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Normal forms for singularities of pedal curves produced by non-singular dual curve germs in S^n

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Abstract For an n-dimensional spherical unit speed curve $\mathbf{r}$ and a given point P, we can define naturally the pedal curve of $\mathbf{r}$ relative to the pedal point P. When the dual curve germs are non-singular, singularity types of such pedal curves depend only on locations of pedal points. In this paper, we give a complete list of normal forms for singularities and locations of pedal points when the dual curve germs are non-singular. As an application of our list, we characterize C^∞ left equivalence classes of pedal curve germs $(I, s_0) \to S^n$ produced by non-singular dual curve germ from the viewpoint of the relation between $\mathcal{L}$ tangent space and $\mathcal{C}$ tangent space.

Keywords normal form · singularity · pedal curve · pedal point · dual curve · map of blow up type

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1 Introduction

Let I be an open interval and S^n be the n-dimensional unit sphere in $\mathbf{R}^{n+1}$. A C^∞ regular map $\mathbf{r} : I \to S^n$ is said to be a *spherical unit speed curve* if each of the following $\mathbf{u}_i(s) \quad (1 \le i \le n-1)$ is inductively well-defined for any $s \in I$ (in other words, each of the following $\kappa_i(s) \quad (1 \le i \le n-1)$ is a positive function), where initial information are $\mathbf{u}_{-1}(s) \equiv \mathbf{0}$, $\mathbf{u}_0(s) = \mathbf{r}(s)$, $\|\mathbf{u}_0'(s)\| \equiv 1$ and $\kappa_0(s) \equiv 0$.

$$\mathbf{u}_i(s) = \frac{\mathbf{u}_{i-1}'(s) + \kappa_{i-1}(s)\mathbf{u}_{i-2}(s)}{\|\mathbf{u}_{i-1}'(s) + \kappa_{i-1}(s)\mathbf{u}_{i-2}(s)\|} \qquad (1 \le i \le n-1)$$

$$\kappa_i(s) = \|\mathbf{u}_{i-1}'(s) + \kappa_{i-1}(s)\mathbf{u}_{i-2}(s)\| \qquad (1 \le i \le n-1)$$

Note that the above inductive conditions for a spherical unit speed curve $\mathbf{r}$ are not so strong restrictions. This is because first by using Thom transversality theorem (for

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instance, see [6]) $(n-2)$ times for any C^∞ regular map $\mathbf{r} : I \to S^n$ we can obtain a sufficiently near C^∞ map $\widetilde{r}$ in $C^\infty(I, S^n)$ with Whitney C^∞ topology such that

$$\widetilde{\mathbf{r}}(s), \frac{d\widetilde{\mathbf{r}}}{ds}(s), \cdots, \frac{d^{n-1}\widetilde{\mathbf{r}}}{ds^{n-1}}(s) \text{ are linearly independent for any } s \in I.$$

Then the so-called *arc-length parameter* gives a C^∞ diffeomorphism $h : I \to I$ such that $\widetilde{\mathbf{r}} \circ h^{-1}$ is a spherical unit speed curve.

For a spherical unit speed curve we see that any two of $\mathbf{u}_i, \mathbf{u}_j \quad (0 \le i, j \le n-1, i \ne j)$ are perpendicular (see §2). Therefore, we can define one more vector $\mathbf{u}_n(s)$ uniquely so that $\{\mathbf{u}_0(s), \mathbf{u}_1(s), \cdots, \mathbf{u}_n(s)\}$ is an orthogonal moving frame and $\det(\mathbf{u}_0(s), \cdots, \mathbf{u}_n(s)) = 1$ for any $s \in I$. The map $\mathbf{u}_n : I \to S^n$ is called the *dual curve* of $\mathbf{r}$ ([1]). By using the dual curve $\mathbf{u}_n$ we define κ_n as follows, where the dot in the center means the scalar product:

$$\kappa_n(s) = \mathbf{u}'_{n-1}(s) \cdot \mathbf{u}_n(s).$$

We see that the dual curve $\mathbf{u}_n$ is non-singular at s if and only if $\kappa_n(s) \ne 0$ (see §2).

