ON $p$-QUASIHYPONORMAL OPERATORS FOR $0<p<1$

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Abstract. For $0<p<1$ the notion of $p$-quasihyponormal operators on a Hilbert space is introduced and studied. It is proved that if $T$ is a $p$-quasihyponormal operator with polar decomposition $T=U|T|$ then the operator $|T|^{1/2}U|T|^{1/2}$ is quasihyponormal for $1/2 \leq p<1$ and it is $(p+(1/2))$-quasihyponormal for $0<p<1/2$.

A bounded linear operator $T$ on a Hilbert space $H$ is said to be hyponormal if

$$\|T^*x\| \leq \|Tx\| \text{ for all } x \in H$$

or equivalently if

$$T^*T - TT^* \geq 0$$

and is said to be quasihyponormal if

$$\|T^*Tx\| \leq \|TTx\| \text{ for all } x \in H$$

or equivalently

$$T^*(T^*T - TT^*)T = T^*T - (T^*T)^* \geq 0$$

(see [5]). For $0<p<1$ $T$ is said to be $p$-hyponormal if

$$(T^*T)^p - (TT^*)^p \geq 0.$$ 

Here $H$ denotes a separable complex infinite dimensional Hilbert space and $B(H)$ the algebra of all bounded linear operators on $H$. Throughout the paper we consider those operators $T$ for which $R(T)$, the range space of $T$, is a closed linear subspace of $H$. We begin with the following definition.

Definition. An operator $T$ on the space $H$ is said to be $p$-quasihyponormal if

$$T^*((T^*T)^p - (TT^*)^p)T \geq 0.$$ 

If $p=1$ then $T$ is quasihyponormal ([5], [6]) and if $p=1/2$ then $T$ is called...
semi-quasihyponormal and for $p=1/4$ $T$ is called quarter-quasihyponormal. Also for $q\leq p$ any $p$-quasihyponormal is $q$-quasihyponormal. It is immediate that every $p$-hyponormal operator is $p$-quasihyponormal but not necessarily conversely. If $T$ is semi-quasihyponormal but not quasihyponormal and if $T=U|T|$ is the polar decomposition of $T$, where $|T|=(T^*T)^{1/2}$, then the operator $T_q=U|T|^2$ is quarter-quasihyponormal but not semi-quasihyponormal.

Let $T$ be a $p$-quasihyponormal operator. Let $T=U|T|$ be the polar decomposition of $T$ and $U$ be unitary and also let

$$\tilde{T}=|T|^{1/2}U|T|^{1/2}.$$ 

Then

(i) $\tilde{T}$ is quasihyponormal for $1/2\leq p<1$,
(ii) $\tilde{T}$ is $(p+(1/2))$-quasihyponormal for $0<p<1/2$.

We begin with the following lemma.

**Lemma 1.** For $T=U|T|$, $R(\tilde{T})\subset R(|T|)$.

**Proof.** As $R(T)$ is assumed to be closed, $R(T^*)$ is closed ([4]). By [2, Theorem 2.2]

$$R(T^*)=R(\sqrt{T^*T})=R(\sqrt{T^*T})\subset R(|T|).$$

This implies that

$$R(T)=R(\sqrt{2T^*T})=R(\sqrt{2T^*T})\subset R(|T|).$$

Thus $R(|T|)$ is closed. Also by [2, Corollary 1] it follows that $R(|T|^{1/2})=R(|T|)$, since $|T|$ is a positive operator and $R(|T|)$ is closed. Therefore

$$R(\tilde{T})\subset R(|T|^{1/2})=R(|T|).$$

Aluthge [1] proved that if $T$ is $p$-hyponormal for $1/2\leq p<1$ and $U$ is unitary, then the operator $\tilde{T}$ is hyponormal. We prove the following for $p$-quasihyponormal operators.

**Theorem 2.** Let $T=U|T|$ be $p$-quasihyponormal; $1/2\leq p<1$, and $U$ be unitary, then $\tilde{T}=|T|^{1/2}U|T|^{1/2}$ is quasihyponormal.

**Proof.** As any $p$-quasihyponormal operator for $1/2\leq p<1$ is semi-quasihyponormal, we have

$$T^*(T^*T)^{1/2}-(TT^*)^{1/2}T \geq 0.$$ 

This implies that

$$|T^*U^*|T|-U|T||U^*|T| \geq 0.$$ 

This is equivalent to

$$|T||U^*|T|-|T||U^*|T| \geq 0.$$
Thus $U^*|T|U-|T|\geq 0$ on $R(|T|)$. Therefore by Lemma it follows that on $R(\tilde{T})$

$$U^*|T|U \geq |T|$$

or equivalently

$$U|T|U^* \leq |T|.$$  

Hence on $R(\tilde{T})$ we have

$$U^*|T|U \geq |T| \geq U|T|U^*.$$ 

Therefore on $R(\tilde{T})$ we get

$$\tilde{T}^*\tilde{T} - \tilde{T}\tilde{T}^* = |T|^{1/2}(U^*|T|U - |T|U^*|T|^{1/2}) \geq 0.$$ 

Hence $\tilde{T}$ is quasihyponormal.

Aluthge has proved that if $T$ is $p$-hyponormal for $0<p<1/2$ and $U$ is unitary then $\tilde{T}$ is $(p+(1/2))$-hyponormal. To see through such a result for $p$-quasihyponormal operators we need the following famous and useful Furuta Inequality [3].

