*

Proof. Define $\Phi(G_\wp) = \mathrm{Emb}(G_\wp, F^2) \subset \mathrm{Emb}(G, F^2)$ for a saturated paneled triangulation $G_\wp$. It is easy to see that $\Phi(G_\wp)$ is saturated and Φ is injective. Conversely, let S be a saturated subset in $\mathrm{Emb}(G, F^2)$. It is obvious that $G_{\wp_S}$ is saturated and $\Phi(G_{\wp_S}) = S$. Thus, Φ is surjective. ■

Let $G_\wp$ be a saturated paneled triangulation over G and G' a triangulation obtained from G by subdividing each face $A \in \wp$ with vertices added inside. Then we say that $G_\wp$ *presents* the panel structure $(G', \wp(G'))$. The following lemma makes this definition meaningful.

LEMMA 12. *Let $G_\wp$ be a saturated paneled triangulation over a triangulation G on a closed surface F^2. Then the panel structures of triangulations presented by $G_\wp$ are all equivalent.*

Proof. Let G' be a refinement of G with only panels subdivided. Then G' includes G as its subgraph. Let $f : G' \to F^2$ be any embedding of G' into F^2 and A a face in $\wp$. Since A contains some vertices of G', $f(\partial A)$ bounds a triangular region A' on F^2 where those vertices are mapped. By the uniqueness of embeddings of 3-connected planar graphs, $f|_{A \cap G}$ extends to a homeomorphism $f' : A \to A'$. This implies that each face contained in A is a panel of G' and hence the panel complex $\mathrm{P}(G')$ occupies the same region as $\mathrm{P}(G)$ does. Since $\mathrm{Fr}(G') = \mathrm{Fr}(G)$, the panel structures $(G', \wp(G'))$ are all equivalent. ■

Note that a panel of $G_\wp$ might not be a panel of G, even if $G_\wp$ is saturated, since the former is artificially assigned. However, there are certain conditions for a face to be a panel of $G_\wp$. It is easy to prove the following lemma. Consider which cycle becomes the link around a vertex v.

LEMMA 13. *Let G be a saturated paneled triangulation on a closed surface F^2 and v a vertex with* $\mathrm{lk}(v) = v_1 \cdots v_n$.

(i) *If v is not skew, then all of vv_iv_{i+1}'s and vv_nv_1 are paneled, that is, v is flat.*

(ii) *If vv_iv_{i+1} is paneled for $i = 2, \ldots n-1$, then vv_1v_2 and vv_nv_1 are paneled.*

(iii) *If vv_iv_{i+1} is paneled for $i = 3, \ldots n-1$, then vv_1v_2 is paneled.* ■

5. Panel-irreducibility

Now we shall consider the "irreducibility" of paneled triangulations, mimicking the irreducible triangulations of a closed surface. However, we need a slight modification on its definition, as follows, to adapt for what we expect.

Let $G_\wp$ be a paneled triangulation with panel $\wp$. An edge uv in $G_\wp$ is said to be *panel-contractible* if it is contractible in the usual sense and if either u or v is flat. Contraction of a panel-contractible edge shrinks it and remove the panels incident to it, if any, from $\wp$ to obtain another paneled triangulation $G_\wp/uv$. If $G_\wp$ has no panel-contractible edge, then $G_\wp$ is said to be *panel-irreducible*.

The following two lemmas show the reason why we define the panel-irreducibility and the panel-contractibility as above.

