ASYMPTOTIC PROPERTIES OF THE SOLUTIONS OF A CLASS OF OPERATOR-DIFFERENTIAL EQUATIONS

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Summary. Some asymptotic properties of the nonoscillating solutions of operator-differential equations of arbitrary order are investigated.

1. Introduction

The goal of the present paper is by means of a single approach to investigate some asymptotic properties of the nonoscillating solutions of differential equations with “maximd”, with distributed delay, with autoregulable deviation, integro-differential equations, etc. To realize this single approach an operator with certain properties is introduced, as well as appropriately chosen operator-differential equations and inequalities. In the paper results obtained in [2]-[8] are generalized.

2. Preliminary notes

Consider the operator-differential equation

$$\sum_{k=1}^{n} c_k(t) [c_{k-1}(t) \cdots [c_0(t)x(t)] \cdots ]']' + \delta F(t, (Ax)(t)) = c(t)$$

for $t \geq t_0$, where $t_0 \in \mathbb{R}$ is a fixed number, $n \geq 1$ is an integer, $\mathcal{A}$ is an operator with certain properties, $\delta = \pm 1$ and

$$c_i \in C^{n-i}([t_0, \infty); (0, \infty)), \quad i=0, 1, \ldots, n-1$$

Introduce the following notation:

$$(L_0x)(t) = c_0(t)x(t)$$

$$(L_i x)(t) = c_i(t)[(L_{i-1} x)(t)]', \quad 1 \leq i \leq n, \quad c_n(t) \equiv 1,$$

where $c_i \in C^{n-i}([t_0, \infty); (0, \infty))$, $0 \leq i \leq n$; $x : [T_X, \infty) \rightarrow \mathbb{R}$, $T_X \geq t_0$. 

Denote by $\mathcal{D}_n$ the set of all functions $x \in C([T_x, \infty); R)$ such that the functions $L_i x (0 \leq i \leq n)$ exist and are continuous in $[T_x, \infty)$.

Definition 1. The function $x$ is said to be a solution of equation (1) if $x \in \mathcal{D}_n$ and $x$ satisfies equation (1) for $t \geq \max \{T_x, T_{Ax}\}$

Definition 2. A given function $u : [t_0, \infty) \rightarrow R$ is said to eventually enjoy the property $P$ if there exists a point $t_{P, u} \geq t_0$ such that for $t \geq t_{P, u}$ it enjoys the property $P$.

Definition 3. The solution $x$ of equation (1) is said to be regular if $\sup \{|x(t)|\} > 0$ eventually.

Definition 4. The regular solution $x$ of equation (1) is said to oscillate if $\sup \{|x(t)|\} = \infty$. Otherwise, the regular solution is said to be nonoscillating.

Introduce the following conditions:

H1. $c_i \in C^{n-i}([t_0, \infty); (0, \infty)),$ $0 \leq i \leq n$.
H2. $\delta = \pm 1$.
H3. $\int^\infty \frac{dt}{c_i(t)} = \infty, \ 1 \leq i \leq n-1$.
H4. $c \in C([t_0, \infty); R)$. 
H5. $A : \mathcal{D}_n \rightarrow C([T_{AX}, \infty); R), \ T_{AX} \geq t_0$.
H6. If $u, v \in \mathcal{D}_n$ and $u(t) \leq v(t)$ for $t \geq t_0$, then $(Au)(t) \leq (Av)(t)$ for $t \geq T_{AX}$
H7. If $u_p, u \in \mathcal{D}_n \ (p = 1, 2, \ldots)$ and $\{u_p\}_{p=1}^\infty$ is a monotone sequence and $\lim_{p \to \infty} u_p(t) = u(t)$ for $t \geq t_0$, then $\lim_{p \to \infty} (Au_p)(t) = (Au)(t)$ for each $t \geq t_0$.
H8. If $u \in \mathcal{D}_n$ and $u$ is an eventually of constant sign and nonzero function, then the function $Au$ is also eventually of constant sign and nonzero, and they are of the same sign.

H9. $F \in C([t_0, \infty) \times (R_+ \cup R_-)), \ R_+ = (0, \infty), \ R_- = (-\infty, 0)$.

Lemma 1 [8]. Let the following conditions hold:
1. Conditions H1-H3 are met.
2. $x \in \mathcal{D}_n, \ x(t) > 0$ for $t \geq T \ (T \geq t_0)$.
3. $(L_n x)(t)$ is of constant sign in $[T, \infty)$.

Then there exists an integer $l$ such that:
1. For $(L_n x)(t) \leq 0$, $n+l$ is an odd number.
2. For $(L_n x)(t) \geq 0$, $n+l$ is an even number.
3. $(-1)^{i+j}(L_j x)(t) \geq 0$ for $i \leq n-1, \ j = l, \ldots, n-1, \ t \geq T$
4. $(L_i x)(t) > 0$ for $i > 1, \ 1 \leq i \leq l-1, \ t \geq T$

Lemma 2 [8]. Let the following conditions hold:
1. Condition H1 is satisfied.
2. $0 < \liminf_{t \to \infty} c_i(t) \leq \limsup_{t \to \infty} c_i(t) < \infty, 1 \leq i \leq n-1.$

