Solution of Stochastic Eigenvalue Problem by Improved Stochastic Inverse Power Method (I-SIPM)

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Abstract Eigenvalue analysis is an important problem in a variety of fields. In structural mechanics in the field of naval architecture and ocean engineering, eigenvalue problems commonly appear in the context of, e.g. vibrations and buckling. In eigenvalue analysis, the physical characteristics are often considered as deterministic, such as mass, geometries, stiffness in the structures. However, in many practical cases, they are not deterministic. Such uncertainties may cause serious problems because the influence of the uncertainties is in general unknown. In order to solve the stochastic eigenvalue problem, in this article, we have proposed two methods. Firstly, the improved stochastic inverse power method (I-SIPM) based on response surface methodology is proposed. The method is different with previous stochastic inverse power method. The minimum eigenvalue and eigenvector of stochastic eigenvalue problems can be evaluated by using the proposed method. Secondly, the stochastic Wielandt deflation method (SWDM) is proposed which can realize to evaluate $i^{\text{th}}(i \geq 1)$ eigenvalues and eigenvectors of stochastic eigenvalue problems. This is very important for solving natural mode and buckling mode analysis problem. Next, two example problems are investigated to show the validity of two new methods compared with a Monte-Carlo simulation, i.e. the vibration problem of a discrete 2-DOF system and the buckling problem of a continuous beam. Finally, the uncertainty estimation for the dynamic damper problem is discussed by using proposed method. The probability of the natural frequency falling into the

range to be avoided is shown when the dynamic damper has a stochastic mass and stiffness.

Keywords Improved stochastic inverse power method (I-SIPM) · Stochastic eigenvalue problem · Polynomial chaos expansion (PCE) · Stochastic Wielandt deflation method (SWDM)

1 Introduction

Eigenvalue analysis is an important problem in a variety of fields. In structural mechanics in the field of naval architecture and ocean engineering, eigenvalue problems commonly appear in the context of, e.g. vibrations and buckling. Currently, the computation of eigenvalues and eigenvectors is well understood for deterministic problems. However, in many practical cases, physical characteristics are not deterministic. There are a lot of uncertainties in the mass, geometries, stiffness and other characteristics in the structures. Such uncertainties may cause serious problems such as resonance occurrence and reduction of buckling load because the influence of the uncertainties is in general unknown. For this reason, eigenvalue analysis when uncertainty exists in the structure is very important for safety assessment of ships and offshore structures.

When the input parameters of a physical problem are expressed by random variables, the derived output will be also probabilistic/random. The methods to compute these random results are in general referred to as stochastic analysis, which focuses on the computation of the statistical characterization (mean, standard deviation, etc.) of the random output. As an important stochastic analysis method, the stochastic finite element method (SFEM) has been recently developed. The efficiency of this method is proved by many studies in engineering problems and summarised in a general review with regards to its development in the past, present and future of SFEM [1] and practical application of SFEM [2]. Nakagiri et al. [3] reported works on uncertainty in shape, material uncertainty and boundary uncertainty, in which the mean value and the variance of stress are obtained by SFEM based on perturbation method. And SFEM based on perturbation is also summarised by Kleiber et al. [4]. Within last two decades, an attractive alternative to the prevalent

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probabilistic techniques was the development of Stochastic Response Surface Methods (SRS) by Ghanem and Spanos [5], where the response is represented by special polynomials (Polynomial Chaos Expansions (PCE)) of the input random variables. It is called as spectral stochastic finite element systems (SSFEM). The subsequent developments are described in a number of key articles [6-9]. However they are mainly concerned with only the random material properties of a structure. Recently, Honda [10] proposed stochastic boundary element method by using the PCE and a Karhunen-Loeve expansion method in which uncertainty of shape of boundary is assumed. Also, in recent years, the authors proposed SFEM based on response surface methodology considering uncertainty in shape of structures, where the validity and feasibility of the proposed method is demonstrated [11-12]. Thus SFEM are well developed for linear structured systems, but are less developed for the stochastic eigenvalue problem. The mathematical base of the iterative inverse power method as an algorithm for solution of the stochastic eigenvalue problem has been discussed in Verhoosel et al. [13]. Sepahvand et al. also discussed the efficiency of the developed method by applying it to two examples [14]. However, the authors found that the method sometimes cannot give satisfactory results. Based on these studies, we improved stochastic inverse power method based on response surface methodology to evaluate uncertainty of minimum eigenvalue and eigenvector, called the improved stochastic inverse power method (I-SIPM). And in order to solve mode problem, we also proposed the stochastic Wielandt deflation method (SWDM) to evaluate $i^{\text{th}} (i > 1)$ eigenvalues and eigenvectors of stochastic eigenvalue problems. The validity of the proposed methods is discussed by comparing the results of the method with the result of Monte-Carlo solution (MCS) solving the deterministic problems. In this paper, the response surface methodology is introduced in section 2. In section 3, statement of the stochastic eigenvalue problem is deduced. I-SIPM is formulated in section 4. SWDM is formulated in section 5. In section 6, three numerical examples are discussed to show the validity of the proposed methods.

2 Response surface methodology

2.1 Concepts of stochastic eigenvalue problem

Fig.1 depicts the general idea of stochastic eigenvalue analysis. The inherent randomness (in the mass (m), stiffness (k), etc.) is treated as an input parameter. The input parameter is defined by probability density function of the random variables, for example k or m . The 'Analysis method (I-SIPM/SWDM)' is formulated by an approximate expression of response surface for eigenvalues (λ) and eigenvectors ($\mathbf{u}$). As result of stochastic eigenvalue analysis, the response ($\lambda, \mathbf{u}$) is obtained as a function of the input random variables. Then, the probability distribution of natural frequency or buckling load is described by using obtained response surface for eigenvalue. And the probability distribution of natural mode or buckling mode can be also described. The stochastic eigenvalue analysis is important to decide the criteria of design parameters, where a probabilistic nature of the parameter should be accounted for.

2.2 The representation of the inherent randomness

In this study, the problem with inherent (input) uncertainty in the mass or stiffness is considered. For example, the uncertainty of stiffness (k_i) in the mass-spring system problem (Fig.3) is assumed as normal random parameter. The parameter k_i is represented by using a standard normal random variable θ_j as follows:

$$k_i(\theta_j) = k_{0i} + k_{1i}\theta_j, \quad (1)$$

where k_{0i} is mean value of the parameter (k_i), and k_{1i} is standard deviation of the parameter, and i is i^{th} spring stiffness, j is j^{th} random variable.

Fig. 1 Conceptual diagram of stochastic eigenvalue problems

2.3 Response surface by polynomial chaos expansion (PCE)

In this study, the stochastic response is approximated as a special polynomial function (Hermite PCE). Thus, a general random response surface $X(\boldsymbol{\theta})$, viewed as a function of the input random variables can be represented by the following equation [15].

$$X(\boldsymbol{\theta}) = \sum_{i=0}^p x_i \Psi_i(\boldsymbol{\theta}), \quad (2)$$

where $\boldsymbol{\theta} = \{\theta_1, \dots, \theta_n\}$. $X(\boldsymbol{\theta})$ is a general form for random response surface, which can represent eigenvalues, $\lambda(\boldsymbol{\theta})$, or eigenvectors, $\mathbf{u}(\boldsymbol{\theta})$. And $\Psi_i(\boldsymbol{\theta})$ denotes the i^{th} basis function represented by Hermite polynomials of the variables ($\boldsymbol{\theta}$). And, x_i are the unknown coefficients that need to be determined to define the response surface. $p+1$ is the number of expansion terms. The orthogonal properties of the Hermite polynomials with respect to a weight function $W(\boldsymbol{\theta})$ as shown below are also important to formulate the stochastic eigenvalue analysis problem.

$$\int_D \Psi_i(\boldsymbol{\theta}) \Psi_j(\boldsymbol{\theta}) W(\boldsymbol{\theta}) d\boldsymbol{\theta} = \langle \Psi_i^2(\boldsymbol{\theta}) \rangle \delta_{ij}, \quad (3)$$

where D is the domain of standard normal random variable ($-\infty < \theta_i < \infty$), δ_{ij} represents the Kronecker-Delta property. When Hermite polynomials are used as basis functions, the weight function $W(\boldsymbol{\theta})$ is the probability density function of the standard normal distribution [15] as follows:

$$W(\boldsymbol{\theta}) = \frac{1}{\sqrt{(2\pi)^n}} e^{-\frac{1}{2}\boldsymbol{\theta}^T \boldsymbol{\theta}}. \quad (4)$$

As an example, orthogonal basis functions and the values of $\langle \Psi_i^2(\boldsymbol{\theta}) \rangle$ for 1-D and 2-D by Hermite polynomials are shown in Table 1 and Table 2 respectively.

These basis functions also satisfy the following equation:

$$\langle \Psi_m(\boldsymbol{\theta}) \rangle = \int_D \Psi_m(\boldsymbol{\theta}) W(\boldsymbol{\theta}) d\boldsymbol{\theta} = 0 \quad \text{for } m = 1, 2, \dots, p. \quad (5)$$

The mean and standard deviation of the stochastic response (Eq. (2)) can be simply calculated as follows:

$$\mu_X = E[X] = \langle X \rangle = x_0, \quad (6)$$

$$\sigma_X = \sqrt{\text{Var}[X]} = \sqrt{\langle (X - E[X])^2 \rangle} = \sqrt{\sum_{i=1}^p x_i^2 \langle \Psi_i(\boldsymbol{\theta})^2 \rangle}. \quad (7)$$

Once the response, $X(\boldsymbol{\theta})$, is determined, we can approximate the mean and standard deviation of the response by using Eq. (6) and Eq. (7). Moreover, we can obtain the probability density function of the response by using large number of realizations of standard normal random variables ($\boldsymbol{\theta}$) and plugging them into Eq. (2).

Table 1 One-Dimensional Hermite polynomial chaoses
(Basis functions)

i	Order of the Basis function	i^{th} Basis function Ψ_i	$\langle \Psi_i^2 \rangle$
0	0	1	1
1	1	θ	1
2	2	$\theta^2 - 1$	2
3	3	$\theta^3 - 3\theta$	6
4	4	$\theta^4 - 6\theta^2 + 3$	24

Table 2 Two-Dimensional Hermite polynomial chaoses
(Basis functions).

i	Order of the Basis function	i^{th} Basis function Ψ_i	$\langle \Psi_i^2 \rangle$
0	0	1	1
1	1	θ_1	1
2		θ_2	1
3	2	$\theta_1 \theta_2$	1
4		$\theta_1^2 - 1$	2
5		$\theta_2^2 - 1$	2
6	3	$\theta_1^2 \theta_2 - \theta_2$	2
7		$\theta_1 \theta_2^2 - \theta_1$	2
8		$\theta_1^3 - 3\theta_1$	6
9		$\theta_2^3 - 3\theta_2$	6

3 Statement of the stochastic eigenvalue problem

The generalized eigenvalue problem can be written as follows:

$$\mathbf{K}\mathbf{u} = \lambda \mathbf{R}\mathbf{u}, \quad (8)$$

where $\mathbf{K}$ is a stiffness matrix with dimension N . $\mathbf{R}$ is mass matrix in vibrations analysis or it is initial stress matrix in buckling analysis with dimension N . The generalized eigenvalue problem may be converted to a standard eigenvalue problem, and

shown as follows:

$$\mathbf{A}\mathbf{u} = \lambda \mathbf{u}, \quad (9)$$

where $\mathbf{A}$ is $\mathbf{R}^{-1}\mathbf{K}$. Note that $\mathbf{R}$ is regular (nonsingular) matrix in general vibration and buckling problems. The objective of solving the deterministic eigenvalue problem is to find a scalar eigenvalue (λ) and the corresponding eigenvector ($\mathbf{u}$) by using the deterministic eigenvalue equation (Eq. (9)).