LEMMA 14. *Let $G_\wp$ be a saturated paneled triangulation and uv a panel-contractible edge of $G_\wp$. Then $G_\wp/uv$ is saturated and the panel structures which $G_\wp/uv$ presents is equivalent to those that $G_\wp$ does.*

Proof. Since uv is panel-contractible, one of them, say u, is a flat vertex in $G_\wp$. Then, we may assume that contraction of uv moves v into u, fixing the position of u on F^2. Then the panel complex $\mathrm{P}(G_\wp/uv)$ occupies the same region as $\mathrm{P}(G_\wp)$ does and $\mathrm{Fr}(G_\wp/uv) = \mathrm{Fr}(G_\wp)$. This implies the panel structures obtained from $G_\wp$ and $G_\wp/uv$ by subdividing their panels are equivalent. ■

LEMMA 15. *Let $G_\wp$ be a saturated paneled triangulation and uv an edge of $G_\wp$. If uv is contractible, but is not panel-contractible, then $G_\wp$ and $G_\wp/uv$ present incongruent panel structures*

Proof. If uv is contractible, but is not panel-contractible, then both u and v are not flat and belong to $\mathrm{Fr}(G_\wp)$. However, uv belongs to $\mathrm{P}(G_\wp)$ since it is contractible and incident to two panels by Lemma 6. Thus, contracting e destroys the homeomorphism type of $\mathrm{P}(G_\wp)$. That is, $\mathrm{P}(G_\wp)$ and $\mathrm{P}(G_\wp/uv)$ are not homeomorphic and hence the lemma follows. ■

The following two theorems show the connection between the paneled triangulations and the panel structures of ordinary triangulations. They are immediate consequences of the above lemmas. For the sake of convenience, we call a panel-irreducible saturated paneled triangulation a *panel-irreducible triangulation* simply, omitting the phrase "saturated paneled".

THEOREM 16. *Every panel structure of a triangulation on a closed surface F^2 is equivalent to those which a panel-irreducible triangulation of F^2 presents.* ■

THEOREM 17. *The panel structures presented by two panel-irreducible triangulations on a closed surface F^2 are equivalent to each other if and only if one of the paneled triangulations is an embedding of the other, equivalent or inequivalent.*

Note that the equivalence over panel structures is defined independently of embeddings of triangulations while the paneled triangulation has a fixed embedding on a closed surface.

Here we shall try to classify the panel structures of triangulations on the projective plane, applying our theory. The following lemma makes it easy to do it, but does not hold for other surfaces, as the proof suggests below.

LEMMA 18. *If $G_\wp$ is a panel-irreducible triangulation on the projective plane, then G is irreducible.*

Proof. It suffices to show that any contractible edge of a paneled triangulation $G_\wp$ on the projective plane is panel-contractible. Let uv be a contractible edge and let $\mathrm{lk}(u) = vu_1 \cdots u_l$ and $\mathrm{lk}(v) = uv_1 \cdots v_r$ be the link of u and v in $G_\wp$ with $u_1 = v_r$ and $v_1 = u_l$.

Since uv is contractible, there is no edge of the form uv_i or vu_j by Lemma 2. On the other hand, if uv is not panel-contractible, then both u and v are not flat and must be skew by Lemma 13. In this case, $\langle \bar{N}(u) \rangle$ and $\langle \bar{N}(v) \rangle$ and nonplanar and there are edges u_iu_j, u_ku_h for some $i < k < j < h$ and v_av_b, v_sv_t for some $a < s < b < t$ by Lemma 1. However, the partial structure $W(u) \cup W(v) + \{u_iu_j, u_ku_h, v_av_b, v_sv_t\}$ of $G_\wp$ cannot be embedded in the projective plane, a contradiction. ■

Since there are only two irreducible triangulations of the projective plane as mentioned in Section 2, it is just a routine to classify the panel-irreducible triangulations of the projective plane. Lemma 13 is useful to do it.

THEOREM 19. *There exist precisely* 15 *panel-irreducible triangulations of the projective plane, up to isomorphism, as given in Figure 2.* ■

In Figure 2, we should identify each antipodal pair of vertices along the boundary of each hexagon to obtain the actual paneled triangulations of the projective plane. Each shaded face is paneled and the integer inside parentheses indicates the number of congruent embeddings of the panel-irreduicble triangulation, up to equivalence. In particular, P7 and P8 are incongruent embeddings of the same panel-irreducible triangulations and present the same panel structure, which generates three inequivalent embeddings. Similarly, P9 and P10 do so.

