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OBSERVING A SOLID ANGLE FROM VARIOUS VIEWPOINTS

By

Yoichi Maeda

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Abstract. Let AOB be a triangle in $\mathbf{R}^3$. When we look at this triangle from various viewpoints, the angle $\angle AOB$ changes its appearance, and its 'visual size' is not constant. In [3], it is proved that the average visual size of $\angle AOB$ is equal to the true size of the angle when viewpoints are chosen at random on the surface of a sphere centered at O. In this paper, a simpler proof of this result is presented. Furthermore, we extend the result to the case of a solid angle in $\mathbf{R}^4$.

Introduction

Let $\angle AOB$ be a fixed angle determined by three points O, A, and B in the three dimensional Euclidean space $\mathbf{R}^3$. When we look at this angle, its appearance changes according to our viewpoint. The *visual angle* of $\angle AOB$ from a viewpoint P is defined as follows:

Definition 1. Let $\angle AOB$ be a fixed angle in $\mathbf{R}^3$. For a viewpoint P, let us denote by

$$\angle_P AOB$$

the dihedral angle of the two faces OAP and OBP of the (possibly degenerate) tetrahedron $POAB$. This angle $\angle_P AOB$ is called the *visual angle* of $\angle AOB$ from the viewpoint P. Its size (measure) is called the *visual size* of $\angle AOB$ from P, and denoted by $\measuredangle_P AOB$.

For an angle with fixed size, its visual size can vary from 0 to π in radians depending on the viewpoint.

For a given angle $\angle AOB$ in $\mathbf{R}^3$, take a random point P distributed uniformly on the unit sphere $\mathbf{S}^2$ centered at O. Then the visual size $\measuredangle_P AOB$ is a random variable, which is called the *random visual size* of $\angle AOB$.

Theorem 1. *For any angle $\angle AOB$, the expected value of the random visual*

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size $\measuredangle_P AOB$ is equal to the true size of $\angle AOB$, that is, $\mathbf{E}(\measuredangle_P AOB) = \measuredangle AOB$.

Thus, when we observe an angle from several viewpoints, each chosen at random, the average visual size is approximately equal to the true size. In [3], We proved this theorem using Santaló's chord theorem (see, [4]). In this paper, we will present a simpler proof of Theorem 1 in Section 1.

For a potential extension of Theorem 1, let us consider 'visual solid angle'. For a tetrahedron $OABC$ in the four dimensional Euclidean space $\mathbf{R}^4$, the triangular cone $\angle(O : \triangle ABC) := \cup_{X \in \triangle ABC} \overrightarrow{OX}$ is called the solid angle with vertex O. The area of the intersection of the unit sphere $\mathbf{S}^3$ with center O and the solid angle $\angle(O : \triangle ABC)$ is called the measure (steradian) of the solid angle $\angle(O : \triangle ABC)$, and it is denoted by $\measuredangle(O : \triangle ABC)$. The visual solid angle of $\angle(O : \triangle ABC)$ from a viewpoint P is defined as follows:

DEFINITION 2. Let $\angle(O : \triangle ABC)$ be a fixed solid angle in $\mathbf{R}^4$. For a viewpoint P, let us denote by

$$\angle_P(O : \triangle ABC)$$

the orthogonal projection of $\angle(O : \triangle ABC)$ into the hyperplane through P and perpendicular to the line PO. This solid angle $\angle_P(O : \triangle ABC)$ is called the *visual solid angle* of $\angle(O : \triangle ABC)$ from the viewpoint P. Its measure is called the *visual measure* of $\angle(O : \triangle ABC)$ from P, and denoted by $\measuredangle_P(O : \triangle ABC)$.

For a solid angle with fixed measure, its visual measure can vary from 0 to 2π in steradians depending on the viewpoint as we will see in Section 2.

For a given solid angle $\angle(O : \triangle ABC)$ in $\mathbf{R}^4$, take a random point P distributed uniformly on the unit sphere $\mathbf{S}^3$ centered at O. Then the visual measure $\measuredangle_P(O : \triangle ABC)$ is a random variable, which is called the *random visual measure* of $\angle(O : \triangle ABC)$.

THEOREM 2. *For any solid angle $\angle(O : \triangle ABC)$, the expected value of the random visual measure $\measuredangle_P(O : \triangle ABC)$ is equal to the true measure of $\angle(O : \triangle ABC)$, that is,* $\mathbf{E}(\measuredangle_P(O : \triangle ABC)) = \measuredangle(O : \triangle ABC)$.