3. $x \in \mathcal{D}_n.$

Then, if one of the following two conditions hold:
1. $L_0x$ is a bounded function in $[T, \infty)$ and $\lim_{t \to \infty}(L_nx)(t)=0.$
2. $L_nx$ is a bounded function in $[T, \infty)$ and $\lim_{t \to \infty}(L_0x)(t) \in R.$

then $\lim_{t \to \infty}(L_i x)(t)=0, 1 \leq i \leq n-1.$

For any function $y \in C([T, \infty); R)$ and for any integer $l, 0 \leq l \leq n$ define the function,

$$(\varphi^l y)(t) = \begin{cases} 
\frac{1}{c_0(t)} \int_t^\infty \frac{1}{c_1(s_1)} \int_{s_1}^\infty \frac{1}{c_2(s_2)} \ldots \int_{s_{n-1}}^\infty \frac{1}{c_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty y(s) ds \, ds_{n-1} \ldots ds_1 
& \text{for } l=0 \\
\frac{1}{c_0(t)} \int_t^\infty \frac{1}{c_1(s_1)} \int_{s_1}^\infty \frac{1}{c_l(s_l)} \ldots \int_{s_{l-1}}^\infty \frac{1}{c_l(s_{l-1})} \int_{s_{l-1}}^\infty y(s) ds \ldots ds_{l-1} \ldots ds_1 
& \text{for } 0 < l \leq n-1 \\
\frac{1}{c_0(t)} \int_t^\infty \frac{1}{c_1(s_1)} \int_{s_1}^\infty \frac{1}{c_l(s_l)} \ldots \int_{s_{l-1}}^\infty \frac{1}{c_l(s_{l-1})} \int_{s_{l-1}}^\infty y(s) ds \ldots ds_{l-1} \ldots ds_1 
& \text{for } l=n
\end{cases}$$

3. **Main results**

**Theorem 1.** Let the following conditions hold:
1. Conditions H1–H9 are met.
2. There exists a function $w$ defined in $[t_0, \infty)$ such that $w \in \mathcal{D}_n$ and $(L_nw)(t)=c(t).$
3. The function $L_0w$ is bounded below in the interval $[t_0, \infty).$
4. There exists a positive solution $y$ of the inequality

$$\delta(L_nx)(t)+F(t, (\Delta x)(t)) \leq \delta c(t)$$

such that $\liminf_{t \to \infty}(L_0y)(t)>0.$
5. $F(t, u)>0$ for $(t, u) \in [t_0, \infty) \times R_+$ and $F(t, u)$ is an increasing function with respect to $u \in R_+.$

Then there exists a positive solution $x$ of equation (1) with the following properties:
1. $\liminf_{t \to \infty}(L_0x)(t)>0$
2. $x(t) \leq y(t)$ eventually.

**Proof.** Let $y(t)>0$ be a solution of inequality (2) in the interval $[T_0, \infty)$ ($T_0 \geq t_0$) and $\liminf_{t \to \infty}(L_0y)(t)>0.$ Then $(\Delta y)(t)>0$ eventually.

Introduce the following notation:
$$w_{0}(t)=w(t)-\frac{1}{c_{0}(t)}\liminf_{t\to\infty}(L_{0}w)(t)$$

$$u(t)=y(t)-w_{0}(t)$$

Then $0<F(t, (\Lambda y)(t))\leq -\delta(L_{n}u)(t)$ teventually, i.e., the function $(L_{n}u)(t)$ is of constant sign for $t\geq T_{0}$. Hence the function $L_{0}u$ is monotone in $[T_{0}, \infty)$. This fact implies the existence of

$$\lim_{t\to\infty}(L_{0}u)(t)\in \mathbb{R}\cup\{-\infty, +\infty\}$$

But $\lim_{t\to\infty}(L_{0}u)(t)=\liminf_{t\to\infty}(L_{0}y)(t)>0$. Thus we obtained that $u$ is an eventually positive function. Let $[\tau, \infty), \tau \geq T_{0}$ be the largest interval in which the function $u$ is positive.

From Lemma 1 it follows that there exists an integer $l$ ($0 \leq l \leq n$) such that

1. $n+l$ is an odd number for $\delta=1$.
2. $n+l$ is an even number for $\delta=-1$.
3. $(-1)^{l+j}(L_{j}u)(t)\geq 0$ for $l \leq n-1; j=l, \ldots, n-1; t \geq \tau$.
4. $(L_{i}u)(t)>0$ for $l>1, 1 \leq i \leq l-1, t \geq \tau^{*}, \tau^{*} \geq \tau$.

Introduce the following notation:

$$T=\begin{cases} \tau & \text{for } l=0 \text{ or } l=1 \\ \tau^{*} & \text{for } l>1 \end{cases}$$

$$K=\begin{cases} \lim_{t\to\infty}(L_{0}u)(t), & l=0 \\ (L_{0}u)(T), & l>0 \end{cases}$$

From condition H1 and the fact that the function $u$ is eventually positive it follows that $K>0$. From (2) we obtain that $-K<(L_{0}w_{0})(t)$ eventually, i.e., $(K/c_{0}(t))+w_{0}(t)>0$ for $t \geq T$.

After a repeated integration of inequality (2) we obtain that

$$y(t)\geq \frac{K}{c_{0}(t)}+w_{0}(t)+(\varphi_{c}{}^{t}(F(\cdot, \iota A y))(t))$$

Let $X$ be the set of all continuous functions $x$ for $t \geq T$ such that

$$\frac{K}{c_{0}(t)}+w_{0}(t)\leq x(t)\leq y(t)$$

For any function $x \in X$ define the function $\bar{x}(t)$:

$$\bar{x}(t)=\begin{cases} x(t), & t \geq T \\ \frac{x(T)}{y(T)}y(t), & T_{0} \leq t \leq T \end{cases}$$

From the definition of $\bar{x}(t)$ it follows that
\[
\frac{K}{c_0(t)} + w_0(t) \leq \overline{x}(t) \leq y(t), \quad t \geq T_0.
\]

Define the operator \( S : X \to E \) by the formula
\[
(Sx)(t) = \frac{K}{c_0(t)} + w_0(t) + (\varphi_c^\ell(F(\cdot, \llcorner 4\overline{x}))(t)
\]
where \( E \) is the set of all continuous functions in \([T_0, \infty)\).

