Periodic solutions for evolution equations in Hilbert spaces

By

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Abstract. We consider the existence of periodic solutions of the problem $g(t, u) \in u' + Au$, where $A$ is a maximal monotone operator defined in a Hilbert space and $g : \mathbb{R} \times H \rightarrow H$ is a Caratheodory function periodic with respect to the first variable.

1. Introduction

In this paper, $H$ is a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$, denote by $\| \cdot \|$ the norm induced by $\langle \cdot, \cdot \rangle$. Let $A : D(A) \subseteq H \rightarrow 2^H$ be a maximal monotone operator, we consider the existence of periodic solutions for nonlinear evolution equations of the form

$$g(t, u) \in \frac{du}{dt} + Au, t \in \mathbb{R}$$

where $g : \mathbb{R} \times H \rightarrow H$ is a Caratheodory function.

If $g$ is Lipschitz or continuous with respect to the second variable, existence of periodic solutions for (1.1) has been studied by many authors. See [3], [5], [8], [9], [11], [13], [14]. When $g$ is a Caratheodory function, periodic solutions for (1.1) has been studied by [12], [14] under other assumptions on $g$. In this paper, we shall give different assumptions on $A$ and $g$ than that of [12], [14]. Now, we state our result.

Theorem 1 Let $A : D(A) \subseteq H \rightarrow 2^H$ be a maximal monotone operator. $g : \mathbb{R} \times H \rightarrow H$ is a Caratheodory operator and $g$ is $T$-periodic with respect to the first variable. Suppose the following conditions are satisfied

1. For some $\lambda > 0$, $J_\lambda = (I + \lambda^{-1}A)^{-1}$ is a compact operator on $H$;

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2. There exist $M_1, M_2 > 0$, such that
\[ \|g(t, v)\| \leq M_1 \|v\| + M_2, \forall (t, v) \in R \times H; \]
\[ (z - g(t, J_{\lambda}v), v) \geq -\frac{a}{\lambda} \|J_{\lambda}v\|, \forall (t, v) \in R \times D(A), z \in A_v; \lambda > 0, \text{ where} a > 0 \text{ is a constant}; \]

Then (1.1) has a $T$-periodic solution.

Our approach is different from that of [12] and [14], the assumptions 1 and 2 are the same as in [12], but 3 is different from (*) of [12]. In [12], $A$ is assumed to be a subdifferential of a lower semicontinuous convex function, but we do not require this condition.

2. Proof of Theorem

Let $L^2(0, T; H)$ be the space of functions $v : [0, T] \rightarrow H$ such that $\int_0^T \|v\|^2 dt < +\infty$. The norm and the inner product of $L^2(0, T; H)$ are denoted by $\|\cdot\|_T$ and $\langle \cdot, \cdot \rangle$ respectively. We identify the function in $L^2(0, T, H)$ with $T$-periodic functions. $W^{1,2}(0, T; H)$ represents the space of functions $v : [0, T] \rightarrow H$ such that $v, v^{(1)} \in L^2(0, T; H)$, where $v^{(1)}$ denotes the 1st derivative in the sense of distribution.

Let $A_{\lambda} = \lambda(I - J_{\lambda})$ for $\lambda > 0$, then $A_{\lambda}x \in AJ_{\lambda}x$ for $x \in H$. It is well known that $J_{\lambda}$ is nonexpansive and $J_{\lambda}x = J_u(J_{\lambda}x + u^{-1}A_{\lambda}x)$ for $\lambda, u > 0$ and $x \in H$ (see [4]).

In this paper, a $T$-periodic solution $u(\cdot)$ of (1.1) means that $u \in W^{1,2}(0, T; H)$, $u(t) \in D(A)$ for almost all $t > 0$ and there exists $v(\cdot) \in L^2(0, T; H)$, such that $v(t) \in Au(t)$ for almost all $t > 0$, and $\langle \frac{dv}{dt} + v(t) - g(t, v), y \rangle = 0$ for all $y \in W^{1,2}(0, T; H)$ verifying $y(0) = y(T)$.