For systems with uncertain parameters, the equation switches to the stochastic standard eigenvalue problem given as follows:

$$\mathbf{A}'(\boldsymbol{\theta})\mathbf{u}'(\boldsymbol{\theta}) = \lambda'(\boldsymbol{\theta})\mathbf{u}'(\boldsymbol{\theta}), \quad (10)$$

where ' is used to indicate involving random variables ($\boldsymbol{\theta}$). $\mathbf{A}'(\boldsymbol{\theta}) = \mathbf{R}'^{-1}(\boldsymbol{\theta})\mathbf{K}'(\boldsymbol{\theta})$, $\mathbf{K}'(\boldsymbol{\theta})$ is a stiffness matrix involving random variables ($\boldsymbol{\theta}$), and $\mathbf{R}'(\boldsymbol{\theta})$ is mass matrix involving random variables ($\boldsymbol{\theta}$) in vibrations analysis or it is initial stress matrix involving random variables ($\boldsymbol{\theta}$) in buckling analysis. Thus the stiffness matrix $\mathbf{K}'(\boldsymbol{\theta})$ and matrix $\mathbf{R}'(\boldsymbol{\theta})$ are considered as stochastic matrix. The stochastic stiffness matrix $\mathbf{K}'(\boldsymbol{\theta})$ can be represented by using limited PCE with $n_1 + 1$ terms of basis functions.

$$\mathbf{K}'(\boldsymbol{\theta}) = \sum_{i=0}^{n_1} \mathbf{K}'_i \Psi_i(\boldsymbol{\theta}). \quad (11)$$

When the uncertain physical characteristics are considered in mass or initial stress matrix, the stochastic matrix $\mathbf{R}'(\boldsymbol{\theta})$ also can be represented by using PCE.

$$\mathbf{R}'(\boldsymbol{\theta}) = \sum_{i=0}^{n_2} \mathbf{R}'_i \Psi_i(\boldsymbol{\theta}). \quad (12)$$

When the parameter $\mathbf{K}'(\boldsymbol{\theta})$ ($\mathbf{R}'(\boldsymbol{\theta})$) is following normal distribution, the n_1 (n_2) is 1. When the parameter $\mathbf{K}'(\boldsymbol{\theta})$ ($\mathbf{R}'(\boldsymbol{\theta})$) is following non-normal distribution, the n_1 (n_2) can be derived by the approximation of the random variable using the method in [12,15]. Thus, $\mathbf{A}'(\boldsymbol{\theta})$ can be represented as follows. See Appendix 1 and Appendix 2 for derivation of the coefficients ($\mathbf{A}'_i$).

$$\mathbf{A}'(\boldsymbol{\theta}) = \sum_{i=0}^{n_3} \mathbf{A}'_i \Psi_i(\boldsymbol{\theta}), \quad (13)$$

where n_3 is the order of expansion terms by the sum of the approximate order of $\mathbf{R}'^{-1}(\boldsymbol{\theta})$ and n_1 . The coefficient matrix $\mathbf{A}'(\boldsymbol{\theta})$ is considered as stochastic by Eq.(13), thus the eigenvalues λ and eigenvectors $\mathbf{u}$ become functions of the random variables such as $\lambda'(\boldsymbol{\theta})$ and $\mathbf{u}'(\boldsymbol{\theta})$. Accordingly, the stochastic eigenvalues response and eigenvectors response can be approximated by using PCE.

$$\lambda'(\boldsymbol{\theta}) = \sum_{i=0}^n \lambda'_i \Psi_i(\boldsymbol{\theta}), \quad (14)$$

$$\mathbf{u}'(\boldsymbol{\theta}) = \sum_{i=0}^n \mathbf{u}'_i \Psi_i(\boldsymbol{\theta}). \quad (15)$$

The purpose of the stochastic eigenvalue problem is to derive unknown PCE coefficients ($\lambda'_i, \mathbf{u}'_i$) of eigenvalues and eigenvectors. Substituting Eq. (13), Eq. (14) and Eq. (15) into Eq. (10) the stochastic generalized eigenvalue equation becomes as follows:

$$\sum_{i=0}^{n_3} \mathbf{A}'_i \Psi_i(\boldsymbol{\theta}) \left(\sum_{j=0}^n \mathbf{u}'_j \Psi_j(\boldsymbol{\theta}) \right) = \sum_{i=0}^n \lambda'_i \Psi_i(\boldsymbol{\theta}) \left(\sum_{j=0}^n \mathbf{u}'_j \Psi_j(\boldsymbol{\theta}) \right), \quad (16)$$

where the coefficient matrix $\mathbf{A}'(\boldsymbol{\theta})$ is approximated by n_3^{th} order polynomial as shown in Eq. (16). Thus, in this study, the order (n) of PCE approximation of Eq. (14)(15) is assumed as n_3 .

Multiplying both sides of Eq. (16) by $\Psi_i(\boldsymbol{\theta})W(\boldsymbol{\theta})$, integrating the equation over the stochastic domain ($\int_D (\text{Eq.}(16))\Psi_i(\boldsymbol{\theta})W(\boldsymbol{\theta})d\boldsymbol{\theta} = \langle (\text{Eq.}(16))\Psi_i(\boldsymbol{\theta}) \rangle$), we have:

$$\begin{aligned} & \sum_{i=0}^{n_3} \sum_{j=0}^{n_3} \mathbf{A}'_i \mathbf{u}'_j \langle \Psi_i(\boldsymbol{\theta}) \Psi_j(\boldsymbol{\theta}) \Psi_i(\boldsymbol{\theta}) \rangle \\ &= \sum_{i=0}^{n_3} \sum_{j=0}^{n_3} \lambda'_i \mathbf{u}'_j \langle \Psi_i(\boldsymbol{\theta}) \Psi_j(\boldsymbol{\theta}) \Psi_i(\boldsymbol{\theta}) \rangle. \end{aligned} \quad (17)$$

Thus the stochastic generalized eigenvalue equation when the $t=0, \dots, n_3$ can be derived as follows:

$$\begin{bmatrix} \mathbf{A}'_{00} & \cdots & \mathbf{A}'_{0n_3} \\ \vdots & \ddots & \vdots \\ \mathbf{A}'_{n_3 0} & \cdots & \mathbf{A}'_{n_3 n_3} \end{bmatrix} \begin{bmatrix} \mathbf{u}'_0 \\ \vdots \\ \mathbf{u}'_{n_3} \end{bmatrix} = \sum_{i=0}^{n_3} \lambda'_i \begin{bmatrix} c_{i00} \mathbf{I} & \cdots & c_{i0n_3} \mathbf{I} \\ \vdots & \ddots & \vdots \\ c_{in_3 0} \mathbf{I} & \cdots & c_{in_3 n_3} \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{u}'_0 \\ \vdots \\ \mathbf{u}'_{n_3} \end{bmatrix}, \quad (18)$$

where, $\mathbf{A}'_i = \sum_{j=0}^{n_3} \mathbf{A}'_j \langle \Psi_i(\boldsymbol{\theta}) \Psi_j(\boldsymbol{\theta}) \Psi_i(\boldsymbol{\theta}) \rangle$, $c_{ijl} = \langle \Psi_i(\boldsymbol{\theta}) \Psi_j(\boldsymbol{\theta}) \Psi_l(\boldsymbol{\theta}) \rangle$. By Eq.(18), we can find that it is difficult to be solved by the conventional method of eigenvalue analysis.

In order to solve the stochastic eigenvalue problem, the Monte Carlo simulation (MCS) is usually used. The stochastic results are obtained by the solution of Eq.(8) again and again for many times. However, MCS needs heavy and a large number of calculations, and the application of MCS to practical problems is sometimes very difficult to get reasonable results. So, it is desirable to solve stochastic eigenvalues (λ'_i) and eigenvectors ($\mathbf{u}'_i$) directly from Eq.(17) or (18). However, at present, there is no algorithm available to solve Eq.(17) or (18) directly. As a numerical algorithm to solve deterministic eigenvalue problems, the inverse power method is developed to compute minimum eigenvalue and eigenvector [16]. The algorithm is early used for evaluating stochastic eigenvalue problems in Verhoosel et al. [13] by which satisfactory results sometimes cannot be obtained in the authors' experience.

In the following section, we improve the algorithm and develop the improved stochastic inverse power method formularization.

4 The formularization of I-SIPM for solution of the stochastic eigenvalue problem

In this section, the improved stochastic inverse power method is proposed. The method is a solution to evaluate the minimum eigenvalue and eigenvector of stochastic eigenvalue problems.

In order to understand the improved stochastic inverse power method easily, firstly, the deterministic inverse power method will be considered.

4.1 The deterministic inverse power method

As shown below, we show the algorithm for the deterministic inverse power method [16]. An approximate deterministic eigenvalue and the corresponding eigenvector are evaluated by using the method to solve $\mathbf{A}\mathbf{u} = \lambda\mathbf{u}$ equation.

In this algorithm q is iterations, $\mathbf{u}^{(0)}$ is initial normalized eigenvector, $\mathbf{A}$ is a matrix by Eq.(9). An eigenvalue $\lambda^{(1)}$ can be found when an initial eigenvector is known by *step1*. Next, an

updated eigenvector $\mathbf{u}^{(1)}$ is found when initial eigenvector $\mathbf{u}^{(0)}$ and eigenvalue $\lambda^{(1)}$ is known in *step2*. We need to normalize the updated eigenvector $\mathbf{u}^{(1)}$ by *step3* so that the length of eigenvector is 1. Finally, the convergence of the iterative method is checked by using *step4*. When the condition $\varepsilon \leq \varepsilon^*$ is satisfied, the iteration will be terminated, and we can obtain an approximate eigenvalue and the corresponding eigenvector.

Algorithm 1 The deterministic inverse power method

Initialize: $\mathbf{u}^{(0)}$

While $\varepsilon > \varepsilon^*$

Step 1: $\lambda^{(q+1)} = (\mathbf{u}^{(q)})^T \mathbf{A} \mathbf{u}^{(q)}$

Step 2: $\mathbf{A} \mathbf{u}^{(q+1)} = \lambda^{(q+1)} \mathbf{u}^{(q)}$

Step 3: $\mathbf{u}^{(q+1)} \rightarrow \frac{\mathbf{u}^{(q+1)}}{\|\mathbf{u}^{(q+1)}\|_{L^2}}$

Step 4: $\varepsilon = \|\mathbf{A} - \lambda^{(q+1)} \mathbf{I}\| \mathbf{u}^{(q+1)}\|_{L^2}$

End While

Result: $\lambda^*, \mathbf{u}^*$

4.2 The improved stochastic inverse power method

In study by Verhoosel et al. [13], the stochastic inverse power method (SIPM) algorithm is proposed. However the correct eigenvalue and eigenvector cannot be obtained when we use the algorithm to solve a simple stochastic eigenvalue problem. Here, we improved *step3* of the method in reference 13, and the improved stochastic inverse power method (I-SIPM) is proposed to obtain a correct eigenvalue and eigenvector. In this section, I-SIPM algorithm is elaborated and the formularization is provided.