For any $i \quad (-1 \le i \le n)$, we put

$$S^i_{\mathbf{u}_i(s)} = (S^n - \{\pm\mathbf{u}_n(s)\}) \cap \sum_{j=-1}^{i} \mathbf{R}\mathbf{u}_j(s).$$

Given a spherical unit speed curve $\mathbf{r} : I \to S^n$, choosing a point P of $S^n - \{\pm\mathbf{u}_n(s) \mid s \in I\}$ gives the map which maps $s \in I$ to the unique nearest point in $S^{n-1}_{\mathbf{u}_{n-1}(s)}$ from P. Such a map is called the *pedal curve* relative to the *pedal point* P for an n-dimensional unit speed curve $\mathbf{r}$ and is denoted by $ped_{\mathbf{r},P}$. Note that since all points in $S^{n-1}_{\mathbf{u}_{n-1}(s)}$ are the nearest points from $\pm\mathbf{u}_n(s)$ the pedal point P for the map-germ $ped_{\mathbf{r},P}$ at s must be outside $\{\pm\mathbf{u}_n(s)\}$.

The purpose of this paper is to show the following.

Theorem 1.1 *Let* $\mathbf{r}$ *be an* n*-dimensional spherical unit speed curve. Let* $s_0 \in I$ *be such that* $\kappa_n(s_0) \ne 0$. *Then the following hold.*

1. *The pedal point* P *is inside* $S^n_{\mathbf{u}_n(s_0)} - S^{n-2}_{\mathbf{u}_{n-2}(s_0)}$ *if and only if the map-germ* $ped_{\mathbf{r},P} : (I, s_0) \to S^n$ *is* C^∞ *left equivalent to the map-germ given by* $s \mapsto (s, 0, \cdots, 0)$.
2. *For any* i $(0 \le i \le n-2)$, *the pedal point* P *is inside* $S^i_{\mathbf{u}_i(s_0)} - S^{i-1}_{\mathbf{u}_{i-1}(s_0)}$ *if and only if the map-germ* $ped_{\mathbf{r},P} : (I, s_0) \to S^n$ *is* C^∞ *left equivalent to the map-germ given by the following:*

$$s \mapsto (\underbrace{s^{n-i}, s^{n-i+1}, \cdots, s^{2n-2i-1}}_{(n-i)\ \textit{elements}}, \underbrace{0, \cdots, 0}_{i\ \textit{elements}}).$$

Here, two map-germs $f, g : (\mathbf{R}, 0) \to (\mathbf{R}^n, 0)$ are said to be C^∞ *left equivalent* if there exist a germ of C^∞ diffeomorphism $h_t : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ such that the identity $g = h_t \circ f$ is satisfied.

As a corollary of theorem 1.1, we can characterize C^∞ left equivalence classes of pedal curve germs $(I, s_0) \to S^n$ with $\kappa_n(s_0) \ne 0$ from the viewpoint of the relation between $\mathcal{L}$ tangent space and $\mathcal{C}$ tangent space. For the definitions of $\mathcal{L}$ tangent space and $\mathcal{C}$ tangent space, see [7] or [9]. Let $\mathcal{O}(1, n)$ be the set of C^∞ map-germs $f : (\mathbf{R}, 0) \to$

$(\mathbf{R}^n,0)$ such that $T\mathcal{L}(f)=T\mathcal{C}(f)$ ($\mathcal{O}$ means "open") with finite codimensions ; and let $\mathcal{P}(1,n)$ be the set of C^∞ map-germs $(\mathbf{R},0)\to(\mathbf{R}^n,0)$ which are C^∞ left equivalent to some pedal curve germ $(I,s_0)\to S^n$ with $\kappa_n(s_0)\neq 0$ ($\mathcal{P}$ means "pedal"). Since any normal form in theorem 1.1 belongs to $\mathcal{O}(1,n)$, any map-germ in $\mathcal{P}(1,n)$ is C^∞ left equivalent to one of normal forms in theorem 1.1 and any map-germ in $\mathcal{O}(1,n)$ is C^∞ left equivalent to one of normal forms in theorem 1.1 [1], we have the following.