Without loss of generality, we assume $0 \in D(A)$ and $0 \in A0$. Let $A : L^2(0, T; H) \rightarrow L^2(0, T; H)$ be as following

\[ Au(\cdot) = \{v(\cdot) \in L^2(0, T; H)|u(t) \in D(A), v(t) \in Au(t); a.e. t \in [0, T]\} \]

It's known that $A : L^2(0, T; H) \rightarrow L^2(0, T; H)$ is maximal monotone (see [1]).

Let $W = \{v : R \rightarrow H, v(t+T) = v(t) \text{ for } t \in R \text{ and } v_{[0,T]} \in W^{1,2}(0, T; H)\}$ endowed with the norm $\|\cdot\|_{1,T}$ of $W^{1,2}(0, T; H)$, i.e. $\|v\|_{1,T}^2 = \|\frac{dv}{dt}\|_T^2 + \|v\|_T^2$ for $v \in W$;
Now, we define an operator $T_n : W \rightarrow W^*$ by

$$
\langle T_n u, v \rangle = \frac{1}{n} \left( \int_0^T \frac{du}{dt}, \frac{d(u - u)}{dt} \right) + \frac{1}{n} \langle u, u \rangle + \left\langle \frac{dv}{dt} - g(t, J_n u), v \right\rangle \quad \text{for } u, v \in W. \quad (2.1)
$$

For a reflexive Banach space $E$, an operator $T : E \rightarrow E^*$ is said to be an operator of $(S_+)$, if $u_n$ converges to $u$ weakly in $E$ and $\lim_{n \to \infty} \langle Tu_n, u_n - u \rangle \leq 0$ imply that $u_n$ converges to strongly in $E$ and $Tu_n$ has a subsequence converging weakly to $Tu$. (see [6])

**Lemma 2.1** For $n \geq 1$, the operator $T_n$ defined by (2.1) is an operator of $(S_+)$. 

**Proof.** For fixed $n \geq 1$, let $(u_i) \subset W$ be a sequence such that $u_i \rightharpoonup u \in W$ weakly in $W$ and

$$
\lim_{i \to \infty} \left[ \frac{1}{n} \left\langle \frac{du_i}{dt}, \frac{d(u_i - u)}{dt} \right\rangle + \frac{1}{n} \langle u_i, u_i - u \rangle + \left\langle \frac{dv_i}{dt} - g(t, J_n u_i), u_i - u \right\rangle \right] \leq 0
$$

Since $J_n$ is compact and nonexpansive, so $J_n u_i$ is relatively compact in $L^2(0, T; H)$, and further that $\{g(t, J_n u_i)\}$ is also relatively compact in $L^2(0, T; H)$. So

$$
\lim_{i \to \infty} \left\langle \frac{dv_i}{dt} - \frac{du_i}{dt}, u_i - u \right\rangle = 0.
$$

Since $\lim_{i \to \infty} \langle \frac{du_i}{dt}, u_i \rangle = \lim_{i \to \infty} \int_0^T \frac{1}{2} \frac{d}{dt} \|u_i\|^2 dt = 0,$

$$
\lim_{i \to \infty} \langle \frac{du_i}{dt}, u_i \rangle = \lim_{i \to \infty} \langle u_i, \frac{du_i}{dt} \rangle = 0.
$$

Therefore, we have

$$
\lim_{i \to \infty} \left[ \frac{1}{n} \left\langle \frac{du_i}{dt}, \frac{d(u_i - u)}{dt} \right\rangle + \frac{1}{n} \langle u_i, u_i - u \rangle \right] \leq 0.
$$

Hence

$$
\lim_{i \to \infty} \int_0^T \frac{1}{n} \left\| \frac{du_i}{dt} - \frac{dv_i}{dt} \right\|^2 dt + \int_0^T \frac{1}{n} \|u_i - u\|^2 dt \leq 0.
$$

So we get $u_i \rightarrow u$ strongly in $W$.