(i). The formularization for Step1

As shown in Algorithm, the Rayleigh quotient is used to update the stochastic eigenvalue based on the normalized stochastic eigenvector $\mathbf{u}^{(q)}$ of the previous iteration.

$$\lambda^{(q+1)} = (\mathbf{u}^{(q)})^T \mathbf{A}' \mathbf{u}^{(q)}. \quad (19)$$

Substituting Eq. (14), Eq. (15) into Eq. (19) the equation becomes as follows:

$$\begin{aligned} & \sum_{i=0}^n \lambda'_i \Psi_i(\boldsymbol{\theta}) \\ &= \left(\sum_{i=0}^n \mathbf{u}_i^{(q)} \Psi_i(\boldsymbol{\theta}) \right)^T \left(\sum_{i=0}^n \mathbf{A}'_i \Psi_i(\boldsymbol{\theta}) \right) \left(\sum_{i=0}^n \mathbf{u}_i^{(q)} \Psi_i(\boldsymbol{\theta}) \right). \end{aligned} \quad (20)$$

Multiplying both sides of Eq. (20) by $\Psi_i(\boldsymbol{\theta})W(\boldsymbol{\theta})$, integrating the equation, we have:

$$\begin{aligned} & \sum_{i=0}^n \lambda'_i \langle \Psi_i(\boldsymbol{\theta}) \Psi_i(\boldsymbol{\theta}) \rangle \\ &= \sum_{i=0}^n \sum_{j=0}^n \sum_{k=0}^n \mathbf{u}_i^{T(q)} \mathbf{A}'_k \mathbf{u}_j^{(q)} \langle \Psi_i(\boldsymbol{\theta}) \Psi_j(\boldsymbol{\theta}) \Psi_k(\boldsymbol{\theta}) \Psi_i(\boldsymbol{\theta}) \rangle. \end{aligned} \quad (21)$$

Then the update equation of the eigenvalue can be shown as follows:

$$\lambda'_i{}^{(q+1)}$$

$$= \frac{1}{\langle \Psi_i^2(\boldsymbol{\theta}) \rangle} \sum_{i=0}^n \sum_{j=0}^n \sum_{k=0}^n \mathbf{u}_i^{r(q)} \mathbf{A}_k^{r(q+1)} \mathbf{u}_j^{r(q)} \langle \Psi_i(\boldsymbol{\theta}) \Psi_j(\boldsymbol{\theta}) \Psi_k(\boldsymbol{\theta}) \Psi_i(\boldsymbol{\theta}) \rangle. \quad (22)$$

The deterministic part ($\lambda_0^{(q+1)}$) of the eigenvalue is calculated when $t=0$, and the stochastic parts ($\lambda_i^{r(q+1)}$) of the eigenvalue can be calculated when $t=1, \dots, n$.

(ii). The formularization for Step2

The stochastic eigenvector is updated by solving the below equation.

$$\mathbf{A}' \mathbf{u}^{r(q+1)} = \lambda^{r(q+1)} \mathbf{u}^{r(q)}. \quad (23)$$

Substituting Eq. (14), Eq. (15) into Eq. (23) the equation becomes as follows:

$$\begin{aligned} & \left(\sum_{i=0}^n \mathbf{A}' \Psi_i(\boldsymbol{\theta}) \right) \left(\sum_{i=0}^n \mathbf{u}_i^{r(q+1)} \Psi_i(\boldsymbol{\theta}) \right) \\ &= \left(\sum_{i=0}^n \lambda_i^{r(q+1)} \Psi_i(\boldsymbol{\theta}) \right) \left(\sum_{i=0}^n \mathbf{u}_i^{r(q)} \Psi_i(\boldsymbol{\theta}) \right). \end{aligned} \quad (24)$$

Multiplying both sides of Eq. (24) by $\Psi_i(\boldsymbol{\theta}) W(\boldsymbol{\theta})$, integrating the equation, we can obtain the following equation.

$$\begin{aligned} & \sum_{i=0}^n \sum_{j=0}^n \mathbf{A}' \mathbf{u}_j^{r(q+1)} \langle \Psi_i(\boldsymbol{\theta}) \Psi_j(\boldsymbol{\theta}) \Psi_i(\boldsymbol{\theta}) \rangle \\ &= \sum_{i=0}^n \sum_{j=0}^n \lambda_i^{r(q+1)} \mathbf{u}_i^{r(q)} \langle \Psi_i(\boldsymbol{\theta}) \Psi_j(\boldsymbol{\theta}) \Psi_i(\boldsymbol{\theta}) \rangle. \end{aligned} \quad (25)$$

Thus when the $t=0, \dots, n$, the update stochastic eigenvector equation can be derived as follows:

$$\begin{bmatrix} \Omega_{00} & \dots & \Omega_{0n} \\ \vdots & \ddots & \vdots \\ \Omega_{n0} & \dots & \Omega_{nn} \end{bmatrix} \begin{bmatrix} \mathbf{u}_0^{r(q+1)} \\ \vdots \\ \mathbf{u}_n^{r(q+1)} \end{bmatrix} = \begin{bmatrix} \zeta_0 \\ \vdots \\ \zeta_n \end{bmatrix}, \quad (26)$$

where Ω_{ii} and ζ_i can be shown as follow:

$$\Omega_{ii} = \sum_{j=0}^n \mathbf{A}'_j \langle \Psi_i(\boldsymbol{\theta}) \Psi_j(\boldsymbol{\theta}) \Psi_i(\boldsymbol{\theta}) \rangle, \quad (27)$$

$$\zeta_i = \sum_{i=0}^n \sum_{j=0}^n \lambda_i^{r(q+1)} \mathbf{u}_i^{r(q)} \langle \Psi_i(\boldsymbol{\theta}) \Psi_j(\boldsymbol{\theta}) \Psi_i(\boldsymbol{\theta}) \rangle. \quad (28)$$

The stochastic eigenvector can be updated by solving the derived Eq. (26). We can update stochastic eigenvector $\mathbf{u}^{r(q)}(\boldsymbol{\theta})$ using Eq.(15) when the coefficients ($\mathbf{u}_0^{r(q)}, \dots, \mathbf{u}_n^{r(q)}$) of stochastic eigenvector is calculated. But derived stochastic eigenvector $\mathbf{u}^{r(q)}$ is not normalized and the length of eigenvector could change when the value of random variables $\boldsymbol{\theta}$ are changed.

(iii). The formularization for Step3

In this section, the improved normalization method is formularized in *step3*, which is different with the previous stochastic inverse power method [13] and is indispensable to achieve solving stochastic eigenvalue problems. The stochastic eigenvector is normalized by using the below equation.

$$\mathbf{u}^{r(q+1)} \rightarrow \frac{\mathbf{u}^{r(q+1)}}{\|\mathbf{u}^{r(q+1)}\|_{L^2_\theta}} \quad (29)$$

Substituting Eq. (15) into Eq. (29) the equation becomes as

follows:

$$\begin{aligned} \mathbf{u}^{r(q+1)} &\rightarrow \frac{\mathbf{u}^{r(q+1)}}{\|\mathbf{u}^{r(q+1)}\|_{L^2_\theta}} \\ &= \frac{\sum_{i=0}^n \mathbf{u}_i^{r(q+1)} \Psi_i(\boldsymbol{\theta})}{\sqrt{\sum_{i=0}^n \sum_{j=0}^n (\mathbf{u}_i^{r(q+1)})^T \mathbf{u}_j^{r(q+1)} \Psi_i(\boldsymbol{\theta}) \Psi_j(\boldsymbol{\theta})}} \end{aligned} \quad (30)$$

It can be seen that the denominator of Eq. (30) involves random variables $\boldsymbol{\theta}$. Though the coefficient vector, $\mathbf{u}_i^{r(q)}$, should be decided for the next iteration based on the normalization equation, it is not possible to solve the above equation directly. Thus, to overcome this problem, the solution is elaborated as follows:

Firstly, we arrange Eq. (30) as follows.

$$\begin{aligned} \mathbf{u}^{r(q+1)} &\rightarrow \frac{\mathbf{u}^{r(q+1)}}{\|\mathbf{u}^{r(q+1)}\|_{L^2_\theta}} \\ &= \frac{\sum_{i=0}^n \mathbf{u}_i^{r(q+1)} \Psi_i(\boldsymbol{\theta})}{\sqrt{\sum_{i=0}^{2n} U_i^{r(q+1)} \Psi_i(\boldsymbol{\theta})}}, \end{aligned} \quad (31)$$

where $U_i^{r(q+1)} = (\mathbf{u}_i^{r(q+1)})^T \mathbf{u}_j^{r(q+1)}$ and the order of expansion becomes $2n$ because two basis functions ($\Psi_i(\boldsymbol{\theta})$ and $\Psi_j(\boldsymbol{\theta})$) are multiplied (see Appendix 2).

Next, in order to overcome the problem of the denominator which involves random variables $\boldsymbol{\theta}$, it is assumed that $1/\sqrt{\sum_{i=0}^{2n} U_i^{r(q+1)} \Psi_i(\boldsymbol{\theta})}$ can be approximated by Hermite PCE as follows.

$$\frac{1}{\sqrt{\sum_{i=0}^{2n} U_i^{r(q+1)} \Psi_i(\boldsymbol{\theta})}} = \sum_{i=0}^{p_1} Z_i^{(q+1)} \Psi_i(\boldsymbol{\theta}), \quad (32)$$

where, $Z_i^{(q+1)}$ are unknown coefficients of approximated polynomial, $\Psi_i(\boldsymbol{\theta})$ are the basis functions, represented by Hermite polynomial, and p_1 is the order of approximated expansion terms and p_1 should be high enough so that the approximation becomes accurate enough. It can be seen that the unknown coefficients $Z_i^{(q+1)}$ need to be determined. Eq. (32) is rewritten as follows.

$$\left(\sum_{i=0}^{p_1} Z_i^{(q+1)} \Psi_i(\boldsymbol{\theta}) \right)^2 \left(\sum_{i=0}^{2n} U_i^{r(q+1)} \Psi_i(\boldsymbol{\theta}) \right) = 1. \quad (33)$$

Multiplying $\Psi_i(\boldsymbol{\theta}) W(\boldsymbol{\theta})$ to both sides of Eq. (33), and integrating the equation, we have:

$$\begin{aligned} & \int_D \sum_{i=0}^{p_1} \sum_{j=0}^{p_1} \sum_{k=0}^{2n} Z_i^{(q+1)} Z_j^{(q+1)} Z_k^{(q+1)} \Psi_i(\boldsymbol{\theta}) \Psi_j(\boldsymbol{\theta}) \Psi_k(\boldsymbol{\theta}) \Psi_i(\boldsymbol{\theta}) W(\boldsymbol{\theta}) d\boldsymbol{\theta} \\ &= \int_D \Psi_i(\boldsymbol{\theta}) W(\boldsymbol{\theta}) d\boldsymbol{\theta}. \end{aligned} \quad (34)$$

This equation can be depicted as follows:

$$\begin{aligned} & \sum_{i=0}^{p_1} \sum_{j=0}^{p_1} \sum_{k=0}^{2n} Z_i^{(q+1)} Z_j^{(q+1)} Z_k^{(q+1)} \langle \Psi_i(\boldsymbol{\theta}) \Psi_j(\boldsymbol{\theta}) \Psi_k(\boldsymbol{\theta}) \Psi_i(\boldsymbol{\theta}) \rangle \\ &= \langle \Psi_i(\boldsymbol{\theta}) \rangle. \end{aligned} \quad (35)$$

This means that the following nonlinear simultaneous equations about the coefficients, $Z_i^{(q+1)}$ and $Z_j^{(q+1)}$, are obtained when $t=0, \dots, p_1$.

$$\begin{aligned} \sum_{i=0}^{p_1} \sum_{j=0}^{p_1} \sum_{k=0}^{2n} Z_i Z_j U_k^{(q+1)} \langle \Psi_i(\boldsymbol{\theta}) \Psi_j(\boldsymbol{\theta}) \Psi_k(\boldsymbol{\theta}) \Psi_0(\boldsymbol{\theta}) \rangle &= \langle \Psi_0(\boldsymbol{\theta}) \rangle, \\ \sum_{i=0}^{p_1} \sum_{j=0}^{p_1} \sum_{k=0}^{2n} Z_i Z_j U_k^{(q+1)} \langle \Psi_i(\boldsymbol{\theta}) \Psi_j(\boldsymbol{\theta}) \Psi_k(\boldsymbol{\theta}) \Psi_1(\boldsymbol{\theta}) \rangle &= \langle \Psi_1(\boldsymbol{\theta}) \rangle, \\ &\vdots \\ \sum_{i=0}^{p_1} \sum_{j=0}^{p_1} \sum_{k=0}^{2n} Z_i Z_j U_k^{(q+1)} \langle \Psi_i(\boldsymbol{\theta}) \Psi_j(\boldsymbol{\theta}) \Psi_k(\boldsymbol{\theta}) \Psi_{p_1}(\boldsymbol{\theta}) \rangle &= \langle \Psi_{p_1}(\boldsymbol{\theta}) \rangle. \end{aligned} \quad (36)$$

Eq. (36) is solved by using Newton's method [17], and the detailed process is indicated in Appendix 3. Thus the unknown coefficients of approximation polynomial, $Z_i^{(q+1)}$, is derived.