The inclusion \( SX \subset X \) is valid since:
1. From the definition of the operator \( S \) it follows that
\[
\frac{K}{c_0(t)} + w_0(t) \leq (Sx)(t), \quad t \geq T.
\]
2. From condition 3 of Theorem 1 and condition H6 we obtain that
\[
y(t) \geq (Sx)(t).
\]

Let \( x_1, x_2 \in X \) and \( 0 < x_1(t) \leq x_2(t) \). From the definition of the operator \( S \) it follows that \( 0 < (Sx_1)(t) \leq (Sx_2)(t) \) for \( t \geq T_0 \), i.e., \( S \) is a monotone increasing mapping of the set \( X \) into itself. Let \( \{x_n(t)\}_{n=0}^\infty \) be a monotone decreasing sequence of elements of the set \( X \) for \( t \geq T \) obtained by the following recurrent formula:
\[
x_{0}(t)=y(t), \quad t \geq T
\]
\[
x_{n}(t)=(Sx_{n-1})(t), \quad t \geq T \quad (4)
\]

Let \( \lim_{n \to \infty} x_n(t) = x(t) \) for \( t \geq T \). Then \( \lim_{n \to \infty} (Ax_n)(t) = (Ax)(t) \). From the Lebesgue dominated convergence theorem we obtain that \( \lim_{n \to \infty} (Sx_n)(t) = (Sx)(t) \) for \( t \geq T \). But from (4) it follows that \( \lim_{n \to \infty} (Sx_n)(t) = x(t) \). Then we obtain that \( (Sx)(t) = x(t) \), i.e., \( x(t) \) is the positive solution sought of equation (1) such that \( \lim_{t \to \infty} (L_0x)(t) > 0 \), \( x(t) \leq y(t) \) eventually. \( \square \)

**Theorem 2.** Let the following conditions hold:
1. Conditions 1 and 2 of Theorem 1 are met.
2. The function \( L_0w \) is bounded above for \( t \geq t_0 \).
3. There exists a negative solution \( y \) of the inequality
\[
\delta(L_nx)(t) + F(t, (Ax)(t)) \geq \delta c(t)
\]
such that \( \lim_{t \to \infty} \sup_{t \geq t_0} (L_0y)(t) < 0 \).
4. \( F(t, u) < 0 \), \( (t, u) \in ([t_0, \infty) \times R_-) \).

Then there exists a negative solution \( x \) of equation (1) with the properties:
1. \( \limsup_{t \to \infty} (L_0 x)(t) < 0. \) \hfill (5)

2. \( x(t) \geq y(t) \) eventually.

The proof of Theorem 2 is carried out along the scheme of the proof of Theorem 1.

**Theorem 3.** Let the following conditions hold:

1. Condition 1 of Theorem 2 is met.
2. There exists \( \lim_{t \to \infty} (L_0 w)(t) \in \mathbb{R}. \)
3. \( F(t, u) > 0 \) for \( (t, u) \in [t_0, \infty) \times \mathbb{R}_+ \), \( F(t, u) < 0 \) for \( (t, u) \in [t_0, \infty) \times \mathbb{R}_- \).

Then each positive (negative) solution of equation (1) enjoys the property

\[
\lim_{t \to \infty} (L_0 x)(t) \in \mathbb{R} \cup \{-\infty, +\infty\}
\] \hfill (6)

**Proof.** Let \( x \) be a positive solution of equation (1) in the interval \([T, \infty), \ T \geq t_0.\)

Introduce the notation:

\[
w_0(t) = w(t) - \frac{1}{c_0(t)} \lim_{t \to \infty} (L_0 w)(t) \\
u(t) = x(t) - w_0(t), \ \ t \geq T
\]

Then \( (L_n u)(t) = -\delta F(t, (\mathcal{A} x)(t)), \) i.e., \( L_n u \) is of constant sign in the interval \([T, \infty).\) This implies that \( L_n u \) is a monotone function for \( t \geq T, \) i.e., there exists

\[
\lim_{t \to \infty} (L_0 u)(t) \in \mathbb{R} \cup \{-\infty, +\infty\}
\]

But \( \lim_{t \to \infty} (L_0 u)(t) = \lim_{t \to \infty} (L_0 x)(t), \) i.e., there also exists

\[
\lim_{t \to \infty} (L_0 x)(t) \in \mathbb{R} \cup \{-\infty, +\infty\}
\]

**Theorem 4.** Let the following conditions hold:

1. Conditions 1, 2 and 5 of Theorem 1 are satisfied.
2. There exists a positive solution \( y \) of the equation

\[
(L_n x)(t) + \delta F(t, (\mathcal{A} x)(t)) = 0
\] \hfill (7)

such that \( \lim_{t \to \infty} (L_0 y)(t) > 0. \)

Then there exists a positive solution \( x \) of equation (1) with the properties:

1. \( \lim_{t \to \infty} (L_0 x)(t) > 0. \)
2. \( x(t) \leq y(t) \) eventually.

**Proof.** Let \( y \) be a positive solution of equation (7) in the interval \([T_0, \infty), \ T_0 \geq t_0 \) and \( \lim_{t \to \infty} (L_0 y)(t) > 0. \)
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Introduce the notation:

\[ w_0(t) = w(t) - \frac{1}{c_0(t)} \lim_{t \to \infty} (L_0 w)(t) \]

\[ u(t) = y(t) + w_0(t) \]

Then \( \lim_{t \to \infty} (L_0 u)(t) = \lim_{t \to \infty} (L_0 y)(t) > 0 \).

Choose a constant \( c \) such that \( 0 < c < \lim_{t \to \infty} (L_0 u)(t) \).