**Lemma 2.2** There exists $R_0 > 0$, such that

$$
\langle T_n u, u \rangle + \langle v, u \rangle + 0, \forall u \in W, \|u\|_{1,T} = R_0 \text{ and } u(\cdot) \in D(A), v(\cdot) \in Au(\cdot).
$$

**Proof.** Suppose $u \in W$, and $u \in D(A), v \in A_u$.

Since

$$
\langle T_n u, u \rangle = \frac{1}{n} \int_0^T \|\frac{du}{dt}\|^2 dt + \frac{1}{n} \int_0^T \|u\|^2 dt
$$
\[
+ \int_0^T \left\langle \frac{du}{dt}, u \right\rangle dt + \int_0^T \left\langle v - g(t, J_n u), u \right\rangle dt
= \frac{1}{n} \|u\|_{1,T}^2 + \int_0^T \left\langle v - g(t, J_n u), u \right\rangle dt
\]

By the assumption 3, we have
\[
\int_0^T \left\langle v - g(t, J_n u), u \right\rangle dt \geq -\frac{a}{n} \int_0^T \|J_n u\| dt.
\]

Since \(0 \in A_0\), so \(\|J_n u\| \leq \|u\|\). Therefore, we have
\[
\int_0^T \left\langle v - g(t, J_n u), u \right\rangle dt \geq -\frac{a}{n} \int_0^T \|u\| dt \geq -\frac{a}{n} \int_0^T \|u\|^2 dt \sqrt{T}
\]

Hence \(T_n u, u \gg + \left\langle v, u \right\rangle \geq \frac{1}{n} \|u\|_{1,T} (\|u\|_{1,T} - a \sqrt{T})\). Let \(R_0 > a \sqrt{T}\), we get the desired result.

In the following, we denote by \(A_\lambda = (\lambda I + \mathcal{A}^{-1})^{-1}, \ R_\lambda = I - \lambda A_\lambda, \lambda > 0\).

**Lemma 2.3** Let \(R_0\) be the same as in Lemma 2.2. Then for each \(n \geq 1\), there exists \(\lambda_0^n > 0\), such that
\[
0 \notin \bigcup_{i \in [0,1]} t(T_n + A_\lambda) + (1 - t)J(\partial B_{R_0}), \forall \lambda \in (0, \lambda_0^n) \quad (2.2)
\]
where \(J : W \rightarrow W^*\) is the dual mapping, \(B_{R_0} = \{u \in W, \|u\|_{1,T} < R_0\}\).

**Proof.** Suppose \((2.2)\) is not true. There exist \(\lambda_i \rightarrow 0^+, t_j \rightarrow t_0, u_j \in \partial B_{R_0}\), with \(u_j \rightharpoonup u_0\) and \(\frac{du_i}{dt} \rightharpoonup \frac{du_0}{dt}\) weakly in \(L^2(0,T;H)\), such that
\[
t_j(T_n u_j + A_\lambda u_j) + (1 - t_j)J u_j = 0 \quad (2.3)
\]

Case(i). \(t_0 = 0\). Since \(A_\lambda 0 = 0\), \(\left\langle A_\lambda u_j, u_j \right\rangle \geq 0\), and \(\left\langle t_j T_n u_j, u_j \right\rangle \rightarrow 0\) as \(j \rightarrow \infty\).

So \(\lim_{i \rightarrow \infty} \left\langle Ju_j, u_j \right\rangle = \lim_{i \rightarrow \infty} \|u_j\|_{1,T}^2 \leq 0\). Therefore, we have \(u_j \rightarrow 0 \in \partial B_{R_0}\), a contradiction.

Case(ii). \(t_0 \neq 0\). Times \((2.3)\) by \(A_\lambda u_j\), we get
\[
t_j \left\langle T_n u_j, A_\lambda u_j \right\rangle + t_j \int_0^T \|A_\lambda u_j\|^2 dt + (1 - t_j) \left\langle Ju_j, A_\lambda u_j \right\rangle = 0
\]
and $\ll Tn u_j, A_{\lambda_j} u_j \gg = \ll \frac{1}{n} \frac{du_j}{dt} , \frac{dA_{\lambda_j} u_j}{dt} \gg + \ll \frac{1}{n} u_j A_{\lambda_j} u_j \gg$

$+ \ll \frac{du_j}{dt} - g(t, Jn u_j), A_{\lambda_j} u_j \gg$.

The monotonicity of $A_{\lambda_j}$ implies that

$\ll \frac{du_j}{dt} , \frac{dA_{\lambda_j} u_j}{dt} \gg \geq 0, \ll u_j A_{\lambda_j} u_j \gg \geq 0$.