Now, we can substitute Eq. (32) into Eq. (30) as follows.

$$\mathbf{u}^{(q+1)} \rightarrow \sum_{i=0}^n \sum_{j=0}^{p_1} \mathbf{u}_i^{(q+1)} Z_j^{(q+1)} \Psi_i(\boldsymbol{\theta}) \Psi_j(\boldsymbol{\theta}) \quad (37)$$

After the right hand side of Eq. (37) is expanded and simplified by the method in Appendix 2, the Eq. (37) can be represented by the basis functions, $\Psi_i(\boldsymbol{\theta})$ as shown below.

$$\mathbf{u}^{(q+1)} \rightarrow \sum_{i=0}^{n+p_1} \mathbf{u}_i^{(q+1)} \Psi_i(\boldsymbol{\theta}) \quad (38)$$

As shown above, the stochastic normalized equation is derived by the polynomial of order $n + p_1$. Because the stochastic normalization equation includes the random variables $\boldsymbol{\theta}$, it is considered that the length of eigenvector should become 1 for the different value of the variables $\boldsymbol{\theta}$. This condition can be satisfied by the above proposed method which is very important to get appropriate results. In references [13] and [14], the correct eigenvalue and eigenvector sometimes cannot be obtained because the stochastic eigenvector is averagely normalized in step3.

(iv). The formularization for Step4

The convergence of the stochastic method is evaluated by the relative change, and the equation is shown as follows:

$$\varepsilon^{(q+1)} = \frac{|V_{\lambda}^{(q+1)} - V_{\lambda}^{(q)}|}{|V_{\lambda}^{(q+1)}|}, \quad (39)$$

where, the $V_{\lambda}^{(q)}$ is the coefficient of variation of the eigenvalue, $V_{\lambda}^{(q)} = \sqrt{\text{Var}[\lambda]} / E[\lambda]$, where $\sqrt{\text{Var}[\lambda]}, E[\lambda]$ is obtained by Eq.(6) and Eq.(7), respectively. The convergence condition definition can be used as a stopping criterion because it will go to zero when the case tends to be convergent. In this improved stochastic inverse power method (I-SIPM), good convergence performance can be obtained using only this convergence criterion.

The above formularized improved stochastic inverse power method is summarized as follows (Algorithm 2):

The minimum eigenvalue can be calculated by using the proposed improved stochastic inverse power method (I-SIPM). Compared with the usual inverse power method, it involves not only the deterministic part ($\lambda_0^{(q+1)}$) of eigenvalue but also involves the

stochastic parts (the stochastic part of order 1 ($\lambda_1^{(q+1)}$), the stochastic part of order 2 ($\lambda_2^{(q+1)}$), $\dots$). And the unknown coefficients ($\mathbf{u}_0^{(p+1)}, \mathbf{u}_1^{(p+1)}, \dots$) of the stochastic eigenvector are derived by solving the updated stochastic eigenvector equation (Eq. (26)).

Algorithm 2 The improved stochastic inverse power method

Initialize: $\mathbf{u}^{(0)}$

While $\varepsilon > \varepsilon^*$

Step 1: $\lambda_i^{(q+1)} = \frac{1}{\langle \Psi_i(\boldsymbol{\theta}) \Psi_i(\boldsymbol{\theta}) \rangle} \sum_{i=0}^n \sum_{j=0}^{p_1} \sum_{k=0}^{2n} \mathbf{u}_i^{(q)} \mathbf{A}_k^{(q+1)} \mathbf{u}_j^{(q)} \langle \Psi_i(\boldsymbol{\theta}) \Psi_j(\boldsymbol{\theta}) \Psi_k(\boldsymbol{\theta}) \Psi_i(\boldsymbol{\theta}) \rangle$

Step 2: $\begin{bmatrix} \Omega_{00} & \dots & \Omega_{0n} \\ \vdots & \ddots & \vdots \\ \Omega_{n0} & \dots & \Omega_{nn} \end{bmatrix} \begin{bmatrix} \mathbf{u}_0^{(q+1)} \\ \vdots \\ \mathbf{u}_n^{(q+1)} \end{bmatrix} = \begin{bmatrix} \zeta_0 \\ \vdots \\ \zeta_n \end{bmatrix}$

Step 3: $\mathbf{u}^{(q+1)} \rightarrow \sum_{i=0}^{n+p_1} \mathbf{u}_i^{(q+1)} \Psi_i(\boldsymbol{\theta})$

Step 4: $\varepsilon^{(q+1)} = \frac{|V_{\lambda}^{(q+1)} - V_{\lambda}^{(q)}|}{|V_{\lambda}^{(q)}|}$

End While

Result: $\lambda^*, \mathbf{u}^*$

5 The formularization of the stochastic Wielandt deflation method

In section 4, the I-SIPM is proposed to evaluate minimum eigenvalue and eigenvector, in which we call mode-1 eigenvalue and eigenvector as $\lambda_{(1)}$, $\mathbf{u}_{(1)}$. However, mode-2, mode-3 and other modes are often important, while these modes cannot be evaluated by the proposed I-SIPM. Thus, as the other purpose of this study, the stochastic Wielandt deflation method (SWDM) is proposed to achieve evaluating the i^{th} ($i > 1$) eigenvalues and eigenvectors of a general stochastic eigenvalue problem. In section 5.1, the deterministic WDM is expounded. In section 5.2, we derive the SWDM formularization.

5.1 The deterministic Wielandt deflation method

In Fig.2, we show the flowchart of the Wielandt deflation method [18]. Generally, if we have the minimum eigenvalue $\lambda_{(1)}$ of $\mathbf{A}$ and corresponding normalized eigenvector $\mathbf{u}_{(1)}$ from the inverse power method, we can make a new matrix $\mathbf{B}$ from the matrix $\mathbf{A}$, where the eigenvalues of $\mathbf{B}$ are identical to all eigenvalues of $\mathbf{A}$ except the minimum eigenvalue in $\mathbf{A}$.

As shown in Fig.2, a new the matrix $\mathbf{B}$ with dimension N can be obtained by *step1* where the first row of the matrix $\mathbf{B}$ is 0. In *step2*, the first row and the first column of matrix $\mathbf{B}$ are deleted to obtain a new matrix $\mathbf{B}_{(1)}$ with dimension $N-1$. Then the matrix $\mathbf{B}_{(1)}$ has eigenvalues $\lambda_{(2)}, \lambda_{(3)}, \dots, \lambda_{(N)}$. Next, the defined new matrix $\mathbf{B}_{(1)}$ is used to the inverse power method to evaluate a new minimum eigenvalue (this is, the 2nd eigenvalue $\lambda_{(2)}$ in $\mathbf{A}$). But we need to obtain an eigenvector with dimension N , because obtained eigenvector is vector with dimension $N-1$. Therefore by *step4*, the 2nd eigenvector $\mathbf{u}_{(2)}$ can be derived. More details about the proof for the new matrix $\mathbf{B}$ identical to all eigenvalues of $\mathbf{A}$ except the minimum eigenvalue $\lambda_{(1)}$ can be found in [18].

5.2 The stochastic Wielandt deflation method

In order to solve a stochastic eigenvalue problem, the stochastic Wielandt deflation method is developed in this section.

Fig.2 Flowchart of the Wielandt deflation method

(i). The formularization for Step1

In step1, a new stochastic matrix $\mathbf{B}'(\theta)$ can be made from the stochastic matrix $\mathbf{A}'(\theta)$ as follows.

$$\mathbf{B}'(\theta) = \mathbf{A}'(\theta) - \frac{1}{u'_{(1),1}(\theta)} \mathbf{u}'_{(1)}(\theta) \mathbf{A}'_1(\theta), \quad (40)$$

where the $\mathbf{B}'$ is a matrix involving random variables (θ) because the matrix $\mathbf{A}'$ and eigenvector $\mathbf{u}'_{(1)}$ involve random variables (θ) . $\mathbf{A}'(\theta)$ and $\mathbf{u}'_{(1)}(\theta)$ is represented by matrix form (See Eq. (13)(15)).

$$\mathbf{A}'(\theta) = \sum_{i=0}^n \begin{bmatrix} A_{11i} & \cdots & A_{1Ni} \\ \vdots & \ddots & \vdots \\ A_{N1i} & \cdots & A_{NNi} \end{bmatrix} \Psi_i(\theta), \quad (41)$$

$$\mathbf{u}'_{(1)}(\theta) = \sum_{i=0}^{n+p_1} \begin{bmatrix} u_{(1),1i} \\ \vdots \\ u_{(1),Ni} \end{bmatrix} \Psi_i(\theta). \quad (42)$$

Then, $u'_{(1),1}(\theta)$ is defined as value in the first row of eigenvector $\mathbf{u}'_{(1)}(\theta)$, $\mathbf{A}'_1(\theta)$ is defined as the first row (row vector) of matrix $\mathbf{A}'(\theta)$. They are shown as follows.

$$u'_{(1),1}(\theta) = \sum_{i=0}^{n+p_1} u_{(1),1i} \Psi_i(\theta), \quad (43)$$

$$\mathbf{A}'_1(\theta) = \left[\sum_{i=0}^n A_{11i} \Psi_i(\theta) \quad \cdots \quad \sum_{i=0}^n A_{1Ni} \Psi_i(\theta) \right]. \quad (44)$$

Substituting Eq. (41), Eq. (42), Eq. (43) and Eq. (44) into Eq. (40) the Eq. (40) becomes as follows:

$$\begin{aligned} \mathbf{B}'(\theta) &= \sum_{i=0}^n \begin{bmatrix} A_{11i} & \cdots & A_{1Ni} \\ \vdots & \ddots & \vdots \\ A_{N1i} & \cdots & A_{NNi} \end{bmatrix} \Psi_i(\theta) \\ &\quad - \frac{1}{\sum_{i=0}^{n+p_1} u_{(1),1i} \Psi_i(\theta)} \left[\sum_{i=0}^{n+p_1} u_{(1),1i} \Psi_i(\theta) \right] \left[\sum_{i=0}^n A_{11i} \Psi_i(\theta) \quad \cdots \quad \sum_{i=0}^n A_{1Ni} \Psi_i(\theta) \right]. \end{aligned} \quad (45)$$

We can find the denominator of Eq.(45) involving random variables. In order to solve this problem, it is assumed that $1/\sum_{i=0}^n u_{(1),1i} \Psi_i(\theta)$ can be approximated by Hermite PCE as follows.

$$\frac{1}{\sum_{i=0}^n u_{(1),1i} \Psi_i(\theta)} = \sum_{u=0}^{p_2} R_u \Psi_u(\theta). \quad (46)$$

Again details about approximation method can be found in Appendix 1.