1. Proof of Theorem 1

Let $\angle AOB$ be an angle of size $\measuredangle AOB$, and let P be a random point on the unit sphere $\mathbf{S}^2$ centered at O in $\mathbf{R}^3$. We may suppose that A and B lie on $\mathbf{S}^2$. Then the spherical distance $\widehat{AB}$ between A and B is equal to $\measuredangle AOB$. (We denote the shortest geodesic connecting A and B, and its length by the same notation $\widehat{AB}$.) Notice that $\measuredangle_P AOB$ is equal to the interior angle $\measuredangle P$ of the spherical triangle $\triangle APB$.

Let us assume that two points A and B are on the equator of $\mathbf{S}^2$. If it is proved that the expected value $\mathbf{E}(\measuredangle_P AOB)$ restricted to any fixed latitude meridian is equal to $\measuredangle AOB$, the proof of Theorem 1 has completed. Hence, in the rest of the proof, let us restrict the random point P to any fixed latitude meridian $L_\phi := \{P \in \mathbf{S}^2 \mid \measuredangle NOP = \phi\}$ where N is the north pole of $\mathbf{S}^2$.

First, let us prove the case of $\measuredangle AOB = 2\pi/n$ where n is an integer greater than 1. Divide the equator into n equal parts,

$$\widehat{A_1A_2} = \widehat{A_2A_3} = \cdots = \widehat{A_{n-1}A_n} = \widehat{A_nA_1} = 2\pi/n.$$

Then, for any point P,

$$\measuredangle_P A_1OA_2 + \measuredangle_P A_2OA_3 + \cdots + \measuredangle_P A_{n-1}OA_n + \measuredangle_P A_nOA_1 = 2\pi. \tag{1}$$

By the rotation with the axis ON and angle $2\pi/n$, the restricted expected value $\mathbf{E}|_{L_\phi}(\measuredangle_P A_2OA_3)$ is equal to $\mathbf{E}|_{L_\phi}(\measuredangle_P A_1OA_2)$, and so on. Therefore, taking the expectation of Equation (1), the linearity of expectation implies that

$$n\mathbf{E}|_{L_\phi}(\measuredangle_P A_1OA_2) = 2\pi. \tag{2}$$

Equation (2) shows that $\mathbf{E}|_{L_\phi}(\measuredangle_P AOB) = \measuredangle AOB$ in the case of $\measuredangle AOB = 2\pi/n$.

In the similar way, we can prove that $\mathbf{E}|_{L_\phi}(\measuredangle_P AOB) = \measuredangle AOB$ in the case of $\measuredangle AOB = q\pi$ where q is a rational number less than 1.

Finally, it is clear that the expected value $\mathbf{E}|_{L_\phi}(\measuredangle_P AOB)$ is a continuous and monotone increasing function of the size of $\angle AOB$. Therefore, we can prove that $\mathbf{E}|_{L_\phi}(\measuredangle_P AOB) = \measuredangle AOB$ in the case of $\measuredangle AOB = r\pi$ where r is a real number less than 1. We have completed the proof of Theorem 1. ■

2. Proof of Theorem 2

Let $\angle(O : \triangle ABC)$ be a solid angle of measure $\measuredangle(O : \triangle ABC)$, and let P be a random point on the unit sphere $\mathbf{S}^3$ centered at O in $\mathbf{R}^4$. We may suppose that A, B and C lie on $\mathbf{S}^3$. Since the tangent space $T_P\mathbf{S}^3$ is orthogonal to the line OP, the visual solid angle $\angle_P(O : \triangle ABC)$ is realized in $T_P\mathbf{S}^3$. Using the fact that for $X \in \mathbf{S}^3$, the orthogonal projection of $\overrightarrow{OX}$ is a vector tangent to the geodesic arc $\widehat{PX}$ at P, $\angle_P(O : \triangle ABC)$ is the solid angle at P of the spherical tetrahedron $PABC$ in $\mathbf{S}^3$. Note that if $\triangle ABC$ is a hemisphere (A, B and C lie on a great circle), then the spherical tetrahedron $PABC$ is a great sphere in $\mathbf{S}^3$, hence, $\measuredangle_P(O : \triangle ABC)$ is equal to 2π for any $P \in \mathbf{S}^3$. In this way, for a solid angle with fixed measure, its visual measure can vary from 0 to 2π in steradians depending on the viewpoint.