So we have

$$\int_0^T \|A_{\lambda_j} u_j\|^2 dt \leq -\frac{1-t_j}{t_j} \ll Ju_j, A_{\lambda_j} u_j \gg - \ll \frac{du_j}{dt} - g(t, Jn u_j), A_{\lambda_j} u_j \gg.$$  

(2.4)

By the assumption 2, we know $(A_{\lambda_j} u_j)$ is bounded in $L^2(0,T;H)$.

Without loss of generality, we may assume $A_{\lambda_j} u_j \rightarrow f_0$ weakly in $L^2(0,T;H)$, $Tn u_j \rightarrow f_1, Ju_j \rightarrow f_2$ weakly in $W^*$ (otherwise, taking a subsequence).

By (2.3), we have

$$t_0 (f_1 + f_0) + (1-t_0) f_2 = 0.$$  

(2.5)

Again, by (2.3), we get

$$\ll Tn u_j, u_j - u_0 \gg + \ll A_{\lambda_j} u_j, u_j - u_0 \gg + \ll \frac{1-t_j}{t_j} Ju_j, u_j - u_0 \gg = 0.$$  

Since $T_n, J$ are operators of $(S_+)$,

so $\lim_{j \rightarrow \infty} \ll A_{\lambda_j} u_j, u_i - u_0 \gg \leq 0$ i.e. $\lim_{j \rightarrow \infty} \ll A_{\lambda_j} u_j, u_i \gg \leq \ll f_0, u_0 \gg$.

(2.6)

By the monotonicity of $A$, we have

$$\ll z - A_{\lambda_j} u_j, v - R_{\lambda_j} u_j \gg \geq 0, v \in D(A), z \in A v;$$

$$\ll z - A_{\lambda_j} u_j, v - u_j + \lambda_j A_{\lambda_j} u_j \gg \geq 0, v \in D(A), z \in A v.$$

Letting $j \rightarrow \infty$, we get

$$\ll f_0, u_0 \gg \geq \lim_{j \rightarrow \infty} \ll A_{\lambda_j} u_j, u_0 \gg \ll -z, v - u_0 \gg + \ll f_0, v \gg , v \in D(A), z \in A v;$$

i.e. $\ll f_0 - u_0 \gg \geq , v \in D(A), z \in A v$.

The maximality of $A$ implies that

$$u_0 \in D(A), \text{ and } f_0 \in A v.$$  

(2.7)
Now, we have $A_{\lambda_j} u_0 \rightarrow A^0 u_0$ strongly in $L^2(0, T; H)$ (see[10], Th23.3). Since $A_{\lambda_j} u_j - A_{\lambda_j} u_0$, $u_j - u_0 \gg \geq 0$, so it follows from (2.3), we get

$$\lim_{j \rightarrow \infty} \ll t_j T_n u_j + (1 - t_j) J u_j, u_j - u_0 \gg \leq 0.$$ 

Both $T_n$ and $I$ are operators of $(S_+).$ So $u_j \rightarrow u_0$ strongly in $W$, and

$$T_n u_j \rightarrow F_1 = T_n u_0, J u_j \rightarrow J u_0 = f_2.$$ 

The maximal monotonicity of $A$ also implies that $A_{\lambda_j} u_j \rightarrow f_0 \in Au_0$. In view of [2.5], we get

$$0 \in t_0(T_n + A)u_0 + (1 - t_o)Jv_o, v_o \in \partial B_{R_0} \cap D(A).$$

It is a contradiction to Lemma 2.2. We complete the proof.

**Lemma 2.4** For each $n \geq 1$, there exists an integer $P_n \geq n$, and $u_n \in W, \|u_n\|_{1,T} < R_0$, such that

$$T_n u_n + A_{1/P_n} u_n = 0,$$ 

where $R_0$ is the same as in lemma 2.3.

**Proof.** By lemma 2.3 and [6], we have

$$\deg(T_n + A_{\lambda}, B_{R_0}, 0) = \deg(J, B_{R_0}, 0) = 1, \forall \lambda \in (0, \lambda_0^0).$$

So $(T_n + A_{\lambda}) u = 0$ has a solution in $B_{R_0}$ for each $\lambda \in (0, \lambda_0^0)$. Taking an integer $P_n \geq n$, such that $\frac{1}{P_n} < \lambda_0^0$, then there is a $u_n \in B_{R_0}$, such that

$$T_n u_n + A_{1/P_n} u_n = 0.$$ 

This ends the proof.