Now, we can substitute Eq. (46) into Eq. (45) as follows.

$$\mathbf{B}'(\theta) = \begin{bmatrix} 0 & \cdots & 0 \\ B_{21} & \cdots & B_{2N} \\ \vdots & \ddots & \vdots \\ B_{N1} & \cdots & B_{NN} \end{bmatrix}, \quad (47)$$

where $B_{ts} = \sum_{i=0}^n A_{tsi} \Psi_i(\theta) - \sum_{i=0}^{n+p_1} \sum_{j=0}^{n+p_1} \sum_{k=0}^n R_u u_{(1),1j} A_{tsk} \Psi_i(\theta) \Psi_k(\theta) \Psi_j(\theta)$. After the right hand side of Eq. (47) is expanded and simplified by the method of Appendix 2, the Eq. (48) can be derived based on the basis functions, $\Psi_i(\theta)$.

$$\mathbf{B}'(\theta) = \sum_{i=0}^{2n+p_1+p_2} \begin{bmatrix} 0 & \cdots & 0 \\ B'_{21} & \cdots & B'_{2N} \\ \vdots & \ddots & \vdots \\ B'_{N1} & \cdots & B'_{NN} \end{bmatrix} \Psi_i(\theta). \quad (48)$$

(ii). The formularization for Step2

In step2, we can obtain the matrix $\mathbf{B}'_{(1)}(\theta)$ with dimension $N-1$ by deleting 0 in the first row and deleting the first column. Then $\mathbf{B}'_{(1)}(\theta)$ has eigenvalues $\lambda'_{(2)}, \lambda'_{(3)}, \dots, \lambda'_{(N)}$.

$$\mathbf{B}'_{(1)}(\theta) = \sum_{i=0}^{2n+p_1+p_2} \begin{bmatrix} B'_{22} & \cdots & B'_{2N} \\ \vdots & \ddots & \vdots \\ B'_{N2} & \cdots & B'_{NN} \end{bmatrix} \Psi_i(\theta). \quad (49)$$

(iii). The formularization for Step3

The minimum eigenvalue $\lambda'_{(2)}(\theta)$ (this is, the 2nd eigenvalue in $\mathbf{A}'(\theta)$) and corresponding eigenvector $\mathbf{v}'_{(2)}(\theta)$ in the matrix

$\mathbf{B}'_{(1)}(\boldsymbol{\theta})$ can be evaluated by using the improved stochastic inverse power method (I-SIPM) in the *step3*.

$$\lambda'_{(2)}(\boldsymbol{\theta}) = \sum_{i=0}^{2n+p_1+p_2} \lambda'_{(2)i} \Psi_i(\boldsymbol{\theta}), \quad (50)$$

$$\mathbf{v}'_{(2)}(\boldsymbol{\theta}) = \sum_{i=0}^{2n+2p_1+p_2} \mathbf{v}'_{(2)i} \Psi_i(\boldsymbol{\theta}). \quad (51)$$

However we can find that the eigenvector, $\mathbf{v}'_{(2)}(\boldsymbol{\theta})$, is a vector with dimension $N-1$. In next step, the vector with dimension $N-1$ will be transformed as a vector with dimension N which is the 2nd eigenvector in the matrix $\mathbf{A}'(\boldsymbol{\theta})$.

(iv). The formularization for Step4

In *step4*, Eq. (52) is used to obtain the eigenvector $\mathbf{u}'_{(2)}(\boldsymbol{\theta})$ of the matrix $\mathbf{A}'(\boldsymbol{\theta})$.

$$\mathbf{u}'_{(2)}(\boldsymbol{\theta}) = \frac{1}{u'_{(1),1}(\boldsymbol{\theta})} \mathbf{u}'_{(1)}(\boldsymbol{\theta}) - \frac{\lambda'_{(1)}(\boldsymbol{\theta}) - \lambda'_{(2)}(\boldsymbol{\theta})}{\mathbf{A}_{1,1} \mathbf{v}'_{(2)}(\boldsymbol{\theta})} \mathbf{v}'_{(2)}(\boldsymbol{\theta}), \quad (52)$$

where $\lambda'_{(1)}(\boldsymbol{\theta})$ is minimum eigenvalue and $\mathbf{u}'_{(1)}(\boldsymbol{\theta})$ is corresponding eigenvector in the matrix $\mathbf{A}'(\boldsymbol{\theta})$ obtained by the I-SIPM shown section 4.2. $\mathbf{v}'_{(2)}(\boldsymbol{\theta})$ is defined as follows.

$$\mathbf{v}'_{(2)}(\boldsymbol{\theta}) = \frac{1}{v'_{(2),1}(\boldsymbol{\theta})} \begin{pmatrix} 0 \\ \mathbf{v}'_{(2)}(\boldsymbol{\theta}) \end{pmatrix}, \quad (53)$$

where $v'_{(2),1}(\boldsymbol{\theta})$ is value in the first row of eigenvector $\mathbf{v}'_{(2)}(\boldsymbol{\theta})$. In this equation, it can be seen that the denominator involves random variables $\boldsymbol{\theta}$ when substituting Eq. (51) into Eq. (53). Thus it will be again approximated by Hermite PCE as follows by the method in Appendix 1.

$$\frac{1}{v'_{(2),1}(\boldsymbol{\theta})} = \sum_{i=0}^{p_3} G_i \Psi_i(\boldsymbol{\theta}). \quad (54)$$

Substituting Eq. (14), Eq. (15), Eq. (43), Eq. (44), Eq. (50), Eq. (51), Eq. (53) and Eq. (54) into Eq. (52), the Eq. (52) becomes as follows.

$$\mathbf{u}'_{(2)}(\boldsymbol{\theta}) = \begin{bmatrix} u'_{(2),1} \\ u'_{(2),2} \\ \vdots \\ u'_{(2),N} \end{bmatrix}, \quad (55)$$

where

$$u'_{(2),1} = \sum_{i=0}^{p_2} \sum_{j=0}^{n+p_1} R_i u_{(1),1j} \Psi_i(\boldsymbol{\theta}) \Psi_j(\boldsymbol{\theta}) \times \left(\sum_{i=0}^n A_{12i} \Psi_i(\boldsymbol{\theta}) + \sum_{i=0}^{p_3} \sum_{j=0}^{2n+2p_1+p_2} \sum_{k=0}^n G_i v_{(2),N-1j} A_{1Nk} \Psi_i(\boldsymbol{\theta}) \Psi_j(\boldsymbol{\theta}) \Psi_k(\boldsymbol{\theta}) \right), \quad (56-1)$$

$$u'_{(2),2} = \sum_{i=0}^{p_2} \sum_{j=0}^{n+p_1} R_i u_{(1),2j} \Psi_i(\boldsymbol{\theta}) \Psi_j(\boldsymbol{\theta}) \times \left(\sum_{i=0}^n A_{12i} \Psi_i(\boldsymbol{\theta}) + \sum_{i=0}^{p_3} \sum_{j=0}^{2n+2p_1+p_2} \sum_{k=0}^n G_i v_{(2),N-1j} A_{1Nk} \Psi_i(\boldsymbol{\theta}) \Psi_j(\boldsymbol{\theta}) \Psi_k(\boldsymbol{\theta}) \right) - \left(\sum_{i=0}^n \lambda_{(1)i}(\boldsymbol{\theta}) \Psi_i(\boldsymbol{\theta}) - \sum_{i=0}^{2n+2p_1+p_2} \lambda_{(2)i}(\boldsymbol{\theta}) \Psi_i(\boldsymbol{\theta}) \right), \quad (56-2)$$

$$u'_{(2),s} = \sum_{i=0}^{p_2} \sum_{j=0}^{n+p_1} R_i u_{(1),sj} \Psi_i(\boldsymbol{\theta}) \Psi_j(\boldsymbol{\theta}) \times \left(\sum_{i=0}^n A_{12i} \Psi_i(\boldsymbol{\theta}) + \sum_{i=0}^{p_3} \sum_{j=0}^{2n+2p_1+p_2} \sum_{k=0}^n G_i v_{(2),N-1j} A_{1Nk} \Psi_i(\boldsymbol{\theta}) \Psi_j(\boldsymbol{\theta}) \Psi_k(\boldsymbol{\theta}) \right) - \sum_{i=0}^{p_3} \sum_{j=0}^{2n+2p_1+p_2} \sum_{k=0}^n G_i v_{(2),s-1j} \lambda_{(1)k}(\boldsymbol{\theta}) \Psi_i(\boldsymbol{\theta}) \Psi_j(\boldsymbol{\theta}) \Psi_k(\boldsymbol{\theta}) - \sum_{i=0}^{p_3} \sum_{j=0}^{2n+2p_1+p_2} \sum_{k=0}^{2n+p_1+p_2} G_i v_{(2),s-1j} \lambda_{(2)k}(\boldsymbol{\theta}) \Psi_i(\boldsymbol{\theta}) \Psi_j(\boldsymbol{\theta}) \Psi_k(\boldsymbol{\theta}), \quad (56-3)$$

where, $s=3, \dots, N$. And they can be simplified to the following equation.

$$\mathbf{u}'_{(2)}(\boldsymbol{\theta}) = \sum_{i=0}^{4n+3p_1+2p_2+p_3} \mathbf{u}'_{(2)i} \Psi_i(\boldsymbol{\theta}). \quad (57)$$

Consequently, the 2nd eigenvalue and eigenvector in $\mathbf{A}'(\boldsymbol{\theta})$ can be obtained by the above proposed method. Also we can understand that the 2nd eigenvalue is approximated by the polynomial of order $2n+p_1+p_2$ in Eq.(50), and the 2nd eigenvector is by the polynomial of order $4n+3p_1+2p_2+p_3$ in Eq.(57).

It is noted that by using the procedure to reduce dimensionality of the matrix ($\mathbf{A}'(\boldsymbol{\theta}) \rightarrow \mathbf{B}'_{(1)}(\boldsymbol{\theta})$, $\mathbf{B}'_{(1)}(\boldsymbol{\theta}) \rightarrow \mathbf{B}'_{(2)}(\boldsymbol{\theta}) \dots (\mathbf{B}'_{(2)}(\boldsymbol{\theta})$ is the matrix with dimension $N-2$)), it is possible to achieve solving m^{th} eigenvalue, $\lambda'_{(m)}(\boldsymbol{\theta})$, and corresponding eigenvector, $\mathbf{u}'_{(m)}(\boldsymbol{\theta})$, of the matrix $\mathbf{A}'(\boldsymbol{\theta})$.

6 Numerical Examples

In this study, we developed I-SIPM program and SWDM program by using C language which can deal with discrete and continuous model considering uncertainty. The validity and feasibility of the proposed methods is discussed by two cases, (1) eigen frequency problem - a simple discrete 2-DOF undamped system with uncertainty in stiffness, and (2) buckling eigenvalue problem - a continuous beam model with uncertainty in size. Finally, (3) the dynamic damper problem is discussed by using proposed method as a more practical example.

6.1 Two degrees of freedom undamped system

A discrete 2-DOF undamped system is considered and shown in Fig.3. The statistical properties of uncertain parameters are given in Table 3. The three springs stiffness $k_1(\theta_1)$, $k_2(\theta_1)$, $k_3(\theta_1)$ obeying same random variable θ_1 , where, $\boldsymbol{\theta} = \{\theta_1\}$. The masses are deterministic with $m_1 = 1\text{kg}$, $m_2 = 2\text{kg}$. The validity of the proposed methods (I-SIPM and SWDM) is discussed by comparing the results of the method with the MCS of the deterministic problems for the same condition.

Fig.3 A discrete 2-DOF undamped system

Table 3 The statistical parameters of the undamped system (N/m)

Uncertain variable	Mean	Standard deviation
$k_1(\theta_1)$	2.5	0.4
$k_2(\theta_1)$	7	1
$k_3(\theta_1)$	1	0.15

The system stiffness matrix involving random variables θ can be obtained as follows.

$$\mathbf{K}'(\theta) = \begin{bmatrix} k_1(\theta_1) + k_3(\theta_1) & -k_3(\theta_1) \\ -k_3(\theta_1) & k_2(\theta_1) + k_3(\theta_1) \end{bmatrix} \quad (58)$$

$$= \begin{bmatrix} 3.5 & -1 \\ -1 & 8 \end{bmatrix} \Psi_0(\theta) + \begin{bmatrix} 0.55 & -0.15 \\ -0.15 & 1.15 \end{bmatrix} \Psi_1(\theta).$$

And the system mass matrix is as follows.

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \Psi_0(\theta) \quad (\Psi_0(\theta)=1) \quad (59)$$

The stochastic eigenvalue problem is represented by Eq.(10), where $\mathbf{R}'(\theta)$ is the system mass matrix and is deterministic form as shown Eq.(59). The 1st stochastic eigenvalue $\lambda'_{(1)}(\theta)$ and corresponding eigenvector $\mathbf{u}'_{(1)}(\theta)$ can be evaluated by using proposed I-SIPM. Also, the 2nd stochastic eigenvalue $\lambda'_{(2)}(\theta)$ and corresponding eigenvector $\mathbf{u}'_{(2)}(\theta)$ can be evaluated by using proposed SWDM.

In Fig. 4, the response surfaces of stochastic eigenvalues ($\lambda'_{(1)}(\theta)$ and $\lambda'_{(2)}(\theta)$) and determinate results are described. As can be seen, a good agreement is observed between the results from the stochastic response surfaces and the reference values (deterministic cases).