Corollary 1.1 $\mathcal{O}(1,n)=\mathcal{P}(1,n)$.

Note that it is impossible to obtain the same result as in corollary 1.1 if we replace $T\mathcal{L}(f)=T\mathcal{C}(f)$ with $T\mathcal{A}(f)=T\mathcal{K}(f)$ in the definition of $\mathcal{O}(1,n)$ since the equality $T\mathcal{A}(f)=T\mathcal{K}(f)$ holds even for $f(s)=(s^3,s^4)$. Thus, in our situation the C^∞ right-left equivalence does not work well, but the C^∞ left equivalence does so. This is a merit since the C^∞ left equivalence is easy to deal with as pointed out in [4]. Furthermore, in our situation we can truncate higher terms quite easily by using Malgrange preparation theorem only one time, we need no calculations by using semigroups as in [2] (see §5). On the other hand, note also that it is impossible in general to characterize $P(1,n)$ as the set of tops of hierarchies of $\mathcal{A}$-simple singularities since normal forms in 2 of theorem 1.1 are $\mathcal{A}$-simple if and only if $n\leq 6$ due to [2]. Thus, it seems that the notion of simple singularity is not suitable for singularities of pedal in general.

In §2 we investigate several properties of the set $\{\mathbf{u}_0(s),\cdots,\mathbf{u}_n(s)\}$. §3 is devoted to factor $ped_{\mathbf{r},P}$ as the composition of the dual curve, the canonical projection and one map $\widetilde{\Psi}_P$. In §4 we show that the map $\widetilde{\Psi}_P$ introduced in §3 is C^∞ right-left equivalent to the blow up of $\mathbf{R}^n$ at the origin. Proof of theorem 1.1 is given in §5.

2 Several properties of the set $\{\mathbf{u}_0(s),\cdots,\mathbf{u}_n(s)\}$

Lemma 2.1 *For any $s\in I$ and any i,k $(-1\leq i<k\leq n-1)$ the following three hold.*

$$\begin{aligned}\mathbf{u}_i(s)\cdot\mathbf{u}_k(s) &= 0,\\ \mathbf{u}_i(s)\cdot\mathbf{u}_k'(s) &= 0 \quad (i<k-1),\\ \mathbf{u}_{k-1}(s)\cdot\mathbf{u}_k'(s) &= -\kappa_k(s).\end{aligned}$$

Proof of lemma 2.1 We show lemma 2.1 by induction on k.

First, by definitions it is trivial that $\mathbf{u}_{-1}(s)\cdot\mathbf{u}_0(s)=0$ and $\mathbf{u}_{-1}(s)\cdot\mathbf{u}_0'(s)=-\kappa_0(s)$.

Next, we assume that for any i,j $(-1\leq i<j<k\leq n-1)$, the following three hold.

$$\begin{aligned}\mathbf{u}_i(s)\cdot\mathbf{u}_j(s) &= 0,\\ \mathbf{u}_i(s)\cdot\mathbf{u}_j'(s) &= 0 \quad (i<j-1),\\ \mathbf{u}_{j-1}(s)\cdot\mathbf{u}_j'(s) &= -\kappa_j(s).\end{aligned}$$

[1] The last assertion on map-germs in $\mathcal{O}(1,n)$ is easily obtained by Gaffney's criterion on $\mathcal{L}$-equivalence (for Gaffney's criterion on $\mathcal{L}$-equivalence, see theorem 2.7 of [9]).