**Proof of Theorem.** By the above Lemmas, for each $n \geq 1$, there exists an integer $P_n \geq n$, $u_n \in W, \|u_n\|_{1,T} < R_0$, such that

$$T_n u_n + A_{1/P_n} u_n = 0.$$ 

(2.8)

We multiply (2.8) by $A_{1/P_n} u_n$ and integrate over $[0, T]$, then we get

$$\int_0^T \langle \frac{1}{n} \frac{du_n}{dt}, \frac{dA_{1/P_n} u_n}{dt} \rangle dt + \int_0^T \langle \frac{1}{n} u_n, A_{1/P_n} u_n \rangle dt + \int_0^T \langle \frac{du_n}{dt} - g(t, J_{n} u_n), A_{1/P_n} u_n \rangle dt + \int_0^T \| A_{1/P_n} u_n \|^2 dt = 0.$$ 

By the monotonicity of $A_{1/P_n}$, we have $\langle \frac{du_n}{dt}, \frac{dA_{1/P_n} u_n}{dt} \rangle \geq 0$, and $\langle u_n, A_{1/P_n} u_n \rangle \geq 0$. 

So we get

\[ \int_{0}^{T} \| A_{\frac{1}{n}} u_{n} \|^2 dt \leq \int_{0}^{T} \langle g(t, J_{n} u_{n}) - \frac{du_{n}}{dt}, A_{\frac{1}{n}} u_{n} \rangle dt. \] (2.9)

Since \( \| u_{n} \|_{1,T} < R_{0} \), and the assumption 2 implies that

\[ \sup_{n \geq 1} \int_{0}^{T} \| A_{\frac{1}{n}} u_{n} \|^2 dt < +\infty. \] (2.10)

Now, we prove

\[ \sup_{n \geq 1} \int_{0}^{T} \| A_{\frac{1}{n}} u_{n} \|^2 dt < +\infty. \] (2.11)

By the monotonicity of \( A \), we have

\[ \ll A_{\frac{1}{n}} u_{n} - A_{\frac{1}{P_{n}}} u_{n}, R_{\frac{1}{n}} u_{n} - R_{\frac{1}{P_{n}}} u_{n} \gg \geq 0, \]
\[ \ll A_{\frac{1}{n}} u_{n} - A_{\frac{1}{n}} u_{n}, u_{n} - \frac{1}{n} A_{\frac{1}{n}} u_{n} - R_{\frac{1}{n}} u_{n} \gg \geq 0, \]
i.e. \[ \ll A_{\frac{1}{n}} u_{n} - A_{\frac{1}{n}} u_{n}, -\frac{1}{n} A_{\frac{1}{n}} u_{n} + \frac{1}{P_{n}} A_{\frac{1}{P_{n}}} u_{n} \gg \geq 0. \]
\[ \ll A_{\frac{1}{n}} u_{n}, A_{\frac{1}{n}} u_{n} \gg \leq \frac{1}{P_{n}} + \frac{1}{n} \ll A_{\frac{1}{n}} u_{n}, A_{\frac{1}{n}} u_{n} \gg \]
\[ \ll -\frac{1}{P_{n}} \ll A_{\frac{1}{P_{n}}} u_{n}, A_{\frac{1}{P_{n}}} u_{n} \gg. \]

So \[ \int_{0}^{T} \| A_{\frac{1}{n}} u_{n} \|^2 dt \leq \left( 1 + \frac{n}{P_{n}} \right) \int_{0}^{T} \langle A_{\frac{1}{n}} u_{n}, A_{\frac{1}{n}} u_{n} \rangle dt - \frac{n}{P_{n}} \int_{0}^{T} \| A_{\frac{1}{P_{n}}} u_{n} \|^2 dt. \]

Since \( n \leq P_{n} \) and by (2.10), we get

\[ \sup_{n \geq 1} \int_{0}^{T} \| A_{\frac{1}{n}} u_{n} \|^2 dt < +\infty. \]

So (2.11) holds.