Let us choose \( T \geq T_\circ \) so that for \( t \geq T \), \( (L_0 u)(t) > c \), \( (L_0 w_0)(t) \leq c \). Then for the function \( \overline{u}(t) = u(t) - \frac{c}{c_0(t)} \) we obtain that \( 0 < \overline{u}(t) \leq y(t), t \geq T \).

Consequently, \( \overline{u}(t) > 0 \) is a solution of the inequality

\[ \delta (L_\circ \overline{u})(t) + F(t, (\mathcal{A} \overline{u})(t)) \leq \delta c(t) \]

Moreover, \( \lim_{t \to \infty} (L_0 \overline{u})(t) = \lim_{t \to \infty} (L_0 u)(t) > 0 \). From \textbf{Theorem 1} it follows that there exists a positive solution \( x \) of equation (1) such that \( \lim_{t \to \infty} (L_0 x)(t) > 0, x(t) \leq \overline{u}(t) \leq y(t) \) eventually. □

**Theorem 5.** Let the following conditions hold:

1. Conditions 1 and 2 of Theorem 3 and condition 4 of Theorem 2 are met.
2. There exists a negative solution \( y \) of equation (7) such that \( \lim_{t \to \infty} (L_0 y)(t) < 0 \).

Then there exists a negative solution \( x \) of equation (1) with the properties:

1. \( \lim_{t \to \infty} (L_0 x)(t) < 0 \).
2. \( x(t) \geq y(t) \) eventually.

The proof of \textbf{Theorem 5} is carried out along the scheme of the proof of \textbf{Theorem 4}.

Consider the operator-differential equation

\[ [c_{n-1}^\circ(t) [c_{n-2}^\circ(t) [\cdots [c_i^\circ(t) x(t)]' \cdots']']' + \partial F(t, (\mathcal{A} x)(t)) = 0 \]  

where \( c_i^\circ \in C^{n-i}([t_0, \infty); R_+) \), \( 0 \leq i \leq n-1 \).

Introduce the following notation:

\[ (L_{i}^\circ x)(t) = c_i^\circ(t) x(t) \]

\[ (L_{i}^\circ x)(t) = c_i^\circ(t) [(L_{i-1}^\circ x)(t)]', \quad i = 1, 2, \cdots, n; \ c_n^\circ(t) = 1. \]

**Theorem 6.** Let the following conditions hold:

1. \( \int_0^\infty \frac{dt}{c_i^\circ(t)} = \infty, \ 1 \leq i \leq n-1 \).
2. \( c_i^\circ(t) \leq c_i(t) \) for \( t \geq t_0 \), \( 0 \leq i \leq n-1 \).
3. Conditions H1-H9 and condition 5 of Theorem 1 are met.
4. There exists a positive solution \( y \) of equation (8) such that \( \lim_{t \to \infty} (L_0^* y)(t) > 0 \).

Then there exists a positive solution \( x \) of equation (7) with the following properties:

1. \( \lim_{t \to \infty} (L_0 x)(t) > 0 \).
2. \( x(t) \leq y(t) \) eventually.

Proof. Let \( y \) be a solution of equation (8) in \( [T_0, \infty) \) for \( T_0 \geq t_0 \) and \( \lim_{t \to \infty} (L_0^* y)(t) > 0 \). Consequently,

\[
(L_n^* y)(t) = -\delta F(t, (A^* y)(t)), \quad \text{i.e., } (L_n^* y)(t) \leq 0 \text{ for } \delta = -1
\]

and

\[
(L_n^* y)(t) \leq 0 \text{ for } \delta = 1, \quad t \geq T_0.
\]

From Lemma 1 it follows that there exists an integer \( l, \quad 0 \leq l \leq n \) such that \( n+l \) is an odd number for \( \delta = -1 \), \( n+l \) is an even number for \( \delta = 1 \) and

\[
(-1)^{l+j} (L_0^* y)(t) \geq 0, \quad t \geq T_0, \quad l \leq n-1, \quad 1 \leq j \leq n-1.
\]

\[
(L_n^* y)(t) > 0, \quad t \geq T_1, \quad T_1 \geq T_0, \quad 1 \leq i \leq l-1.
\]

Introduce the following notation:

\[
T = \begin{cases} T_0, & l = 0 \text{ or } l = 1 \\ T_1, & l > 1 \end{cases}
\]

\[
K = \begin{cases} \lim_{t \to \infty} (L_0^* y)(t), & l = 0 \\ (L_0^* y)(T), & l > 0 \end{cases}
\]

Then for \( t \geq T \) we obtain that

\[
y(t) \geq \frac{K}{c_0(t)} + (\varphi_{\ell} F(\cdot, A y))(t)
\]

But from condition 2 of Theorem 6 it follows that

\[
y(t) \geq \frac{K}{c_0(t)} + (\varphi_{\ell} F(\cdot, A y))(t)
\]

Consider the set \( X \) of all continuous functions \( x \) in \( [T, \infty) \) such that \( K/c_0(t) \leq x(t) \leq y(t) \) and define

\[
x(t) = \begin{cases} x(t), & t \geq T \\ \frac{x(T)}{y(T)} & t \leq T \end{cases}
\]

for each function \( x \in X \).