Lawrencenko's result mentioned in the introduction follows immediately from this classification of panel structures for the projective plane.

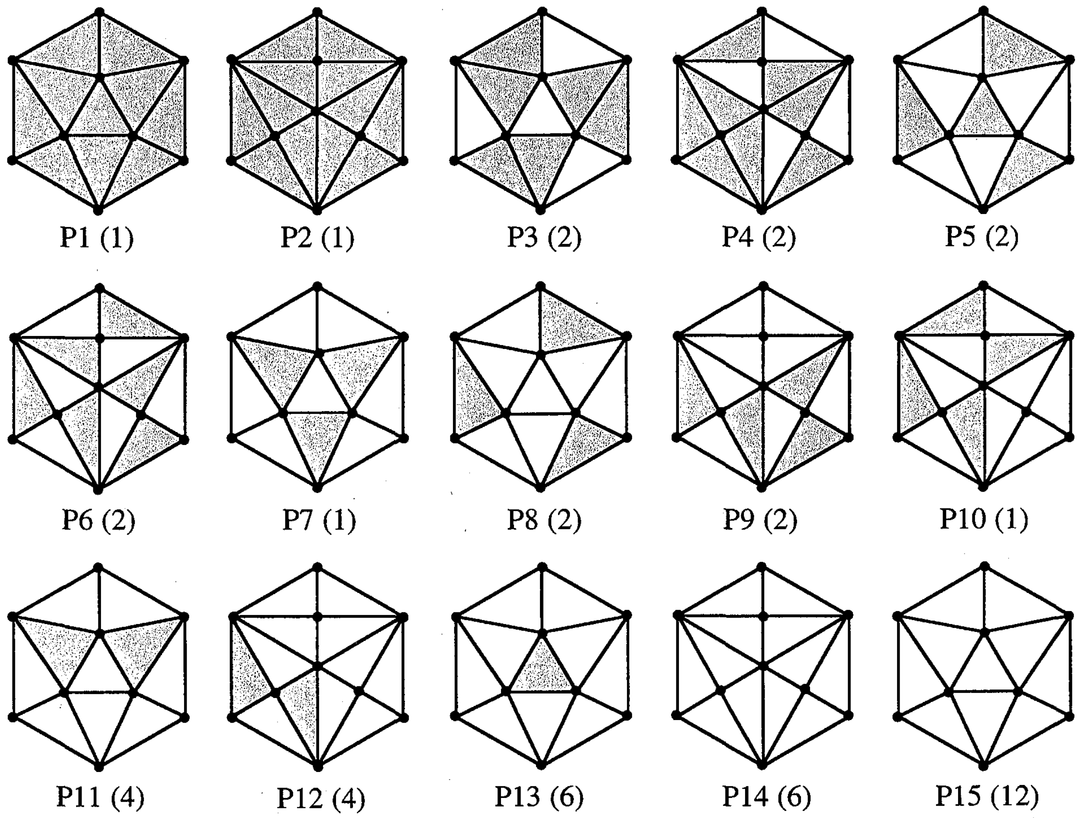

Figure 2: Panel-irreducible triangulations of the projective plane

COROLLARY 20. (Lawrencenko [8]) *Every triangulation on the projective plane admits precisely* 1, 2, 3, 4, 6 *and* 12 *embeddings, up to equivalence.*

6. Finiteness of panel structures

In this section, we shall show the finiteness of panel-irreducible triangulations and that of panel structures in number. First, we begin with some technical lemmas.

Let $\chi(F^2)$ denote the Euler characteristic of a closed surface F^2. The *Euler genus* of a graph G is defined as the minimum value of $2 - \chi(F^2)$ taken over all closed surfaces F^2 where G is embeddable, and is denoted by $\bar{\gamma}(G)$. It is easy to see that if G is a triangulation on F^2, then $\bar{\gamma}(G) = 2 - \chi(F^2)$. Miller [10] has shown the semi-additivity of this Euler genus, as follows.