Under this assumption, we see that

$$\begin{aligned}\mathbf{u}_{k-2}(s)\cdot\mathbf{u}_k(s) &= \frac{1}{\kappa_k(s)}\mathbf{u}_{k-2}(s)\cdot(\mathbf{u}'_{k-1}(s)+\kappa_{k-1}(s)\mathbf{u}_{k-2}(s))\\ &= \frac{1}{\kappa_k(s)}(-\kappa_{k-1}(s)+\kappa_{k-1}(s))=0\end{aligned}$$

and for $i<k, i\neq k-2$ we see that

$$\begin{aligned}\mathbf{u}_i(s)\cdot\mathbf{u}_k(s) &= \frac{1}{\kappa_k(s)}\mathbf{u}_i(s)\cdot(\mathbf{u}'_{k-1}(s)+\kappa_{k-1}(s)\mathbf{u}_{k-2}(s))\\ &= \frac{1}{\kappa_k(s)}(0+0)=0.\end{aligned}$$

Next, under the same assumption we see that for any $i \quad (0\leq i<k-1)$

$$\begin{aligned}\mathbf{u}_i(s)\cdot\mathbf{u}'_k(s) &= -\mathbf{u}'_i(s)\cdot\mathbf{u}_k(s)\\ &= -(\kappa_{i+1}(s)\mathbf{u}_{i+1}(s)-\kappa_i(s)\mathbf{u}_{i-1}(s))\cdot\mathbf{u}_k(s)\\ &= -(0+0)=0\end{aligned}$$

and in the case that $i=k-1$ we see

$$\begin{aligned}\mathbf{u}_{k-1}(s)\cdot\mathbf{u}'_k(s) &= -\mathbf{u}'_{k-1}(s)\cdot\mathbf{u}_k(s)\\ &= -(\kappa_k(s)\mathbf{u}_k(s)-\kappa_{k-1}(s)\mathbf{u}_{k-2}(s))\cdot\mathbf{u}_k(s)\\ &= -(\kappa_k(s)+0)=-\kappa_k(s).\end{aligned}$$

Of course, $\mathbf{u}_{-1}(s)\cdot\mathbf{u}'_k(s)=0$ holds under no assumption.

Therefore, lemma 2.1 is proved by induction. □

Lemma 2.1 shows that $\{\mathbf{u}_0(s),\cdots,\mathbf{u}_n(s)\}$ is an orthogonal moving frame.

Lemma 2.2 *For any $s\in I$ the following two hold.*

1. $\mathbf{u}'_{n-1}(s)=-\kappa_{n-1}(s)\mathbf{u}_{n-2}(s)+\kappa_n(s)\mathbf{u}_n(s)$,
2. $\mathbf{u}'_n(s)=-\kappa_n(s)\mathbf{u}_{n-1}(s)$.

Proof of lemma 2.2 First we show 2 of lemma 2.2. By definition, for any $i \quad (i<n-1)$

$$\mathbf{u}'_i(s)\cdot\mathbf{u}_n(s)=(\kappa_{i+1}(s)\mathbf{u}_{i+1}(s)-\kappa_i(s)\mathbf{u}_{i-1}(s))\cdot\mathbf{u}_n(s)=0.$$

Thus, we have that $\mathbf{u}_i(s)\cdot\mathbf{u}'_n(s)=0$. Combining this result with $\mathbf{u}'_n(s)\cdot\mathbf{u}_n(s)=0$ implies that we may put $\mathbf{u}'_n(s)=\alpha(s)\mathbf{u}_{n-1}(s)$. Then,

$$\alpha(s)=\mathbf{u}_{n-1}(s)\cdot\mathbf{u}'_n(s)=-\mathbf{u}'_{n-1}(s)\cdot\mathbf{u}_n(s)=-\kappa_n(s).$$