Define the operator \( S: X \to C([t_0, \infty); \mathbb{R}) \) by the formula

\[
(Sx)(t) = \frac{K}{c_0(t)} + (\varphi_{\ell} F(\cdot, A x))(t)
\]
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It is immediately verified that

\[
\frac{K}{c_0(t)} \leq (Sx)(t) \leq y(t), \quad t \geq t_0, \text{ i.e., } S : X \to X
\]

Let \( x_1, x_2 \in X \) and \( x_1(t) \leq x_2(t), \quad t \geq T \). Then

\[(Sx_1)(t) \leq (Sx_2)(t), \quad t \geq T.
\]

Consider the convergent sequence \( \{x_k(t)\}_{k=0}^{\infty}, \quad t \geq T \) such that

\[x_0(t) = y(t),\]

\[x_k(t) = (Sx_{k-1})(t), \quad K = 1, 2, \ldots\]

i.e. the sequence \( \{x_k(t)\}_{k=0}^{\infty} \) is decreasing for \( t \geq T \). If \( x(t) = \lim_{n \to \infty} x_n(t) \) then from the Lebesgue dominated convergence theorem it follows that \( x(t) = (Sx)(t) \), i.e., \( x(t) \) is a positive solution of equation (7) with the properties (9).

**Theorem 7.** Let the following conditions hold:

1. Conditions 1, 2 and 3 of Theorem 6 and condition 4 of Theorem 2 are met.
2. There exists a negative solution \( y \) of equation (8) with the properties \( \lim_{t \to \infty} (L_0^*y)(t) < 0 \).

Then there exists a negative solution \( x \) of equation (7) with the following properties:

1. \( \lim_{t \to \infty} (L_0^*x)(t) < 0 \).
2. \( x(t) \geq y(t) \) eventually.

The proof of Theorem 7 is carried out along the scheme of the proof of Theorem 6.

**Theorem 8.** Let the following conditions hold:

1. Conditions H1, H2, H4–H9 and condition 5 of Theorem 1 are met.
2. There exists a positive solution \( y \) of the inequality

\[
\delta(L_n^*x)(t) + F(t, (dx)(t)) \leq 0
\]

such that \( \lim_{t \to \infty} (L_0^*y)(t) > 0 \).

Then there exists a positive solution \( x \) of equation (7) with the following properties:

\[
\lim_{t \to \infty} (L_0^*x)(t) > 0
\]

\[
0 < (L_i x)(t) \leq (L_i y)(t) \text{ eventually, } 0 \leq i \leq l - 1, \quad l \geq 1
\]

\[
0 \leq (-1)^{l+i}(L_i x)(t) \leq (-1)^{l+i}(L_i y)(t) \text{ eventually, } l \leq i \leq n - 1, \quad l \leq n - 1,
\]

(10)
where \( l \) is an integer, \( 0 \leq l \leq n \), such that \( n + l \) is odd for \( \delta = 1 \) and \( n + l \) is even for \( \delta = -1 \).

**Theorem 8** is a corollary of **Theorem 1** and Lemma 1.

**Theorem 9.** Let the following conditions hold:
1. Conditions H1, H2, H4–H9 and condition 4 of **Theorem 2** are met.
2. There exists a negative solution \( y \) of the inequality

\[
\delta(L_n x)(t) + F(t, (Ax)(t)) \geq 0
\]

such that \( \lim_{t \to \infty}(L_0 y)(t) < 0 \).

Then there exists a negative solution \( x \) of equation (7) with the properties:

\[
(L_1 y)(t) \leq (L_1 x)(t) \quad \text{eventually, } \quad l > 0, \quad 0 \leq i \leq l - 1.
\]

\[
\lim_{t \to \infty}(L_0 x)(t) < 0
\]

\[
(-1)^{l+i}(L_1 y)(t) \leq (-1)^{l+i}(L_1 x)(t) \leq 0 \quad \text{eventually, } \quad l \leq n - 1, \quad l \leq i \leq n - 1.
\]

where \( l \) is an integer (\( 0 \leq l \leq n \)) such that \( n + l \) is odd for \( \delta = 1 \) and \( n + l \) is even for \( \delta = -1 \).

**Theorem 9** is a corollary of **Theorem 2** and Lemma 1.

**Theorem 10.** Let the following conditions hold:
1. Conditions 1, 2 and 3 of **Theorem 1** are met.
2. There exists a positive solution \( y \) of equation (1) such that

\[
\lim_{t \to \infty}(L_0 y)(t) > 0.
\]

Then there exists a positive solution \( x \) of equation (7) with the following properties:

1. \( \lim_{t \to \infty}(L_0 x)(t) > 0. \)
2. \( x(t) \leq y(t) \quad \text{eventually.} \)

**Proof.** Let \( y \) be a positive solution of equation (1) in \([T_0, \infty), \quad T_0 \geq t_0 \) such that

\[
\lim_{t \to \infty}(L_0 y)(t) > 0.
\]

Introduce the following notation:

\[
w_0(t) = w(t) - \frac{1}{c_0(t)} \lim_{t \to \infty}(L_0 y)(t)
\]

\[
v(t) = y(t) - w_0(t)
\]

Then \( \lim_{t \to \infty}(L_0 v)(t) = \lim_{t \to \infty}(L_0 y)(t) > 0. \) From the fact that \( \lim_{t \to \infty}(L_0 v)(t) > 0 \) it follows that we can choose a constant \( c \) such that \( 0 < c < \lim_{t \to \infty}(L_0 v)(t) \). Choose
$T \geq T_0$ so that for $t \geq T$ the following inequalities be valid

$$ (L_0v)(t) > c > 0 \quad \text{and} \quad (L_0w)(t) \geq -c $$

If we denote $\bar{v}(t) = v(t) - (c/c_0(t))$ for $t \geq T$, then we obtain that

$$ 0 < \bar{v}(t) \leq y(t), \quad t \geq T $$

Then $\delta(L_0\bar{v})(t) + F(t, (_{c}A\bar{v})(t)) \leq 0$. Since $\lim_{t \to \infty}(L_0\bar{v})(t) = \lim_{t \to \infty}(L_0v)(t) - c > 0$, then from Theorem 8 it follows that there exists a positive solution $x$ of equation (7) for which $\lim_{t \to \infty}(L_0x)(t) > 0$ and $x(t) \leq \bar{v}(t) \leq y(t)$.