Fig. 5 and Fig. 6 show probability density distributions of stochastic eigenvalues obtained through the proposed methods (The 1st stochastic eigenvalue obtained through I-SIPM, and the 2nd stochastic eigenvalue obtained through SWDM) and the reference distribution obtained by MCS respectively. This has been obtained by using 10000 realization of the standard random variable θ_1 in the obtained response surface of the eigenvalue. As can be seen, the results of proposed methods agree well with the results from the Monte Carlo simulation (MCS) of 10000 samples. Also, as shown in the Table in Fig.5, we find that statistical properties are reasonably well estimated by the proposed I-SIPM when compared with those obtained from determinate (MCS) result. And we can also find from Fig.6 that statistical properties of the second eigenvalue by the proposed SWDM have a good agreement with the MCS result. Note that the mean and the standard deviation of "I-SIPM_PCE/SWDM_PCE" in the Tables of Fig.5 and Fig.6 are calculated by Eq. (6) and Eq. (7), which also takes almost same value.

Fig. 7 and Fig. 8 show the response surfaces for the components $\{u'_{(1),1}, u'_{(1),2}\}^T$ and $\{u'_{(2),1}, u'_{(2),2}\}^T$ of eigenvectors $\mathbf{u}'_{(1)}(\theta)$ and $\mathbf{u}'_{(2)}(\theta)$ respectively. They showed good agreement between proposed two methods and reference values. And the probability density distributions are also shown in the Fig.9 and Fig.10. As can be seen, the reasonable results are shown for solving eigenvectors problem. Thus, from the results in this section we can conclude the proposed I-SIPM is valid for evaluating 1st eigenvalue and eigenvector. And the validity of the proposed SWDM is also demonstrated for evaluating 2nd eigenvalue and eigenvector.

It is noted that the approximate expressions of the stochastic minimum eigenvalue response surface and the stochastic eigenvector components response surface are 5th ($n=5$) and 10th ($n=5$ and $p_1=5$) order PCE, respectively. The approximate expressions of the stochastic 2nd eigenvalue response surface and the stochastic eigenvector components response surface are 20th

($n=5$, $p_1=5$ and $p_2=5$) and 50th ($n=5$, $p_1=5$, $p_2=5$ and $p_3=5$) order PCE, respectively. However, only the 3rd order approximate expressions of the response surfaces are shown in Fig.4, Fig.7 and Fig.8, because the influence is small after 4th order.

Fig. 4 The response surface of frequencies

Fig.5 The probability density distributions of eigenvalue $\lambda'_{(1)}(\theta)$

Fig.6 The probability density distributions of eigenvalue $\lambda'_{(2)}(\theta)$

Fig. 7 The response surface of eigenvector components: $\mathbf{u}'_{(1)}(\theta)$

Fig. 8 The response surface of eigenvector components: $\mathbf{u}'_{(2)}(\theta)$

Fig.9 The probability density distributions of eigenvector components: $\mathbf{u}'_{(1)}(\theta)$

Fig.10 The probability density distributions of eigenvector components: $\mathbf{u}'_{(2)}(\theta)$

6.2 A continuous beam

In this section, a continuous free-standing beam problem (buckling problem shown in Fig.11 (a)) is considered to prove that (i) the proposed method can deal with continuous problem, (ii) the method can solve buckling problem, and (iii) the method is valid for solving multi degree of freedom problem.

The buckling problem of a beam is discussed considering beam section dimension H with uncertainty, which is solved by the developed I-SIPM. The mesh is shown in Fig. 11(b) and the beam section dimension is shown in Fig. 11(c), and the depth H is assumed to be following Gaussian distribution and obeying random variable θ_1 , where $\theta = \{\theta_1\}$. In the rule of JIS G3192[19] for tolerance of H-sections, the tolerance of depth (H) is $\pm 3\text{mm}$ when over 400mm in width (B). In this problem, we define the mean of depth as 418mm and the standard deviation as 1mm, so that 3 times of the standard deviation (3σ) becomes equal to 3mm. The statistical properties of uncertain parameter H are given in Table 4. It is also assumed that the concentrated load $P=1\text{ N/mm}^2$, Young's modulus $E=206000\text{ N/mm}^2$, the length of beam $l=5000\text{mm}$, and fixed boundary condition is applied to the boundary Γ_1 . The moment of inertia of area about the stronger axis is considered, assuming that the column deflection is supported to its weaker direction. The validity of the proposed method (I-SIPM) is discussed by comparing the results of the deterministic problem for the same condition.

Fig.11 Free-standing beam

Table 4 The statistical parameters of free-standing beam

Uncertain parameter	Mean(H_0)	Std. dev.(H_1)
$H(\theta_1)$	418mm	1mm

The parameter depth, H , is represented using a standard normal random variable θ_1 as follows by the above condition.

$$H(\theta_1) = H_0 + H_1\theta_1, \quad (60)$$

where H_0 is mean with 418mm, and H_1 is standard deviation with 1mm. The cross section A and the moment of inertia of area I are related to the depth H . In order to construct the element stiffness matrix and the element initial stress matrix considering the uncertainty in depth H , A and I involving uncertainty in depth H is treated as follows.

For H-shaped steel, the equation of cross section A is given as follows.

$$A = BH - bh. \quad (61)$$

The cross section $A'(\theta)$ with the uncertain parameter is derived by substituting Eq. (60) into Eq. (61).

$$A'(\theta) = A_0\Psi_0(\theta) + A_1\Psi_1(\theta), \quad (62)$$

where $A_0 = BH_0 - bh$, $A_1 = BH_1$.

The moment of inertia of area I is given as follows.

$$I = \frac{BH^3 - bh^3}{12}. \quad (63)$$

Substituting Eq. (60) into Eq. (63), we can derive the moment of inertia of area $I'(\theta)$ with uncertain parameter.

$$I'(\theta) = I_0\Psi_0(\theta) + I_1\Psi_1(\theta) + I_2\Psi_2(\theta) + I_3\Psi_3(\theta), \quad (64)$$

where, $I_0 = (BH_0^3 - bh^3 + 3BH_0H_1^2)/12$, $I_1 = (BH_0^2H_1 + BH_1^3)/4$, $I_2 = BH_0H_1^2/4$, $I_3 = BH_1^3/12$.

Thus, the element stiffness matrix involving random variables θ can be derived as follows by the polynomial of 3rd order.

$$\mathbf{K}^e(\theta) = \sum_{i=0}^3 \begin{bmatrix} EA_i/l & 0 & 0 & -EA_i/l & 0 & 0 \\ 0 & 12EI_i/l^3 & 6EI_i/l^2 & 0 & -12EI_i/l^3 & 6EI_i/l^2 \\ 0 & 6EI_i/l^2 & 4EI_i/l & 0 & -6EI_i/l^2 & 2EI_i/l \\ -EA_i/l & 0 & 0 & EA_i/l & 0 & 0 \\ 0 & -12EI_i/l^3 & -6EI_i/l^2 & 0 & 12EI_i/l^3 & -6EI_i/l^2 \\ 0 & 6EI_i/l^2 & 2EI_i/l & 0 & -6EI_i/l^2 & 4EI_i/l \end{bmatrix} \Psi_i(\theta) \quad (65)$$

And the element initial stress matrix is as follows. ($\Psi_0(\theta)=1$)

$$\mathbf{K}_G^e = P \begin{bmatrix} 1/l & 0 & 0 & -1/l & 0 & 0 \\ 0 & 6/5l & 1/10 & 0 & -6/5l & 1/10 \\ 0 & 1/10 & 2/15 & 0 & -1/10 & -1/30 \\ -1/l & 0 & 0 & 1/l & 0 & 0 \\ 0 & -6/5l & -1/10 & 0 & 6/5l & -1/10 \\ 0 & 1/10 & -1/30 & 0 & -1/10 & 2/15 \end{bmatrix} \Psi_0(\theta) \quad (66)$$

By assembly of element system, we can derive a global equation system which has 24 degrees of freedom. The eigenvalue is derived by solving Eq.(10) in which $\mathbf{R}'(\theta)$ is initial stress matrix with deterministic (Eq.(66)) and $\mathbf{K}'(\theta)$ is stiffness matrix with uncertainty (Eq.(65)).

In Fig.12, we show the response surface of the buckling load. A good agreement is observed between the results from the stochastic response surface and the reference values by usual theoretical formula. The approximate expression of the response surface is 5th order PCE. However, only the 1st order approximate expression of the response surface is shown in Fig.12, because the coefficients of approximation polynomial are 0 after the 2nd order. In Fig.13, the probability density distributions are also shown to prove that the results are well consistent with MCS results. Also, as shown in Table 5, we find that the statistics are reasonably well estimated by the proposed I-SIPM when compared with those obtained from MCS. Note that the mean and the standard deviation of "I-SIPM-PCE" in the Table 5 are calculated by Eq. (6) and Eq. (7), which also takes almost same value.

From the results in this section, it can be concluded that the proposed methodology can effectively evaluate (i) the continuous problem, (ii) the buckling problem, and (iii) the

multi degree of freedom problem. And we investigated the computational time of the proposed method compared with that of MCS for this example. Then we found that the computational time is 2s for solving 1st mode by using the proposed I-SIPM. On the other hand, the computational time of MCS is shown which is taking 2.2 hours by the deterministic IPM of 10000 samples.

Fig. 12 The response surface of the buckling load

Table 5 Statistics of the buckling load ($\times 10^7$ N)

	<i>I – SIPM</i>	<i>MCS</i>	<i>I – SIPM _PCE</i>
<i>Mean</i>	2.028	2.026	2.028
<i>Std.dev.</i>	0.036	0.036	0.036

Fig. 13 The probability density distributions of the buckling load

6.3 The dynamic damper

The validity and feasibility of the proposed methods is concluded by the previous two examples for vibrations problem and buckling problem. In this section, we will discuss a dynamic damper problem as a more practical example in which stiffness and mass are considered as uncertain parameters, the probability of the natural frequency falling into the range to be avoided of

ship structure is studied by using the developed I-SIPM.

6.3.1 Modelling and condition

In large ship structures, the resonance of the superstructure of a ship to engine or propeller exciting force is an important problem. In order to reduce the resonance response of the superstructure, the dynamic damper is attracted as a counterplan of avoiding resonance [20].

The main structure A and the dodger B are shown in Fig.14. A simple model of two degrees of freedom (Fig.15 (a)) is proposed to model the superstructure of ship and the dodger in Fig.14. And when the dynamic damper attached on the dodger B is considered, we can use a simple model of the three degrees of freedom to model the system where k_3 and m_3 represent the dynamic damper system (Fig.15 (b)).

Fig.14 Superstructure

Fig.15 Superstructure (Simple model)

In this example, it is considered that first order natural frequency of two degrees of freedom (Fig.15 (a)) is close to fourth order exciting frequency component of the main engine. So the system will produce serious vibration with large amplitude. And we can understand that when the dynamic damper is attached (three degrees of freedom (Fig.15 (b)), the first order natural frequency will be decreased. Then, it has possibility of that first order natural frequency of three degrees of freedom is close to the third order exciting frequency component of the main engine (That is, the first order natural frequency is probably entering the range between third order and fourth order exciting frequency

component of the main engine.). Thus, we discuss the problem to avoid third and fourth order exciting frequencies from the main engine.

As problem conditions, it is assumed that the third order exciting frequency component of the main engine is 4Hz (3rdorder×80rpm/60), and the fourth order exciting frequency component of the main engine is 5.33Hz (4thorder×80rpm/60) with main engine normal revolution of 80rpm[21]. In order to avoid large response, it is necessary to avoid ranges of the third order and fourth order exciting frequency of the main engine which are assumed as 3.8Hz~4.2Hz (±5% of 4Hz), 5.07Hz~5.60Hz (±5% of 5.33Hz), respectively.