LEMMA 21. (Miller [10]) *If two graphs G_1 and G_2 have at most two common vertices, then* $\bar{\gamma}(G_1 \cup G_2) \geq \bar{\gamma}(G_1) + \bar{\gamma}(G_2)$. ■

LEMMA 22. *Let G be a triangulation on a closed surface F^2 with minimum degree at least 4. Then there exists an independent set X in G such that*

$$|X| \geq \frac{1}{15}(V(G) + \chi(F^2)).$$

Proof. Let V_i be the set of vertices of degree i. Take three disjoint independent sets X_4, X_5 and X_6 so that they are maximal in V_4, $V_5 - N(X_4)$ and $V_6 - N(X_4) - N(X_5)$ in order. Then it is clear that $X = X_4 \cup X_5 \cup X_6$ is independent.

Put $A_i = V_i \cap N(X_4)$ for $i = 5$ and 6, and $B_6 = V_6 \cap N(X_5)$. Since $\bar{N}(x) \subset V_4 \cup A_5 \cup A_6$ and $\bar{N}(x) = 5$ for each vertex $x \in X_4$, if $5|X_4| < |V_4| + |A_5| + |A_6|$, then $(V_4 \cup A_5 \cup A_6) - (N(X_4) \cup X_4) = V_4 - (N(X_4) \cup X_4)$ would not be empty and contain a vertex $x' \in V_4 - (N(X_4) \cup X_4)$ so that $X \cup \{x'\}$ is independent in V_4. This is contrary to the maximality of X_4. Thus, we have

$$|X_4| \geq \frac{1}{5}(|V_4| + |A_5| + |A_6|).$$

Similarly, we can obtain the following inequalities.

$$|X_5| \geq \frac{1}{6}(|V_5| - |A_5| + |B_6|), \quad |X_6| \geq \frac{1}{7}(|V_6| - |A_6| - |B_6|).$$

Therefore,

$$\begin{aligned} |X| = |X_4 \cup X_5 \cup X_6| &\geq \frac{1}{5}|V_4| + \frac{1}{6}|V_5| + \frac{1}{7}|V_6| + \frac{1}{30}|A_5| + \frac{2}{35}|A_6| + \frac{1}{42}|B_6| \\ &\geq \frac{|V_4|}{5} + \frac{|V_5|}{6} + \frac{|V_6|}{7} \geq \frac{1}{15}(3|V_4| + 2|V_2| + |V_6|). \end{aligned}$$

On the other hand, we have the following well-known formula, which is easily derived from Euler's formula on F^2:

$$\sum_{i \geq 4}(6-i)|V_i| = -|V(G)| + \sum_{i \geq 4}(7-i)n_i = 6\chi(F^2)$$

Thus,

$$3|V_4| + 2|V_5| + |V_6| \geq |V(G)| + 6\chi(F^2).$$

Substituting this to the previous inequality, we obtain the lower bound for $|X|$ as in the lemma. ∎

LEMMA 23. *Let G be a triangulation on a closed surface F^2 with $\chi(F^2) = 2 - r$. If the closed neighborhood of each vertex in G induces a nonplanar graph, then*

$$|V(G)| \leq 171r - 72.$$

Proof. Let X be a maximum independent set in G and put $Y = N(X) - X$. By Lemma 22, $|X| \geq (|V(G)| + \chi(F^2))/15 = (|V(G)| - 6r + 12)/15$ since G has no vertex of degree 3, whose closed neighborhood induces K_4. We construct a maximal subset

$S = \{v_1, v_2, \dots, v_s\}$ in X, adding v_j for $j = 1, 2, 3, \dots$ with the following condition as far as possible:

$$\left| N(v_j) \cap \bigcup_{1 \le i < j} N(v_i) \right| \le 2$$

Let H_j be the subgraph in G induced by $\bigcup_{1 \le i \le j} \bar{N}(v_i)$.