**Theorem 11.** Let the following conditions hold:
1. Conditions 1, 2, 3 and 4 of Theorem 2 are valid.
2. There exists a negative solution $y$ of equation (1) such that $\lim_{t \to \infty}\sup(L_0y)(t) < 0$.

Then there exists a negative solution $x$ of equation (7) with the following properties:
1. $\lim_{t \to \infty}(L_0x)(t) < 0$.
2. $x(t) \geq y(t)$ eventually.

The proof of Theorem 11 is carried out along the scheme of the proof of Theorem 10.

4. Some particular realizations of the operator $A$

1. Let $(Ax)(t) = \max_{s \in K(t)} x(s)$, where $M(t) = [p(t), q(t)]$ is a compact subset of the interval $[t_0, \infty)$, $t \geq t_0$ and $\lim_{t \to \infty}p(t) = \infty$, $p(t) \leq q(t)$ for $t \geq t_0$, $p, q \in C([t_0, \infty); R)$.

We shall prove that for the so defined operator conditions H5-H8 are satisfied. In fact, if $0 < x(t) \leq y(t)$ for $t \geq t_0$, then it is immediately verified that $0 < (Ax)(t) = \max_{s \in K(t)} x(s) \leq \max_{s \in K(t)} y(s) = (A\cdot)(t)$ and $x(t), (Ax)(t) > 0$ for $t \geq t_0$.

Let $x, x_k \in C([t_0, \infty); R), k = 0, 1, \ldots, x_k(t) \leq x(t)$ or $x_k(t) \geq x(t)$ and $\lim_{k \to \infty}x_k(t) = x(t)$.

We shall prove that $\lim_{k \to \infty}\max_{s \in K(t)} x_k(s) = \max_{s \in K(t)} x(s)$.

To this end we shall use the inequality

$$ \max_{s \in K(t)} x(s) - \max_{s \in K(t)} y(s) \leq \max_{s \in K(t)} [x(s) - y(s)] \quad \text{(cf. [9])}. $$

From the fact that $x_k(t) \to x(t)$ for $t \geq t_0$ it follows that for each $\varepsilon > 0$ there exists $k_0 > 0$ such that if $k \geq k_0$, then $|x_k(t) - x(t)| < \varepsilon$ for $t \geq t_0$.

Then
\[
\max_{s \in M(t)} x(s) - \max_{s \in M(t)} x(s) = \begin{cases} 
\max_{s \in M(t)} x(s) - \max_{s \in M(t)} x(s), & x(s) \geq x(s) \\
\max_{s \in M(t)} x(s) - \max_{s \in M(t)} x(s), & x(s) \leq x(s) \\
\max_{s \in M(t)} [x(s) - x(s)] < \epsilon \\
\max_{s \in M(t)} [x(s) - x(s)] < \epsilon
\end{cases}
\]

If \( x \in \mathcal{D} \), then \( \Delta x \in C([T_{Ax}, \infty); R) \) (cf. [1]).

**Example 1.** Consider the differential equation
\[
(t^{-1}x'(t))' + \frac{1}{2}t^{-3} \max_{s \in [t, t+1]} x(s) = -t^{-2}, \quad t \geq 1
\]  
and the differential inequality
\[
(t^{-1}x'(t))' + \frac{1}{2}t^{-3} \max_{s \in [t, t+1]} x(s) \leq -t^{-2}, \quad t \geq 1
\]
Here \( \Delta x(t) = \max_{s \in [t, t+1]} x(s) \). The functions \( c_0(t) = 1, \ c_1(t) = t^{-1}, \ F(t, u) = (1/2)t^{-1}u \)
and \( c(t) = t^{-3}, \ w(t) = -t \) for \( t \geq 1 \) satisfy the conditions of Theorem 1 and \( y(t) = 4t > 0 \) is a solution of (14). Then there exists a positive solution \( x \) of equation (13) with the properties (3). For instance, \( x(t) = 2t \) is such a solution.

**Example 2.** Consider the differential equation
\[
(t^{-1}x'(t))' + \frac{1}{2}t^{-3} \max_{s \in [t, t+1]} x(s) = t^{-2}, \quad t \geq 1
\]  
and the differential inequality
\[
(t^{-1}x'(t))' + \frac{1}{2}t^{-3} \max_{s \in [t, t+1]} x(s) \geq t^{-2}, \quad t \geq 1
\]
Here \( \Delta x(t) = \max_{s \in [t, t+1]} x(s) \). The functions \( c_0(t) = 1, \ c_1(t) = t^{-1}, \ F(t, u) = (1/2)t^{-1}u \)
and \( c(t) = t^{-3}, \ w(t) = -t \) for \( t \geq 1 \) satisfy the conditions of Theorem 2 and \( y(t) = -4t < 0 \) is a solution of (16). Then there exists a negative solution \( x \) of equation (15) with the properties (5). For instance, \( x(t) = -2t \) is such a solution.

**Example 3.** Consider the differential equation
\[
[e^{-t}[e^{-t}[e^{-t}[e^t x(t)']']']'] - 4e^{-t} \max_{s \in [t-1, t]} x(s) = 0, \quad t \geq 1
\]  
and the differential inequality
\[
[e^{-t}[e^{-t}[e^{-t}[e^t x(t)']']']'] - 4e^{-t} \max_{s \in [t-1, t]} x(s) \leq 0, \quad t \geq 1
\]
Here \( \Delta x(t) = \max_{s \in [t-1, t]} x(s) \). The functions \( c_0(t) = e^t, \ c_1(t) = c_2(t) = e^{-t}, \ c_3(t) = e^{-3t}, \ F(t, u) = e^{-3t}u \)
and \( M(t) = [t-1, t] \) satisfy the conditions of Theorem 8 and \( y(t) \)
$e^{t}$ is a solution of (18) such that $\lim_{t \to \infty} e^{t} = \infty$. Then there exists a positive solution $x$ of equation (17) with the properties (10). For instance, $x(t) = e^{t}$ is such a solution.