Table 6 The parameters of model (Deterministic parameters)

Structure	The spring stiffness (N/m)	The mass (kg)
The main structure A	$k_1 = 7.27 \times 10^8$	$m_1 = 1.8 \times 10^5$
The dodger B	$k_2 = 2.31 \times 10^7$	$m_2 = 2.0 \times 10^4$

The parameters of model are assumed in Table 6 which is a practical assumption by references [22][23]. As a result of the coupled vibration of the main structure and the dodger, the first and the second order natural frequencies are calculated to be 5.30Hz and 10.33Hz, respectively. The first order natural frequency is completely within the range of fourth order exciting frequency of the main engine, very possibly causing excessive vibration of the superstructure. To avoid this resonance, the two cases are discussed in the following: (1) The spring stiffness of the dynamic damper is assumed as including random variable θ_1 and the mass is deterministic value in section 6.3.2. (2) The spring stiffness of the dynamic damper is assumed as including a random variable θ_1 , and the mass of dynamic damper is assumed as including another random variable, θ_2 , in section 6.3.3.

6.3.2 The spring stiffness of dynamic damper with uncertainty

In this case, the spring stiffness with uncertainty is discussed through different values of the mass of the dynamic damper. In Table 7, the statistical properties of two cases are given. In both cases, the spring stiffness of the dynamic damper is assumed to following Gaussian distribution and obeying random variable θ_1 . Also in case (1), the mass is deterministic value with $\mu = 1/110$, and in case (2), the mass is deterministic value with $\mu = 1/40$. Note that μ is defined as $\mu = m_3/m_2$. The general relationship between spring stiffness and mass of dynamic damper can be found from the dynamic damper theory [24].

Table 7 The statistical parameters of the dynamic damper

Case	The spring stiffness k_3 (N/m) (Uncertain parameters)		The mass m_3 (kg) ($m_3 = m_2 \mu$) (deterministic value)
	Mean ($k_{3\mu}$)	Standard deviation ($k_{3\sigma}$)(C.V. is 8%)	
(1)	1.995×10^5	1.596×10^4	$180(\mu = 1/110)$
(2)	5.542×10^5	4.434×10^4	$500(\mu = 1/40)$

Here, the system stiffness matrix involving random variables θ can be derived as follows.

$\mathbf{K}'(\theta)$

$$= \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3(\theta_1) & -k_3(\theta_1) \\ 0 & -k_3(\theta_1) & k_3(\theta_1) \end{bmatrix} \\ = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_{3\mu} & -k_{3\mu} \\ 0 & -k_{3\mu} & k_{3\mu} \end{bmatrix} \Psi_0(\theta) \\ + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_{3\sigma} & -k_{3\sigma} \\ 0 & -k_{3\sigma} & k_{3\sigma} \end{bmatrix} \Psi_1(\theta), \quad (67)$$

where, $k_{3\mu}$ is mean of the spring stiffness of the dynamic damper and $k_{3\sigma}$ is standard deviation. And the system mass matrix is represented as follows.

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \Psi_0(\theta) \quad (\Psi_0(\theta) = 1) \quad (68)$$

The matrix $\mathbf{A}'(\theta)$ in Eq.(10) can be obtained by using Eq.(67) and Eq.(68), and we solve Eq.(10) by using proposed I-SIPM. When the spring stiffness k_3 is considered as uncertain parameter, the first order natural frequency will be described as probability distribution. So the stochastic eigenvalue can be evaluated to observe the probability of the natural frequency falling into the range to be avoided (hereinafter called “probability of resonant high response”). In Fig.16, the probability density distributions of first order natural frequency are shown for the case (1) and the case (2). As can be seen, in both cases, the mean values of the first order natural frequency are moving to the range of third order exciting frequency of the main engine and are out of the range of the fourth order exciting frequency of the main engine, in which the mean values are a theoretical value when the spring stiffness k_3 is deterministic value ($k_3 = k_{3\mu}$). However they are more closer to the range of fourth order exciting frequency of the main engine, while the resonance due to the third order exciting frequency of the main engine is completely avoided.

In case (1), when the mass is 180kg, observing the blue histogram in Fig.16, we can see that the resonance due to the fourth order exciting frequency cannot be avoided because the first order natural frequency has a probabilistic nature and most of the right of the distribution overlaps the fourth order exciting frequency range. The probability of resonant high response reaches up to 44.8%. This is very dangerous because of a large probability of the natural frequency falling into the range to be avoided for this practical problem.

In case (2), we attempt to increase the mass of the dynamic damper to 500kg to discuss the probability of resonant high response. Here the probability density distribution is shown in Fig.16. Observing the red histogram, we can see that the probability density distribution of the first order natural frequency moves to more left when the mass ratio μ becomes larger compared with case(1). And the probability becomes smaller for entering the range of the fourth order exciting frequency of the main engine. It should be clear that the probability of resonant high response becomes a small value, 0.2%.

So we can understand that when the mass ratio μ becomes larger, the probability density distribution of the first order natural

frequency moves closer to the range of the third order exciting frequency of the main engine compared with a small mass ratio, and the probability of resonant high response becomes smaller. As to probability of the natural frequency falling into the range to be avoided and economic cost synthetically, the appropriate selection of the mass is necessary.

Regarding the second order natural frequency, the probability of resonant high response can be evaluated in the same manner using SWDM. We have confirmed that the natural frequency is sufficiently above the range of fourth order exciting frequency of the main engine, and the probability of resonant high response is very small, both in case (1) and in case (2).

6.3.3 Both the spring stiffness and the mass of dynamic damper with uncertainty

In this case, both the spring stiffness and the mass with uncertainty are discussed through different mean values of the mass of the dynamic damper. The statistical properties of two cases are given in Table 8. In both cases, the spring stiffness of the dynamic damper is assumed to following Gaussian distribution and obeying random variable θ_1 , and the mass is also assumed to following Gaussian distribution represented by another random variable, θ_2 . Also in case(1), the mean of the mass is considered as 180kg ($\mu=1/110$). In case(2), the mean of the mass is considered as 500kg ($\mu=1/40$). It is noted that in these two cases, the two random variables $\theta=\{\theta_1, \theta_2\}$ is used to solve the stochastic eigenvalue problem.

Table 8 The statistical parameters of the dynamic damper

Case	The spring stiffness k_3 (N/m)		The mass (kg)	
	(Uncertain parameters)		(Uncertain parameters)	
	Mean ($k_{3\mu}$)	Standard deviation ($k_{3\sigma}$)(C.V. is 8%)	Mean ($m_{3\mu}$)	Standard deviation ($m_{3\sigma}$)(C.V. is 5%)
(1)	1.995×10^5	1.596×10^4	180 ($\mu=1/110$)	9
(2)	5.542×10^5	4.434×10^4	500 ($\mu=1/40$)	25

Here, the system stiffness matrix involving random variables θ is same as Eq.(67) except that two dimensional basis functions are used ($\Psi_0(\theta)=1, \Psi_1(\theta)=\theta_1, \Psi_2(\theta)=\theta_2, \dots$, by Table 2). The system mass matrix involving random variable θ is derived as follows because the mass is also uncertain parameter.

$$\mathbf{M}'(\theta) = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3(\theta_2) \end{bmatrix} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_{3\mu} \end{bmatrix} \Psi_0(\theta) + \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_{3\sigma} \end{bmatrix} \Psi_2(\theta), \quad (69)$$

Fig.16 The probability density distributions of first order frequency. (The spring stiffness with uncertainty)

Fig. 17 The response surface by I-SIPM (Case(2))

Fig. 18 The response surface by deterministic values (Case(2))

Fig.19 The probability density distributions of first order frequency. (Both the spring stiffness and the mass with uncertainty)

where $m_{3\mu}$ is mean of the mass of the dynamic damper and $m_{3\sigma}$ is standard deviation. The $\mathbf{R}'(\theta)$ in Eq.(10) is mass matrix, $\mathbf{M}'(\theta)$. The matrix $\mathbf{A}'(\theta)$ can be derived by to solve $\mathbf{A}'(\theta) = \mathbf{R}'^{-1}(\theta)\mathbf{K}'(\theta)$, where, $\mathbf{R}'^{-1}(\theta)$ is derived by Appendix 1 and the simplification of $\mathbf{R}'^{-1}(\theta)\mathbf{K}'(\theta)$ is done by the method in Appendix 2. The stochastic eigenvalue can be evaluated using proposed I-SIPM to solve Eq.(10).

Firstly, we will discuss validity of the proposed I-SIPM for the system with two random variables, $\theta = \{\theta_1, \theta_2\}$. In Fig.17, the response surface by I-SIPM is shown for case(2), while that obtained by many deterministic analyses is shown in Fig.18. As can be seen, a good agreement is observed between the results from the stochastic response surface and deterministic results in the most of the areas. Though they have some inconsistent results

in edges (for example: θ_1 is around 5 and θ_2 is around -5, θ_1 is around -5 and θ_2 is around 5.), if we see Table 9, it does not affect the statistical properties because the probabilities of occurrence are very low in the edges.

Fig.19 also showed the probability density distributions of first order natural frequency obtained from the I-SIPM (red graph) and MCS (gray graph) of 10000 samples in case(2). They have good agreement from the two results for the two random variables problem. In Fig.19, we can also find the probability density distribution of first order natural frequency in case(1) by using I-SIPM.

Table 9 Statistical properties of case (2) (Hz)

	<i>I – SIPM</i>	<i>MCS</i>
<i>Mean</i>	4.883	4.882
<i>Std.dev.</i>	0.118	0.116

In case(1), the probability of resonant high response is 45.3%. As can be seen, the probability becomes larger for entering the range of the fourth order exciting frequency of the main engine compared with case(1) in section 6.3.2 because both the spring stiffness k_3 and the mass m_3 are considered as uncertain parameters.

In case(2), from the observation of the red histogram in Fig.19, we can see that the probability density distribution of the first order natural frequency moves to the range of the third order exciting frequency of the main engine when the mass ratio μ becomes larger. The probability of resonant high response becomes 1.99%, which is significantly reduced compared with case(1). And we can also be see the probability is larger compared with case(2) in section 6.3.2.

From the results in this section, it can be concluded that the proposed I-SIPM is valid for the system with two random variables $\theta = \{\theta_1, \theta_2\}$. And by applying the proposed method to the dynamic damper problem, it is possible to evaluate the probability of resonant high response, and to propose an effective countermeasure to reduce the probability of resonant high response by appropriate selection of the mass and the spring of the dynamic damper.

7 Concluding remarks

1. In this study, the stochastic eigenvalue problem is formulated. And in order to solve a stochastic eigenvalue problem, as a numerical analysis algorithm, the improved stochastic inverse power method (I-SIPM) based on response surface methodology is formulated by an Hermite PCE. The method is different with the previous stochastic inverse power method. The minimum eigenvalue and eigenvector of stochastic eigenvalue problems can be validly evaluated by using the proposed method.
2. As another purpose of this study, the stochastic Wielandt deflation method (SWDM) is proposed and formulated to evaluate $i^{\text{th}}(i>1)$ eigenvalues and eigenvectors of stochastic eigenvalue problems. Bases on the I-SIPM, the proposed SWDM can successfully evaluate i^{th} eigenvalues and eigenvectors.
3. In discrete 2-DOF example, the 1st and 2nd eigenvalues and eigenvectors are computed by the proposed I-SIPM and SWDM. The validity and feasibility of the proposed I-SIPM and SWDM are demonstrated for the eigen frequency problem in which the uncertainty exists in spring stiffness. The probabilistic characteristics (the response surface and the probability distribution of the response) can be accurately estimated by the proposed methods.
4. As an example of buckling eigenvalue problem, a continuous free-standing beam is considered with uncertainty of beam section dimension, H . The validity and feasibility of the proposed I-SIPM are demonstrated that (i) the proposed method can deal with continuous problem, (ii) the method can solve buckling problem, and (iii) the method is valid to solve multi degree of freedom problem. We can find that computational time is significantly reduced evidently by using proposed method compared with MCS with 10000 samples.
5. In the final example, the practicability of the proposed method is proved by the problem of resonance avoidance by the application of the dynamic damper. When the spring stiffness is considered as uncertain parameter, the probability of the natural frequency falling into the range to be avoided is obtained. It is understood that the probability of resonant high response could be significantly reduced by appropriate selection of the mass of the dynamic damper. And it is understood that the probability of resonant high response becomes larger when the mass is also considered as uncertain parameter. By the dynamic damper example, the validity and feasibility of the proposed I-SIPM is demonstrated to solve the two variables problem $\theta = \{\theta_1, \theta_2\}$ in which the uncertainty exists in both the spring stiffness and the mass.
6. In the proposed I-SIPM, the degree of freedom is increased compared to the deterministic IPM. However the overall results including response surface and probability distribution of the response can be obtained by solving the proposed method only once.
It is considered that the computational cost becomes higher by increase of random variables or increase of order of approximate expression. However, by using the proposed method, it is possible to obtain approximate response surface which is theoretically reasonable from the viewpoint of response surface approximation concept, which is different from the Monte Carlo simulation (MCS) with conventional eigenvalue analysis.
7. The proposed methods are very important for solving stochastic vibrations and buckling problem for safety assessment of ship and offshore structures. As a future work, the proposed methods can be used to deal with the shell structure with initial imperfection and can be also used to deal with multi degrees of freedom problem which consider uncertainty in the practical large structures such as ships and offshore structures.