**Theorem A.** If $A$ and $B$ are bounded self-adjoint operators such that $A \geq B \geq 0$. Then

$$(B^r A^p B^r)^{1/2} \geq B^{(p+2r)/q}$$

and

$$A^{(p+2r)/q} \geq (A^r B^p A^r)^{1/q}$$

hold for each $r \geq 0$, $p \geq 0$, $q \geq 1$ such that $(1+2r)q \geq p+2r$.

**Theorem 3.** Let $T=U|T|$ be $p$-quasihyponormal, $0<p<1/2$ and $U$ be unitary. Then $\tilde{T} = |T|^{1/2}U|T|^{1/2}$ is $(p+(1/2))$-quasihyponormal.

**Proof.** We have only to employ the ingenious proof of Theorem 2 in [1] based on Theorem A. Since $T$ is $p$-quasihyponormal, therefore

$$T^*((T^*T)^p -(TT^*)^p)T \geq 0.$$ 

This implies that

$$|T|U^*|T|^p|U|^{1/2} - |T|^p U|T|U^*|T|^{1/2} \geq 0.$$ 

This is equivalent to

$$|T|U^*|T|^p|U| - |T|^p U|T| |T|^{1/2} \geq 0.$$ 

Thus on $R(|T|)$

$$U^*|T|^p U \geq |T|^p.$$ 

By Lemma it follows that on $R(\tilde{T})$

$$U^*|T|^p U \geq |T|^p$$

or equivalently

$$U|T|^p U^* \leq |T|^p.$$ 

Hence on $R(\tilde{T})$, we have
$U^*|T|^{2p}U \geqq |T|^{z_p} \geqq U|T|^{2p}U^*$.

Let $A=U^*|T|^{2p}U$, $B=|T|^{z_p}$ and $C=U|T|^{z_p}U^*$. Then using Theorem A, we get that on $R(\mathcal{T})$, we have

$$(\mathcal{T}^*\mathcal{T})^{p+1/2}=(|T|^{1/2}U^{*}|T|U|T|^{1/2})^{p+1/2}$$

$$=(B^{1/4p}A^{1/2p}B^{1/4p})^{p+1/2}$$

and

$$(\mathcal{T}\mathcal{T}^*)^{p+1/2}=(|T|^{1/2}U|T|U^{*}|T|^{1/2})^{p+1/2}$$

$$=(B^{1/4p}C^{1/2p}B^{1/4p})^{p+1/2}$$

$$\geqq B^{(1/2p+2/4p)(p+1/2)}=B^{1+1/z_p}$$

$$\leqq B^{(1/2p+2/4p)(p+1/2)}=B^{1+1/z_p}$$

Hence on $R(\mathcal{T})$

$$(\mathcal{T}^*\mathcal{T})^{p+1/2}\geqq(\mathcal{T}\mathcal{T}^*)^{p+1/2}.$$ 

This implies that

$$\mathcal{T}^*((\mathcal{T}^*\mathcal{T})^{p+1/2}-(\mathcal{T}\mathcal{T}^*)^{p+1/2})\mathcal{T}\geqq 0.$$ 

Hence $\mathcal{T}$ is $(p+(1/2))$-quasihyponormal.

As a consequence of Theorems 2 and 3, we obtain

**Corollary 4.** If $T$ is a $p$-quasihyponormal operator for $0<p<1/2$, then the operator $|T|^{1/2}\mathcal{U}|T|^{1/2}$ is quasihyponormal, where $\mathcal{T}=|T|^{1/2}U|T|^{1/2}$ and $\mathcal{T}=\mathcal{U}|\mathcal{T}|$ is the polar decomposition of $\mathcal{T}$.

Finally we give an example to show that the class of $p$-hyponormal operators is properly contained in the class of $p$-quasihyponormal operators.

**Example 5.** Let $K$ be the direct sum of a denumerable number of copies of $H$. For given positive operators $A$ and $B$ defined on $H$, define the operator $T_{A,B}$ on $K$ as follows

$$T_{A,B}(x_1, x_2, \ldots) = (0, Ax_1, Ax_2, \ldots, Ax_n, Bx_{n+1}, \ldots).$$

The operator $T_{A,B}$ is $p$-hyponormal if and only if $B^{2p} \geqq A^{2p}$ and is $p$-quasihyponormal if and only if $AB^{2p}A \geqq A^{2(p+1)}$.

Let $H$ be a two-dimensional Hilbert space with

$$A=\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix} 29 & 12 \\ 12 & 5 \end{bmatrix}$$

and let $p=1/2$. Then

$$B^{2p} - A^{2p} = B - A = \begin{bmatrix} 25 & 12 \\ 12 & 5 \end{bmatrix}$$
which is not positive. Therefore $T_{A,B}$ is not semihyponormal. But

$$AB^{p}A-A^{2(p+1)}=ABA-A^{3}$$

$$=\begin{bmatrix} 464 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 64 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 400 & 0 \\ 0 & 0 \end{bmatrix}$$

which is positive. Therefore $T_{A,B}$ is semi-quasihyponormal.

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References


[3] T. Furuta, $A \geq B \geq 0$ assures $(B^{p}A^{p}B^{r})^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)/q \geq p+2r$, Proc. Amer. Math. Soc., 101 (1987), 85-88.


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