Since \( \| u_{n} \|_{1,T} < R_{0}, n \geq 1 \), so \[ \int_{0}^{T} \| \frac{du_{n}}{dt} \|^2 dt < R_{0}^{2}, \] then it follows that

\[ \int_{0}^{T} \| \frac{d(J_{n} u_{n})}{dt} \|^2 dt \leq R_{0}^{2}, n \geq 1 \] (2.12)

By (2.12), we get

\[ \sup\{ \| J_{n} u_{n}(t) \| : n \geq 1, 0 \leq t \leq T \} < +\infty \] (2.13)

Let us remark that \( A_{\frac{1}{n}} u_{n}(t) = A_{n} u_{n}(t), t \in [0, T], \) (2.11) becomes that

\[ \int_{0}^{T} \| A_{n} u_{n}(t) \|^2 dt < +\infty \] (2.14)
We show that \( \{ J_n u_n \} \) is relatively compact in \( L^2(0, T; H) \). Let \( \epsilon > 0 \). By (2.12) and (2.13), using the same argument of [12], there exists an integer \( m_0 > 0 \) such that
\[
\| J_n u_n(t) - J_n u_n(s) \|^2 < \frac{\epsilon}{6T}, \forall n \geq 1, |t - s| < \frac{2T}{m_0}.
\] (2.15)

On the other hand, there exists \( D > 0 \) such that
\[
\inf \{ \| A_n u_n(\tau) \| : t \leq \tau \leq t + \frac{T}{m_0} \} < D, \forall n \geq 1, 0 \leq t \leq (1 - \frac{1}{m_0})T.
\]

We now choose \( \{ t_{m,n} : 1 \leq n; 1 \leq m \leq m_0 \} \subset [0, T] \) such that \( \frac{(m-1)T}{m} \leq t_{m,n} \leq \frac{mT}{m_0} \) and \( \| A_n u_n(t_{m,n}) \| \leq D, \forall n \geq 1, 1 \leq m \leq m_0 \).

For fixed \( n_1 \geq 1 \), we have
\[
J_n u_n(t_{m,n}) = J_{n_1} (J_n u_n(t_{m,n}) + n_1^{-1} A_n u_n(t_{m,n})). \forall n \geq 1.
\]

By the assumption 1, we know \( \{ J_n u_n(t_{m,n}) \} \) is relatively compact.
Without loss of generality, we may assume \( \{ J_n u_n(t_{m,n}) \} \) is a convergent sequence for all \( 1 \leq m \leq m_0 \), (otherwise, taking a subsequence).
Again, by (2.15), one can easily see \( \{ J_n u_n \} \) is convergent in \( L^2(0, T; H) \).

Let \( J_n u_n \to u \) strongly in \( L^2(0, T; H) \). (2.16)

Since \( A_n u_n = n(u_n - J_n u_n) \), by (2.14), (2.16), we know
\[
u_n \to u \) strongly in \( L^2(0, T; H) \). (2.17)

Since \( A_{\frac{1}{n}} u_n = P_n(u_n - R_{\frac{1}{n}} u_n) \), by (2.10), (2.17), we know
\[
R_{\frac{1}{n}} u_n \to u \) strongly in \( L^2(0, T; H) \). (2.18)

In view of (2.10), we may assume
\[
A_{\frac{1}{n}} u_n \to z \) weakly in \( L^2(0, T; H) \). (2.19)

By (2.18), (2.19) and the maximal monotonicity of \( A \), we know \( z \in Au \) i.e. \( z(t) \in Au(t) \), a.e. \( t \in [0, T] \).

Since
\[
\ll T_n u_n, v \gg + \ll A_{\frac{1}{n}} u_n, v \gg = 0, \forall v \in W;
\] (2.20)
and \( \| u_n \|_{1,T} < R_0, \frac{1}{n} \frac{du_n}{dt} \to 0 \) strongly in \( L^2(0, T; H) \).

Letting \( n \to \infty \) in (2.20), by (2.16),(2.17),(2.19), we get
\[
\ll \frac{du}{dt} - g(t, u), v \gg + \ll z, v \gg = 0, \forall v \in W.
\]

So \( u \) is a \( T \)-perodic solution of (1.1). We complete the proof.
3. Examples

In this section, we give some examples that satisfy our condition 3.