First suppose that $|S| \ge (|V(G)| + 72)/171$. By the assumption in the theorem, $\tilde{\gamma}(\langle \bar{N}(v_i) \rangle) \ge 1$. Since H_i and $\langle \bar{N}(v_{i+1}) \rangle$ share at most two vertices, we have

$$\tilde{\gamma}(H_s) = \tilde{\gamma}\left(\bigcup_{i=1}^{s} \langle \bar{N}(v_i) \rangle \right) \ge \sum_{i=1}^{s} \tilde{\gamma}(\langle \bar{N}(v_i) \rangle) \ge |S| \ge \frac{|V(G)| + 72}{171}$$

by Lemma 21. Since H_s is a subgraph of G and is embedded in F^2, $r = \tilde{\gamma}(G) \ge \tilde{\gamma}(H_s)$. Thus, we obtain that $|V(G)| \le 171r - 72$.

Now suppose that $|S| < (|V(G)| + 72)/171$. Put $T = N(S) \cap Y$ and let M be the subgraph in G with $V(M) = X \cup T$ and $E(M) = \{xy \in E(G) | x \in X, y \in T\}$. Since M is embedded in F^2, we have

$$|V(M)| - |E(M)| + |F(M)| \ge 2 - r.$$

Since M is bipartite, $4|F(M)| \le 2|E(M)|$ and hence we have

$$|V(M)| - \frac{1}{2}|E(M)| \ge 2 - r.$$

By the maximality of S, each vertex $v \in X - S$ has at least three neighbors in T and there are at least $|T|$ edges between S and T. Hence $|E(M)| \ge 3(|X| - |S|) + |T|$. Substituting this inequality to the above, we obtain

$$|X| + |T| - \frac{1}{2}(3(|X| - |S|) + |T|) \ge 2 - r, \quad \text{or} \quad -|X| + |T| + 3|S| \ge 4 - 2r.$$

Since $|X| \ge (|V(G)| - 6r + 12)/15$, $|T| \le 6|S|$ and $|S| < (|V(G)| + 72)/171$, then we have

$$-\frac{1}{15}(|V(G)| - 6r + 12) + 9 \cdot \frac{|V(G)| + 72}{171} > 4 - 2r$$

This implies that $|V(G)| < 171r - 72$. ∎

These three lemmas are based on Nakamoto and Ota's arguments in [11] to show the finiteness of irreducible triangulations of a closed surface. In fact, they have proved that an irreducible triangulation G on a closed surface F^2 has at most $171r - 72$ vertices with $r = 2 - \chi(F^2)$. This is an immediate consequence of the above lemma since the closed neighborhood of each vertex in an irreducible triangulation induces a nonplanar subgraph by Lemma 3.

THEOREM 24. *There exist only finitely many panel-irreducible triangulations of a closed surface, up to isomorphism.*

Proof. Let $G_\wp$ be a panel-irreducible triangulation of a closed surface F^2 and let v be a vertex of $G_\wp$. If v is flat, then each edge incident to v is not contractible and $\langle \bar{N}(v) \rangle$

is nonplanar by Lemma 3. If v is not flat, then it is skew by Lemma 13 and $\langle \bar{N}(v) \rangle$ is nonplanar by Lemma 1. Thus, $G_\wp$ satisfies the condition in Lemma 23 and hence $|V(G_\wp)|$ is bounded by a constant, which depends on only F^2. This implies the finiteness of panel-irreducible triangulations of F^2. ∎

By this theorem and Lemma 16, we can conclude the following two immediately.

COROLLARY 25. *There exist only finitely many panel structures of triangulations on a closed surface, up to equivalence.*

COROLLARY 26. *There is a constant $\tau = \tau(F^2)$, for each closed surface F^2, such that any triangulation on F^2 has at most τ twistable vertices.*

Although a twistable vertices is skew, there might be a skew vertex which is flat. Is there a constant bound for the number of skew vertices?

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