For the proof of Theorem 2, we prepare several subsets of $\mathbf{S}^3$. Let

$$
\begin{aligned}
S_0 &:= \{(x,y,z,w) \in \mathbf{S}^3 \mid w = 0\ \} \text{ (great sphere in } \mathbf{S}^3),\\
S_1 &:= \{(x,y,z,w) \in \mathbf{S}^3 \mid w = w_0\} \text{ (small sphere in } \mathbf{S}^3),\\
C_0 &:= \{(x,y,z,w) \in S_0 \mid z = 0\ \} \text{ (great circle in } \mathbf{S}^3),\\
C_1 &:= \{(x,y,z,w) \in S_1 \mid z = z_0\} \text{ (small circle in } \mathbf{S}^3).
\end{aligned}
$$

In the following argument, we assume that three points A, B and C lie on S_0 without loss of generality. Similarly as the proof of Theorem 1, it is enough to prove that for any $w_0 \in [-1,1]$, the restricted expected value of $\mathbf{E}(\measuredangle_P(O : \triangle ABC))$ to S_1 is equal to the true solid angle $\measuredangle(O : \triangle ABC)$.

The proof of Theorem 2 is similar to that of the Girard's formula in spherical geometry([1] pp.278-279, [2] p.51).

Now, we will define a sector-like solid angle:

$$
\begin{aligned}
\angle(O : A\text{-sec.}) &:= \angle(O : \triangle ABC) \cup \angle(O : \triangle A^*BC),\\
\angle(O : B\text{-sec.}) &:= \angle(O : \triangle ABC) \cup \angle(O : \triangle AB^*C),\\
\angle(O : C\text{-sec.}) &:= \angle(O : \triangle ABC) \cup \angle(O : \triangle ABC^*).
\end{aligned}
$$

where A^*, B^* and C^* are the antipodal points of A, B and C, respectively. (Notice that if $B' \in \widehat{AB} \cup \widehat{BA^*}$, $C' \in \widehat{AC} \cup \widehat{CA^*}$, then $\angle(O : \triangle ABC) \cup \angle(O : \triangle A^*BC) = \angle(O : \triangle AB'C') \cup \angle(O : \triangle A^*B'C')$. Hence $\angle(O : A\text{-sec.})$ depends only on the "lune" ABA^*CA.)

Lemma 3. *For a given point V on S_0, let V^* be the antipodal point. Two great circles on S_0 meeting at an angle θ at V bound a solid angle $\angle(O : V\text{-sec.})$. Then,*

$$\mathbf{E}|_{S_1}(\measuredangle_P(O : V\text{-}sec.)) = \measuredangle(O : V\text{-}sec.).$$

Proof. Without loss of generality, we can assume that $V = (0,0,1,0)$ and $V^* = (0,0,-1,0)$ on S_0. Divide the great circle C_0 into n equal parts,

$$\widehat{A_1A_2} = \widehat{A_2A_3} = \cdots = \widehat{A_{n-1}A_n} = \widehat{A_nA_1} = 2\pi/n.$$

Then,

$$
\begin{aligned}
&\measuredangle(O : \triangle VA_1A_2) = \measuredangle(O : \triangle VA_2A_3)\\
&\qquad = \cdots = \measuredangle(O : \triangle VA_{n-1}A_n) = \measuredangle(O : \triangle VA_nA_1) = 2\pi/n.
\end{aligned}
$$

For any point $P \in S_1$,

$$\measuredangle_P(O : \triangle VA_1A_2) + \measuredangle_P(O : \triangle VA_2A_3) \\ + \cdots + \measuredangle_P(O : \triangle VA_{n-1}A_n) + \measuredangle_P(O : \triangle VA_nA_1) = 2\pi,$$

since the visual measure of a hemisphere is equal to 2π. Now, let us restrict the random point P to the small circle C_1 for any fixed $z_0 \in [-1, 1]$. By the rotation with the matrix

$$\begin{pmatrix} \cos 2\pi/n & -\sin 2\pi/n & 0 & 0 \\ \sin 2\pi/n & \cos 2\pi/n & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

the restricted expected value $\mathbf{E}|_{C_1}(\measuredangle_P(O : \triangle VA_2A_3))$ is equal to $\mathbf{E}|_{C_1}(\measuredangle_P(O : \triangle VA_1A_2))$, and so on. Therefore,