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Appendix 1

Approximation method of $1/\sum_{i=0}^n Q_i \Psi_i(\theta)$

Here we show the approximation method. When denominator of $1/\sum_{i=0}^n Q_i \Psi_i(\theta)$ involves random variables (θ) , the orthogonality of basis functions cannot be used in the formulation. In this study, to overcome this problem, it is assumed that $1/\sum_{i=0}^n Q_i \Psi_i(\theta)$ can be approximated by Hermite PCE as follows.

$$\frac{1}{\sum_{i=0}^n Q_i \Psi_i(\theta)} = \sum_{u=0}^m A_p \Psi_u(\theta), \quad (70)$$

where, A_p are unknown coefficients of approximated polynomial, $\Psi_u(\theta)$ are the basis functions, represented by Hermite polynomial, and m is the order of expansion terms, m should be high enough so that the approximation becomes accurate enough.

The unknown coefficients, A_p , can be decided as follows:

Firstly, Eq. (70) is rewritten as follows:

$$\sum_{i=0}^n Q_i \Psi_i(\theta) \times \sum_{u=0}^m A_p \Psi_u(\theta) = 1. \quad (71)$$

Multiplying $\Psi_i(\theta)W(\theta)$ to both sides of Eq. (71), and integrating the equation, we have:

$$\begin{aligned} \int_{-\infty}^{\infty} \sum_{i=0}^n Q_i \Psi_i(\theta) \times \sum_{u=0}^m A_p \Psi_u(\theta) \Psi_i(\theta) W(\theta) d\theta \\ = \int_{-\infty}^{\infty} 1 \times \Psi_i(\theta) W(\theta) d\theta. \end{aligned} \quad (72)$$

This equation can be depicted as follows:

$$\sum_{i=0}^n \sum_{u=0}^m Q_i A_p \langle \Psi_i(\theta) \Psi_u(\theta) \Psi_i(\theta) \rangle = \langle \Psi_i(\theta) \rangle. \quad (73)$$

This means that the following simultaneous equations about the coefficients, A_p , are obtained.

$$\begin{bmatrix} \hat{Q}_{00} & \hat{Q}_{01} & \cdots & \hat{Q}_{0m} \\ \hat{Q}_{10} & \hat{Q}_{11} & \cdots & \hat{Q}_{1m} \\ \vdots & \vdots & \vdots & \vdots \\ \hat{Q}_{lm} & \hat{Q}_{2m} & \cdots & \hat{Q}_{mm} \end{bmatrix} \begin{bmatrix} A_{p_0} \\ A_{p_1} \\ \vdots \\ A_{p_m} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (74)$$

where $\hat{Q}_{iu} = \sum_{i=0}^n Q_i \langle \Psi_i(\theta) \Psi_u(\theta) \Psi_i(\theta) \rangle$. It is noted that $\langle \Psi_i(\theta) \Psi_u(\theta) \Psi_i(\theta) \rangle$ can be evaluated by the following equation.

$$\langle \Psi_i(\theta) \Psi_u(\theta) \Psi_i(\theta) \rangle = \int_D \Psi_i(\theta) \Psi_u(\theta) \Psi_i(\theta) W(\theta) d\theta. \quad (75)$$

Then the unknown coefficients of approximation polynomial, A_p , is derived by the above simultaneous equations.

Appendix 2

Derivation of coefficients, C_k , for

$$\sum_{i=0}^n \sum_{j=0}^n A_i B_j \Psi_i(\theta) \Psi_j(\theta) = \sum_{k=0}^m C_k \Psi_k(\theta).$$

In order to easy to understand, here the order n of expansion terms is defined as 2 and $\theta = \{\theta_1\}$ is considered to explain the derivation process. The expansion equation can be shown as follows.

$$\begin{aligned} \sum_{i=0}^2 \sum_{j=0}^2 A_i B_j \Psi_i(\theta) \Psi_j(\theta) = \\ (A_0 \Psi_0(\theta) + A_1 \Psi_1(\theta) + A_2 \Psi_2(\theta)) (B_0 \Psi_0(\theta) + B_1 \Psi_1(\theta) + B_2 \Psi_2(\theta)). \end{aligned} \quad (76)$$

Expand right of Eq.(76), we have:

$$\begin{aligned} \sum_{i=0}^2 \sum_{j=0}^2 A_i B_j \Psi_i(\theta) \Psi_j(\theta) = \\ = A_0 B_0 \Psi_0(\theta) \Psi_0(\theta) + A_0 B_1 \Psi_0(\theta) \Psi_1(\theta) + A_0 B_2 \Psi_0(\theta) \Psi_2(\theta) \\ + A_1 B_0 \Psi_1(\theta) \Psi_0(\theta) + A_1 B_1 \Psi_1(\theta) \Psi_1(\theta) + A_1 B_2 \Psi_1(\theta) \Psi_2(\theta) \\ + A_2 B_0 \Psi_2(\theta) \Psi_0(\theta) + A_2 B_1 \Psi_2(\theta) \Psi_1(\theta) + A_2 B_2 \Psi_2(\theta) \Psi_2(\theta), \end{aligned} \quad (77)$$

and referring to Table 1, we can understand $\Psi_0(\theta) = 1$, $\Psi_0(\theta) = \theta_1$, $\Psi_0(\theta) = \theta_1^2 - 1, \dots$. Substituting basis functions into Eq.(77), it can show as follows:

$$\begin{aligned} \sum_{i=0}^2 \sum_{j=0}^2 A_i B_j \Psi_i(\theta) \Psi_j(\theta) \\ = A_0 B_0 + A_0 B_1 \theta_1 + A_0 B_2 (\theta_1^2 - 1) \\ + A_1 B_0 \theta_1 + A_1 B_1 \theta_1^2 + A_1 B_2 (\theta_1^3 - \theta_1) \\ + A_2 B_0 (\theta_1^2 - 1) + A_2 B_1 (\theta_1^3 - \theta_1) + A_2 B_2 (\theta_1^4 - 2\theta_1^2 + 1). \end{aligned} \quad (78)$$

Simplify Eq.(78), we can obtain Eq.(79).

$$\begin{aligned} \sum_{i=0}^2 \sum_{j=0}^2 A_i B_j \Psi_i(\theta) \Psi_j(\theta) \\ = A_0 B_0 - A_0 B_2 - A_2 B_0 + A_2 B_2 \\ + (A_0 B_1 + A_1 B_0 - A_1 B_2 - A_2 B_1) \theta_1 \\ + (A_0 B_2 + A_1 B_1 + A_2 B_0 - 2A_2 B_2) \theta_1^2 \\ + (A_1 B_2 + A_2 B_1) \theta_1^3 \\ + A_2 B_2 \theta_1^4. \end{aligned} \quad (79)$$

We rewrite Eq.(79). So the $\sum_{i=0}^n \sum_{j=0}^n A_i B_j \Psi_i(\theta) \Psi_j(\theta)$ can be represented by basis functions, $\Psi_i(\theta)$.

$$\begin{aligned} \sum_{i=0}^2 \sum_{j=0}^2 A_i B_j \Psi_i(\theta) \Psi_j(\theta) \\ = (A_0 B_0 + A_1 B_1 - A_2 B_2 + 3A_2 B_2) \Psi_0(\theta) \\ + (A_0 B_1 + A_1 B_0 + 2A_1 B_2 + 2A_2 B_1) \Psi_1(\theta) \\ + (A_0 B_2 + A_1 B_1 + A_2 B_0 + 4A_2 B_2) \Psi_2(\theta) \\ + (A_1 B_2 + A_2 B_1) \Psi_3(\theta) \\ + A_2 B_2 \Psi_4(\theta). \end{aligned} \quad (80)$$

Then, C_k is derived as follows, and the order m is 4.

$$C_0 = A_0 B_0 + A_1 B_1 - A_2 B_2 + 3A_2 B_2, \quad (81-1)$$

$$C_1 = A_0 B_1 + A_1 B_0 + 2A_1 B_2 + 2A_2 B_1, \quad (81-2)$$

$$C_2 = A_1 B_2 + A_2 B_1, \quad (81-3)$$

$$C_4 = A_2 B_2. \quad (81-4)$$

Thus, $\sum_{i=0}^n \sum_{j=0}^n A_i B_j \Psi_i(\theta) \Psi_j(\theta) = \sum_{k=0}^m C_k \Psi_k(\theta)$ is derived. This derivation method is implemented in the developed program for arbitrary n .

Appendix 3

Firstly, the Eq. (36) with Eq.(5) ($\langle \Psi_0(\theta) \rangle = 1, \langle \Psi_{i \neq 1}(\theta) \rangle = 0$) is rewritten as follows in the form $\mathbf{F}(\mathbf{Z}) = \mathbf{0}$ where $\mathbf{Z}$ is a vector

containing the set $\{Z_0, Z_1, \dots, Z_{p_i}\}$.

$$\begin{aligned} f_1(Z_0, Z_1, \dots, Z_{p_i}) &= \sum_{i=0}^{p_i} \sum_{j=0}^{p_i} \sum_{k=0}^{2n} Z_i Z_j U_k^{(q+1)} \langle \Psi_i(\theta) \Psi_j(\theta) \Psi_k(\theta) \Psi_0(\theta) \rangle - 1 \\ f_2(Z_0, Z_1, \dots, Z_{p_i}) &= \sum_{i=0}^{p_i} \sum_{j=0}^{p_i} \sum_{k=0}^{2n} Z_i Z_j U_k^{(q+1)} \langle \Psi_i(\theta) \Psi_j(\theta) \Psi_k(\theta) \Psi_1(\theta) \rangle \\ &\vdots \\ f_{p_i+1}(Z_0, Z_1, \dots, Z_{p_i}) &= \sum_{i=0}^{p_i} \sum_{j=0}^{p_i} \sum_{k=0}^{2n} Z_i Z_j U_k^{(q+1)} \langle \Psi_i(\theta) \Psi_j(\theta) \Psi_k(\theta) \Psi_{p_i}(\theta) \rangle \end{aligned} \quad (82)$$

$\mathbf{Z} = \{Z_0, Z_1, \dots, Z_{p_i}\}$ are obtained by using following the iterative equation. More details about the Newton's method can be found in reference [16].

$$\mathbf{Z}^{(q_i+1)} = \mathbf{Z}^{(q_i)} + \delta(\mathbf{Z}^{(q_i)}), \quad (83)$$

where q_i is iteration number and $\delta(\mathbf{Z}^{(q_i)}) = -\mathbf{J}(\mathbf{Z}^{(q_i)})^{-1} \mathbf{F}(\mathbf{Z}^{(q_i)})$, $\mathbf{J}(\mathbf{Z}^{(q_i)})$ is depicted as follows.

$$\mathbf{J}(\mathbf{Z}^{(q_i)}) = \begin{pmatrix} \frac{\partial f_1(\mathbf{Z}^{(q_i)})}{\partial Z_0} & \dots & \frac{\partial f_1(\mathbf{Z}^{(q_i)})}{\partial Z_0} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{p_i+1}(\mathbf{Z}^{(q_i)})}{\partial Z_0} & \dots & \frac{\partial f_{p_i+1}(\mathbf{Z}^{(q_i)})}{\partial Z_0} \end{pmatrix}. \quad (84)$$

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