Next, we show 1 of lemma 2.2. By similar arguments as in the proof of 2 of lemma 2.2 we may put $\mathbf{u}'_{n-1}(s)=\beta(s)\mathbf{u}_{n-2}(s)+\kappa_n(s)\mathbf{u}_n(s)$. Then, lemma 2.1 and 2 of lemma 2.2 show that $\beta(s)=-\kappa_{n-1}(s)$. □

By lemma 2.2, we see that the dual curve $\mathbf{u}_n(s)$ is non-singular if and only if $\kappa_n(s) \neq 0$ and we obtain the following Serret Frenet type formula.

$$\begin{pmatrix} \mathbf{u}_0'(s) \\ \mathbf{u}_1'(s) \\ \mathbf{u}_2'(s) \\ \vdots \\ \mathbf{u}_{n-2}'(s) \\ \mathbf{u}_{n-1}'(s) \\ \mathbf{u}_n'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1(s) & 0 & \cdots & 0 & 0 & 0 \\ -\kappa_1(s) & 0 & \kappa_2(s) & \ddots & 0 & 0 & 0 \\ 0 & -\kappa_2(s) & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & \kappa_{n-1}(s) & 0 \\ 0 & 0 & 0 & \ddots & -\kappa_{n-1}(s) & 0 & \kappa_n(s) \\ 0 & 0 & 0 & \ddots & 0 & -\kappa_n(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_0(s) \\ \mathbf{u}_1(s) \\ \mathbf{u}_2(s) \\ \vdots \\ \mathbf{u}_{n-2}(s) \\ \mathbf{u}_{n-1}(s) \\ \mathbf{u}_n(s) \end{pmatrix}$$

By using the Serret Frenet type formula again and again, we obtain the following lemma 2.3.

Lemma 2.3 *For any* $i \quad (0 \leq i \leq n-2)$*, we have the following.*

1. $\mathbf{u}_i(s) \cdot \frac{d^j \mathbf{u}_n}{ds^j}(s) = 0 \quad (1 \leq j \leq n-i-1)$,
2. $\mathbf{u}_i(s) \cdot \frac{d^{n-i} \mathbf{u}_n}{ds^{n-i}}(s) = (-1)^{n-i} \prod_{j=0}^{n-i-1} \kappa_{n-j}(s)$.

3 Explicit formula for the pedal curve relative to P

Let $\mathbf{r}$ be an n-dimensional spherical unit speed curve and let P be any point in $S^n - \{\pm \mathbf{u}_n(s) \mid s \in I\}$. By using the orthogonal frame $\{\mathbf{u}_0(s), \cdots, \mathbf{u}_n(s)\}$, we may decompose P as

$$P = \sum_{i=0}^{n} (P \cdot \mathbf{u}_i(s)) \mathbf{u}_i(s).$$

Lemma 3.1

$$ped_{\mathbf{r},P}(s) = \frac{1}{\sqrt{1 - (P \cdot \mathbf{u}_n(s))^2}} (P - (P \cdot \mathbf{u}_n(s)) \mathbf{u}_n(s)).$$

Proof of lemma 3.1 For any $s \in I$, by subtracting $(P \cdot \mathbf{u}_n(s))\mathbf{u}_n(s)$ from P we obtain the vector $P - (P \cdot \mathbf{u}_n(s))\mathbf{u}_n(s)$ in $\mathbf{R}^{n+1}$ which is positive scalar multiple of $ped_{\mathbf{r},P}(s)$. Normalizing this vector gives the right hand side of the formula in lemma 3.1, which must be the vector $ped_{\mathbf{r},P}(s)$. □

By this formula, we have the following.