Example 1 Let $A : \mathbb{R}^1 \to 2^{\mathbb{R}^1}$ be as following

$$Ax = \begin{cases} 
    x + 1, & x > 0; \\
    [0, 1], & x = 0; \\
    x & x < 0.
\end{cases}$$

Through a direct simple calculation, we get

$$J_\lambda x = \begin{cases} 
    \frac{\lambda x - 1}{\lambda + 1}, & x > \frac{1}{\lambda}; \\
    0, & x \in [0, \frac{1}{\lambda}]; \\
    \frac{\lambda x}{\lambda + 1}, & x < 0
\end{cases}$$

Let $g(t, x) : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$ be as following

$$g(t, x) = |\sin t| x + |\cos x|, \quad (t, x) \in \mathbb{R}^1 \times \mathbb{R}^1.$$ 

Then

$$(z - g(t, J_\lambda x), x) = \begin{cases} 
    x^2 \left(1 - \frac{\lambda}{\lambda + 1} |\sin t|\right) + \left(\frac{\lambda+2}{\lambda+1} - |\cos \frac{\lambda x}{\lambda+1}|\right), & x > \frac{1}{\lambda}; \\
    x^2, & x \in [0, \frac{1}{\lambda}]; \\
    0, & x = 0; \\
    (1 - \frac{\lambda}{\lambda + 1} |\sin t|)x^2 - x|\cos \frac{\lambda x}{\lambda+1}|, & x < 0.
\end{cases}$$

So we have $(z - g(t, J_\lambda x), x) \geq 0 > -\frac{a}{\lambda} |J_\lambda x|$, $a > 0$ is any constant.

Example 2 Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial \Omega$. $Au = -\Delta u$, $D(A) = \{u \in H_0^1(\Omega) : Au \in L^2(\Omega)\}$; $g(t, u) : \mathbb{R} \times L^2(\Omega) \to L^2(\Omega)$ is defined as following $g(t, u) = \lambda_1 u - \frac{\nu}{1 + \|u\|^2} |\sin t|$, $(t, u) \in \mathbb{R} \times L^2(\Omega)$; where $\lambda_1$ is the first eigenvalue of $-\Delta$ under the Dirichlet boundary condition, $\| \cdot \|$ is the $L^2$-norm.

It's obvious that

$$\langle Au - g(t, u), u \rangle \geq \frac{\|u\|^2}{1 + \|u\|^2} |\sin t|, \quad u \in D(A).$$

For any $a > 0$, $C_{t_0} > 0$, we can not prove

$$\langle Au - g(t, u), u \rangle \geq a\|u\|^2 - C_0$$

So $A$ and $g$ do not satisfy the third condition in [12], but we have

$$\langle Au - g(t, J_\lambda u), u \rangle = \|\nabla u\|^2 - \lambda_1 \langle J_\lambda u, u \rangle + \frac{\langle J_\lambda u, u \rangle}{1 + \|J_\lambda u\|^2} |\sin t|, \quad u \in D(A);$$
and $0 \leq \langle J_{\lambda}u, u \rangle \leq |u|^2$.

So $\langle Au - g(t, J_{\lambda}u), u \rangle \geq \frac{\langle J_{\lambda}u,u \rangle}{1+||J_{\lambda}u||^2} |\sin t| \geq -\frac{a}{\lambda}\|J_{\lambda}u\|$ for arbitrary constant $a > 0$.

**Remark 1** In [12], condition (3) was replaced by $(\ast)$: $\langle z-g(t,v), v \rangle \geq a\|v\|^2 - b$, \(\forall v \in D(A), z \in Av\) where $a, b$ are positive constants;

For appropriately large $v$ in $D(A)$, we have

$$a\|v\|^2 - b \geq c > 0$$

For $v \in D(A)$, we have $J_{\lambda}v \rightarrow v$ as $\lambda \rightarrow +\infty$. So if $(\ast)$ holds, then for appropriately large $v$ in $D(A)$ and sufficiently large $\lambda$, 3 holds.

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**References**


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