$$n\mathbf{E}|_{C_1}(\measuredangle_P(O : \triangle VA_1A_2)) = 2\pi.$$

This implies that $\mathbf{E}|_{C_1}(\measuredangle_P(O : \triangle VA_1A_2)) = \measuredangle(O : \triangle VA_1A_2)$ in the case of $\measuredangle A_1VA_2 = 2\pi/n$. Similar arguments in the proof of Theorem 1 show that $\mathbf{E}|_{C_1}(\measuredangle_P(O : \triangle VA_1A_2)) = \measuredangle(O : \triangle VA_1A_2)$ in the case of $\measuredangle VA_1A_2 = \measuredangle VA_2A_1 = \pi/2$ and $\measuredangle A_1VA_2 \in (0, \pi)$.

In the next place, the equation $\mathbf{E}|_{C_1}(\measuredangle_P(O : \triangle VA_1A_2) = \measuredangle(O : \triangle VA_1A_2)$ implies that

$$\mathbf{E}|_{S_1}(\measuredangle_P(O : \triangle VA_1A_2) = \measuredangle(O : \triangle VA_1A_2).$$

Finally, since $\angle(O : V\text{-sec.}) = \angle(O : \triangle VA_1A_2) \cup \angle(O : \triangle V^*A_1A_2)$,

$$\mathbf{E}|_{S_1}(\measuredangle_P(O : V\text{-sec.})) = \measuredangle(O : V\text{-sec.}).$$

We have completed the proof of Lemma 3. ■

For a general solid angle $\angle(O : \triangle ABC)$, we prepare three sector-like solid angles $\angle(O : A\text{-sec.})$, $\angle(O : B\text{-sec.})$ and $\angle(O : C\text{-sec.})$. Then, using the same technique of the proof of Girard's formula in spherical geometry,

$$\measuredangle(O : \triangle ABC) = \{\measuredangle(O : A\text{-sec.}) + \measuredangle(O : B\text{-sec.}) + \measuredangle(O : C\text{-sec.}) - 2\pi\}/2. \quad (3)$$

In the same way,

$$\measuredangle_P(O : \triangle ABC) = \{\measuredangle_P(O : A\text{-sec.}) + \measuredangle_P(O : B\text{-sec.}) + \measuredangle_P(O : C\text{-sec.}) - 2\pi\}/2. \quad (4)$$

Taking the expectation of Equation (4) on S_1, Equations (3) and (4) and Lemma 3 imply that

$$
\begin{aligned}
&\mathbf{E}|_{S_1}(\measuredangle_P(O : \triangle ABC)) \\
&= \mathbf{E}|_{S_1}(\measuredangle_P(O : A\text{-sec.}) + \measuredangle_P(O : B\text{-sec.}) + \measuredangle_P(O : C\text{-sec.}) - 2\pi)/2 \\
&= \{\measuredangle(O : A\text{-sec.}) + \measuredangle(O : B\text{-sec.}) + \measuredangle(O : C\text{-sec.}) - 2\pi\}/2 \\
&= \measuredangle(O : \triangle ABC).
\end{aligned}
$$

It is trivial that we can relax the restriction from S_1 to the whole space $\mathbf{S}^3$. We have completed the proof of Theorem 2. ■

Finally, as a degenerated case, let us consider the case that a three dimensional being such as ourselves observes a solid angle from various viewpoints. This special case corresponds with the case that our viewpoints P is in S_0, and the tangent space $T_P\mathbf{S}^3$ degenerates to two dimensional plane. According as the three tangent vectors lie on a half plane or not, the visual measure takes 0 or 2π. Since for any $w_0 \in [-1, 1]$, $\mathbf{E}|_{S_1}(\measuredangle_P(O : \triangle ABC)) = \measuredangle(O : \triangle ABC)$, so especially,
$\mathbf{E}|_{S_0}(\measuredangle_P(O : \triangle ABC)) = \measuredangle(O : \triangle ABC)$. This fact indicates that when we observe an solid angle from several viewpoints in $\mathbf{R}^3$, each chosen at random, the average visual measure is approximately equal to the true measure of the solid angle.

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Department of Mathematics,
Faculty of Sciences,
Tokai University,
1117 Kitakaname, Hiratsuka,
Kanagawa 259-1292,
JAPAN.
E-mail address: `maeda@keyaki.cc.u-tokai.ac.jp`