**Example 4.** Consider the differential equation

$$[e^{-t}[e^{-t}[e^{-t}[e^{t}, x(t)]]]']' - 4e^{-3t} \max_{s \in [t, t+1]} x(s) = 0, \quad t \geq 1$$

and the differential inequality

$$[e^{-t}[e^{-t}[e^{-t}[e^{t}, x(t)]]]']' - 4e^{-3t} \max_{s \in [t, t+1]} x(s) \leq 0, \quad t \geq 1$$

Here $(Ax)(t) = \max_{s \in [t, t+1]} x(s)$. The functions $c_0(t) = e^{t}$, $c_1(t) = e^{-t}$, $c_2(t) = e^{-2t}$, $c_3(t) = e^{-t}$, $F(t, u) = 4e^{-3}u$ satisfy the conditions of Theorem 9. Moreover, $y(t) = -e^{t}$ is a solution of inequality (20) such that $\lim_{t \to \infty} (L_0 y)(t) < 0$. Then there exists a negative solution $x$ of equation (19) with the properties (11). For instance, $x(t) = -e^{t}$ is such a solution.

For $n = 4$ and $\delta = -1$ we obtain that $l = 2$. Then it is immediately verified that

$$(L_i y)(t) \leq (L_i x)(t) < 0, \quad i = 0, 1,$$

$$(-1)^{2+i}(L_i y)(t) \leq (-1)^{2+i}(L_i x)(t) < 0, \quad i = 2, 3; \quad t \geq 1$$

**Example 5.** Consider the differential equations

$$[t^{-1}[t^{-1}x'(t)]']' - 3t^{-5} \max_{s \in [t-1, t]} x(s) = -3t^{-3}, \quad t \geq 2$$

$$[t^{-1}[t^{-1}x'(t)]']' - 3t^{-5} \max_{s \in [t-1, t]} x(s) = 0, \quad t \geq 2$$

Here $(Ax)(t) = \max_{s \in [t-1, t]} x(s)$. The functions $c_0(t) = 1$, $c_1(t) = c_2(t) = t^{-1}$, $F(t, u) = 3t^{-5}u$, $c(3) = 3t^{-3}$, $w(t) = (3/4)[t^{2} \ln t - (t^{2}/2)]$ satisfy the conditions of Theorem 10. Moreover, $y(t) = t^{2} > 0$ is a solution of equation (21) such that $\inf_{t \geq 2} y(t) = 4 > 0$. Then there exists a solution $x$ of equation (22) with the properties (12). For instance, $x(t) = t$ is a solution of equation (22) for which $\lim_{t \to \infty} (L_0 x)(t) = \infty$ and $x(t) = t \leq t^{2} = y(t)$ for $t \geq 2$.

2. Let $(Ax)(t) = x(g(t))$, where $g \in C([t_0, \infty); R)$, $\lim_{t \to \infty} g(t) = \infty$.

It is immediately verified that for the operator considered conditions H5–H8 are met.

**Example 6.** Consider the differential equation

$$[e^{-t}[e^{-t}[e^{-t}[e^{t}, x(t)]]]']' + 2e^{-4t} x(2t) = 2e^{-2t}, \quad t \geq 2$$

(23)
and the differential inequality

$$[e^{-t}[e^{-t}x'(t)]']' + 2e^{-2t}x(2t) \leq 2e^{-2t}, \quad t \geq 2$$

(24)

Here \((Ax)(t) = x(2t)\). The functions \(c_0(t) = 1, c_1(t) = c_2(t) = e^{-t}, w(t) = t, F(t, u) = 2e^{-4t}u, c(t) = -2e^{-2t}\) satisfy the conditions of \(\text{Theorem 1}\). Moreover, \(y(t) = te^t\) is a solution of inequality \(24\) such that \(\lim_{t \to \infty} y(t) > 0\). Then there exists a positive solution \(x\) of equation \(23\) with the properties \(3\).

For instance, \(x(t) = e^t\) is a solution of equation \(23\), for which \(\lim_{t \to \infty} (L_0x)(t) > 0\) and \(x(t) = e^t \leq te^t = y(t)\) for \(t \geq 2\).

**Example 7.** Consider the differential equation

$$[e^{-t}[e^{-t}x'(t)]']' + 2e^{-4t}x(2t) = -2e^{-2t}, \quad t \geq 2$$

(25)

and the differential inequality

$$[e^{-t}[e^{-t}x'(t)]']' + 2e^{-4t}x(2t) \geq -2e^{-2t}, \quad t \geq 2$$

(26)

Here \((Ax)(t) = x(2t)\). The functions \(c_0(t) = 1, c_1(t) = c_2(t) = e^{-t}, w(t) = -t, F(t, u) = 2e^{-4t}u, c(t) = -2e^{-2t}\) satisfy the conditions of \(\text{Theorem 2}\). Moreover, \(y(t) = -te^t\) is a solution of inequality \(26\). Then there exists a negative solution \(x\) of equation \(25\) with the properties \(5\).

For instance, \(x(t) = -e^t\) is such a solution.

**Example 8.** Consider the differential equation

$$[t^{-1}[t^{-2}x(t)]']' + t^{-4}x(3t^2) = 6t^{-4}, \quad t \geq 1$$

(27)

Here \((Ax)(t) = x(3t^2)\). The functions \(w(t) = 2t, c(t) = 6t^{-4}, F(t, u) = t^{-4}u > 0\) for \(u \in R_+\), \(c_0(t) = t^{-2}, c_1(t) = t^{-1}\) satisfy the conditions of \(\text{Theorem 3}\). Then each positive solution of equation \(27\) enjoys the property \(6\). For instance, \(x(t) = t\) is such a solution for which \(\lim_{t \to \infty} (x(t)/t^2) = 0\).