Lemma 3.2

$$ped_{\mathbf{r},P}'(s) = 0 \quad \Longleftrightarrow \kappa_n(s) = 0 \ \textit{or} \ P \in S^{n-2}_{\mathbf{u}_{n-2}(s)}.$$

Proof of lemma 3.2 By differentiating $ped_{\mathbf{r},P}$ and using lemmata 2.2 and 3.1, we have the following.

$$\begin{aligned} & ped'_{\mathbf{r},P}(s) \\ = & -\kappa_n(s)\frac{(P\cdot \mathbf{u}_n(s))(P\cdot \mathbf{u}_{n-1}(s))}{(1-(P\cdot \mathbf{u}_n(s))^2)^{\frac{3}{2}}}\sum_{i=0}^{n-1}(P\cdot \mathbf{u}_i(s))\mathbf{u}_i(s) \\ & +\kappa_n(s)\frac{1}{(1-(P\cdot \mathbf{u}_n(s))^2)^{\frac{1}{2}}}\Big(\,(P\cdot \mathbf{u_n(s)})\mathbf{u}_{n-1}(s)+(P\cdot \mathbf{u}_{n-1}(s))\mathbf{u}_n(s)\Big).\end{aligned}$$

Since $\{\mathbf{u}_0(s),\cdots,\mathbf{u}_n(s)\}$ is an orthogonal frame, we see that $ped'_{\mathbf{r},P}(s)=0$ if and only if $\kappa_n(s)=0$ or $P\in S^{n-2}_{\mathbf{u}_{n-2}(s)}$. □

Let P be a point of $S^n-\{\pm\mathbf{u}_n(s)\mid s\in I\}$. We consider the following C^∞ map $\mathit{\Psi}_P : S^n-\{\pm P\}\to S^n$:

$$\mathit{\Psi}_P(\mathbf{x})=\frac{1}{\sqrt{1-(P\cdot\mathbf{x})^2}}(P-(P\cdot\mathbf{x})\mathbf{x}).$$

We see that the image $\mathit{\Psi}_P(S^n-\{\pm P\})$ is inside the open hemisphere centered at P. Let this open hemisphere, the set $\pi(S^n-\{\pm P\})$ be denoted by X_P, B_P respectively, where $\pi : S^n\to P^n(\mathbf{R})$ is the canonical projection. Note that X_P is C^∞ diffeomorphic to the n-dimensional open ball $\{(x_1,\cdots x_n)\mid \sum_{i=1}^n x_i^2<1\}$.

Since $\mathit{\Psi}_P(\mathbf{x})=\mathit{\Psi}_P(-\mathbf{x})$, $\mathit{\Psi}_P$ induces the map $\widetilde{\mathit{\Psi}}_P : B_P\to X_P$. Then, lemma 3.1 shows that $ped_{\mathbf{r},P}$ is factored into three maps in the following way.

$$ped_{\mathbf{r},P}(s)=\widetilde{\mathit{\Psi}}_P\circ\pi\circ\mathbf{u}_n(s).$$

4 Map of blow up type

Let $p : B\to\mathbf{R}^n$ be the blow up of $\mathbf{R}^n$ centered at the origin.

Lemma 4.1 *Let P be a point of $S^n-\{\pm\mathbf{u}_n(s)\}$. Then, there exist C^∞ diffeomorphisms $h_s : B_P\to B$ and $h_t : X_P\to\mathbf{R}^n$ such that the equality $h_t\circ\widetilde{\mathit{\Psi}}_P\equiv p\circ h_s$ is satisfied.*

By lemma 4.1, it is reasonable to call $\widetilde{\mathit{\Psi}}_P$ a *map of blow up type.*

Proof of lemma 4.1 By a suitable rotation of S^n if necessary, we may assume that $P=(0,\cdots,1)$. For any i $(1\le i\le n)$ and any $(x_1,\cdots,x_{n+1})\in S^n-\{\pm P\}$ with $x_i\neq 0$ we put

$$\varphi_{P,i}(\pi(x_1,\cdots,x_{n+1}))=(\frac{x_1}{x_i},\cdots,\frac{x_{i-1}}{x_i},-\tan(\lambda)x_i,\frac{x_{i+1}}{x_i},\cdots,\frac{x_n}{x_i}),$$

where $\lambda=\sin^{-1}(x_{n+1})$ $(-\frac{\pi}{2}<\lambda<\frac{\pi}{2})$. Then, we see easily that for any i,j $(1\le i,j\le n)$ the following equality holds

$$\varphi_{P,j}\circ\varphi_{P,i}^{-1}\equiv\varphi_j\circ\varphi_i^{-1},$$

where $\{(U_1,\varphi_1)\cdots,(U_n,\varphi_n)\}$ is the standard atlas for the blowing up $p : B\to\mathbf{R}^n$. Thus, the set

$$\{(U_{P,1},\varphi_{P,1}),\cdots,(U_{P,n},\varphi_{P,n})\}$$

can be an atlas for $\pi(S^n - \{\pm P\})$, where $U_{P,i} = \{\pi(x_1, \cdots, x_{n+1}) \,|\, x_i \neq 0\}$.

Next, we express our map $\widetilde{\Psi}_P$ by using euclidean coordinates $(u_1, \cdots, u_n)$. Since we have assumed $P = (0, \cdots, 0, 1)$, for $\mathbf{x} = (x_1, \cdots, x_n, \sin(\lambda))$ we have

$$\frac{1}{\sqrt{1-(P\cdot\mathbf{x})^2}}(P - (P\cdot\mathbf{x})\mathbf{x}) = (-\tan(\lambda)x_1, \cdots, -\tan(\lambda)x_n, \cos(\lambda))$$

and therefore for any $i \quad (1 \leq i \leq n)$ we have

$$q \circ \widetilde{\Psi}_P \circ \varphi_{P,i}^{-1}(u_1, \cdots, u_n) = (u_1u_i, \cdots, u_{i-1}u_i, u_i, u_{i+1}u_i, \cdots, u_nu_i),$$

where $q : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ is the canonical projection.

Since this expression is completely the same as that of the blow up by using the standard coordinate system $(U_i, \varphi_i) \quad (1 \leq i \leq n)$ and the restriction $q|_{X_P} : X_P \to q(X_p)$ is a C^∞ diffeomorphism, we see that lemma 4.1 is proved for $\widetilde{\Psi}_P|_{U_{P,i}}$ and $p|_{U_i}$. Thus, in order to finish the proof of lemma 4.1 it suffices to show that for any $i, j \quad (1 \leq i, j \leq n)$ the equality

$$\varphi_i^{-1} \circ \varphi_{P,i}(\pi(x_1, \cdots, x_{n+1})) = \varphi_j^{-1} \circ \varphi_{P,j}(\pi(x_1, \cdots, x_{n+1}))$$

holds for $\pi(x_1, \cdots, x_{n+1}) \in U_{P,i} \cap U_{P,j}$. This holds since we have already checked that the patching relations for our $\{(U_{P,i}, \varphi_{P.i})\}_{1\leq i\leq n}$ are completely the same as for the standard atlas of B. $\square$

5 Proof of theorem 1.1

Since $\{S^n_{\mathbf{u}_{n(s_0)}} - S^{n-2}_{\mathbf{u}_{n-2}(s_0)}, S^{n-2}_{\mathbf{u}_{n-2}(s_0)} - S^{n-3}_{\mathbf{u}_{n-3}(s_0)}, \cdots, S^0_{\mathbf{u}_0(s_0)} - S^{-1}_{\mathbf{u}_{-1}(s_0)}\}$ gives a stratification of $S^n - \{\pm\mathbf{u}_n(s_0)\}$, "if parts" of 1, 2 of theorem 1.1 follows from "only if parts" of 1, 2 of theorem 1.1. Thus, we show only "only if parts" in the following.