**Example 9.** Consider the differential equation

$$[e^{-t}[e^{t}x(t)]']' - 2e^{-t}x(2t) = 0, \quad t \geq 2$$

(28)

and the differential inequality

$$-[e^{-t}[e^{t}x(t)]']' + 2e^{-t}x(2t) \geq 0, \quad t \geq 2.$$

(29)

Here \((Ax)(t) = x(2t)\). The functions \(c_0(t) = e^t, c_1(t) = e^{-t}, F(t, u) = 2e^{-t}u\) satisfy the conditions of \(\text{Theorem 3}\). Moreover, \(y(t) = -e^{2t}\) is a solution of inequality \(29\) such that \(\lim_{t \to \infty} (L_0y)(t) = -\infty\). Then there exists a negative solution \(x\) of equation \(28\) with the properties \(11\).
For instance, \( x(t) = -e^t \) is such a solution.

**Example 10.** Consider the differential equations
\[
[t^{-4}t^{-1}x'(t)]' - 4t^{-1}x(t^2) = -4t^{-3}, \quad t \geq 1, \\
[t^{-4}t^{-1}x'(t)]' - 4t^{-1}x(t^2) = 0, \quad t \geq 1.
\]
Here \( (Ax)(t) = x(t^2) \). The functions \( c_i(t) = t^{-1} \), \( c_2(t) = t^{-2} \), \( F(t, u) = 4t^{-1}u \), \( C(t) = -4t^{-3} \), \( w(t) = (2/3)t^3 \) satisfy the conditions of [Theorem 10]. Moreover, \( y(t) = t^2 \) is a solution of equation (30) such that \( \lim \inf_{t \to \infty} y(t) > 0 \). Then there exists a positive solution \( u \) of equation (31) for which \( \lim_{t \to \infty} (L_0 x)(t) = \infty \), \( x(t) = t \leq t^2 = y(t), \quad t \geq 1 \).

3. Let \( (Ax)(t) = \int_{t-a}^{t} k(t, s) x(s)ds \) where \( a \) is a positive constant, \( k \in C([t_0+a, \infty)^2; (0, \infty)) \) and there exists a constant \( c > 0 \) such that \( k(t, s) \leq c \) eventually.

We shall prove that for the operator considered conditions H5–H8 are met. If \( 0 < x(t) \leq y(t) \), then
\[
(Ay)(t) - (Ax)(t) = \int_{t-a}^{t} k(t, s)[y(s) - x(s)]ds \geq 0
\]
It is immediately verified that conditions H5 and H9 hold. \( x_k, x \in D_a, k = 0, 1, \ldots, \lim_{k \to \infty} x_k(t) = x(t) \), i.e., for any \( \varepsilon > 0 \) and each fixed number \( t \geq t_0 \) there exists \( k_0 > 0 \) such that for \( k \geq k_0 \) we have \( |x_k(t) - x(t)| < \varepsilon/(ca) \).

Then
\[
\lim_{k \to \infty} (Ax_k)(t) = (Ax)(t).
\]

**Example 11.** Consider the differential equation
\[
[e^{-t}e^{-t}x'(t)]' + \int_{t-1}^{t} e^{-t}x(s)ds = \frac{e^2 - 1}{2e^2} e^t, \quad t \geq 1.
\]
and the differential inequality
\[
[e^{-t}e^{-t}x'(t)]' + \int_{t-1}^{t} e^{-t}x(s)ds \leq \frac{e^2 - 1}{2e^2} e^t, \quad t \geq 1.
\]
Here \( (Ax)(t) = \int_{t-1}^{t} e^{-t}x(s)ds \). The functions \( c_i(t) = 1, \ c_2(t) = e^{-t} \), \( F(t, u) = u \)
\( k(t, s) = e^{-t}, \ w(t) = (e^2 - 1)/(12e^3)e^{3t} \) satisfy the conditions of [Theorem 1]. Moreover, \( y(t) = t, \ e^t > 0 \) is a solution of inequality (33) such that \( \lim \inf_{t \to \infty} (L_0 y)(t) > 0 \). Then there exists a positive solution \( x \) of equation (32) with the properties (3).

For instance, \( x(t) = e^t \) is such a solution.

**Example 12.** Consider the differential equation
\[
[e^{-t}[e^{-t}x'(t)]']' + \int_{t-1}^{t}e^{s-t}x(s)ds = \frac{1-e^{2}}{2e^2}e^{t}, \quad t \geq 1.
\] (34)
and the differential inequality
\[
[e^{-t}[e^{-t}x'(t)]']' + \int_{t-1}^{t}e^{s-t}x(s)ds \geq \frac{1-e^{2}}{2e^2}e^{t}, \quad t \geq 1
\] (35)
Here \((Ax)(t)=\int_{t-1}^{t}e^{s-t}x(s)ds\). The functions \(c_0(t)=1, \ c_1(t)=c_2(t)=e^{-t}, \ F(t, u)=u\)
\(k(t, s)=e^{s-t} \leq 1, \ c(t)=\frac{1-e^{g}}{2e^2}e^{t}, \ w(t)=\frac{1-e^{g}}{12e^2}e^{3t}\),
satisfy the conditions of Theorem 2. Moreover, \(y(t)=-te^{t}\) is a solution of inequality (35).

Then there exists a negative solution \(x\) of equation (34) with the properties (5).

For instance, \(x(t)=-e^{t}\) is such a solution.

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**References**

ASYMPTOTIC PROPERTIES OF THE SOLUTIONS

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