[*Proof of "only if part" of 1*] By lemma 3.2, $ped'_{\mathbf{r},P}(s_0) \neq 0$ in this case. Thus, the map-germ $ped_{\mathbf{r},P}(s_0)$ is non-singular. $\square$

[*Proof of "only if part" of 2*] By a suitable rotation of S^n if necessary, we may assume that $P = (0, \cdots, 0, 1) \in \mathbf{R}^{n+1}$. Then, since $P \in S^i_{\mathbf{u}_i(s_0)} - S^{i-1}_{\mathbf{u}_{i-1}(s_0)}$, by a furthermore suitable rotation of S^n if necessary we may assume that $\mathbf{u}_n(s_0) = (1, 0, \cdots, 0)$, $\mathbf{u}_{n-1}(s_0) = (0, 1, 0, \cdots, 0)$, $\cdots$, $\mathbf{u}_{i+1}(s_0) = (\underbrace{0, \cdots, 0}_{(n-i-1) \text{ elements}}, 1, \underbrace{0, \cdots, 0}_{(i+1) \text{ elements}})$; and $\mathbf{u}_j(s_0) \quad (0 \leq j \leq i)$ have the following form

$$\mathbf{u}_j(s_0) = (\underbrace{0, \cdots, 0}_{(n-i) \text{ elements}}, \underbrace{a_{(n-i)j}, \cdots, a_{nj}}_{(i+1) \text{ elements}}),$$

where $a_{ni} \neq 0$.

By lemma 2.3, we see that the following three hold for component function-germs $u_{0n}, \cdots, u_{nn}$ of the map-germ $\mathbf{u}_n = (u_{0n}, \cdots, u_{nn}) : (I, s_0) \to S^n$.

1. For any $j \quad (0 \leq j \leq n-i-1)$, the lowest degree of non-zero terms of u_{jn} is j.
2. For any $j \quad (n-i \leq j \leq n-1)$, the lowest degree of non-zero terms of u_{jn} is more than or equal to $n-i$.

3. The lowest degree of non-zero terms of u_{nn} is $n-i$.

Therefore, by lemma 4.1 we see that the following two hold for component function-germs $\psi_1, \cdots, \psi_n$ of the map-germ $(q \circ \widetilde{\Psi}_P \circ \varphi_{P,1}^{-1}) \circ (\varphi_{P,1} \circ \pi \circ \mathbf{u}_n) : (I, s_0) \to \mathbf{R}^n$.

1. For any $j \quad (1 \leq j \leq n-i)$, the lowest degree of non-zero terms of ψ_j is $n-i+j-1$,
2. For any $j \quad (n-i+1 \leq j \leq n)$, the lowest degree of non-zero terms of ψ_j is $2n-2i$.

Let $\mathcal{E}_1$ be the set of all C^∞ function germs with one variable $(\mathbf{R}, 0) \to \mathbf{R}$, m_1 be its subset consisting of all function-germs with zero constant terms. Then, $m_1^{n-i}\mathcal{E}_1$ is a finitely generated $\mathcal{E}_1$-module. We put $f(t) = t^{n-i}$ and apply the Malgrange preparation theorem (for instance, see [3], [6], [9]) to $m_1^{n-i}\mathcal{E}_1$ and f. Then we see that for any function-germ $g \in m_1^{n-i}\mathcal{E}_1$ there exists a certain C^∞ function-germ ψ such that

$$g(t) = \psi(t^{n-i}, \cdots, t^{2n-2i-1}).$$

Thus, for our map-germ $ped_{\mathbf{r},P} : (I, s_0) \to (S^n, ped_{\mathbf{r},P}(s_0))$ there exists a germ of C^∞ diffeomorphism $h_t : (S^n, ped_{\mathbf{r},P}(s_0)) \to (\mathbf{R}^n, 0)$ such that

$$h_t \circ ped_{\mathbf{r},P}(s) = (\underbrace{(s-s_0)^{n-i}, \cdots, (s-s_0)^{2n-2i-1}}_{(n-i) \text{ elements}}, \underbrace{0, \cdots, 0}_{i \text{ elements}}).$$

$\square$

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