AN LIL FOR RANDOM WALKS WITH TIME STATIONARY RANDOM DISTRIBUTION FUNCTION

By

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Summary. Let $\mathcal{F}$ be a family of distribution functions and let $\nu$ be a stationary ergodic probability measure on $\mathcal{F}$. Now for each $\omega=(F_{1}^{\omega}, F_{2}^{\omega}, \cdots) \in \mathcal{F}$, we define a probability measure $P_{\omega}$ on $(\mathcal{R}^{\infty}_{1}, \mathcal{B}^{\infty}_{1})$ so that $P_{\omega}:=\Pi_{n=1}^{\infty}F_{n}^{\omega}$. Let $X_{n}:\mathcal{R}^{\infty}_{1}\rightarrow\mathcal{R}$ be the coordinate functions $X_{n}(x)=x_{n}$, $x=(x_{n})$. In this paper we study LIL for partial sums of $\{X_{n}\}$ with respect to $P_{\omega}$ and as a special case of above model we also study LIL for interchangeable process.

1. Introduction

Let $\mathcal{F}$ be a set of distributions on $\mathcal{R}^{1}$ with the topology of weak convergence, and let $\mathcal{A}$ be the $\sigma$-field generated by the open sets. We denote by $\mathcal{F}_{1}^{\infty}$ the space consisting of all infinite sequence $(F_{1}, F_{2}, \cdots), F_{n} \in \mathcal{F}$, and $\mathcal{R}_{1}^{\infty}$ the space consisting of all infinite sequences $(x_{1}, x_{2}, \cdots)$ of real numbers. Take the $\sigma$-field $\mathcal{A}^{\infty}_{1}$ to be the smallest $\sigma$-field of subsets of $\mathcal{F}_{1}^{\infty}$ containing all finite-dimensional rectangles and take $\mathcal{B}_{1}^{\infty}$ to be the Borel $\sigma$-field of $\mathcal{R}_{1}^{\infty}$. Let $\omega=(F_{1}^{\omega}, F_{2}^{\omega}, \cdots)$ be the coordinate process in $\mathcal{F}_{1}^{\infty}$ and $\nu$ its distribution on $\mathcal{A}^{\infty}_{1}$. Let $\theta$ be the coordinate shift: $\theta(\omega)=\omega'$ with $F_{n}^{\omega'}=F_{n+1}^{\omega}$, $k=1, 2, \cdots$. On $(\mathcal{R}_{1}^{\infty}, \mathcal{B}_{1}^{\infty})$ we also define the shift transformation $\sigma: \mathcal{R}_{1}^{\infty}\rightarrow \mathcal{R}_{1}^{\infty}$ by $\sigma(x_{1}, x_{2}, \cdots)=(x_{2}, x_{3}, \cdots)$. $\nu$ is called stationary if for every $A \in \mathcal{A}^{\infty}_{1}$, $\nu(\theta^{-1}(A))=\nu(A)$ and we let $\pi$ be its marginal distribution. Let $\tilde{\omega}$ be the $\sigma$-field of invariant sets in $\mathcal{B}_{1}^{\infty}$, that is, $\tilde{\omega} = \{B | \sigma^{-1}(B) = B, B \in \mathcal{B}_{1}^{\infty}\}$. For each $\omega$ define a probability measure $P_{\omega}$ on $(\mathcal{R}_{1}^{\infty}, \mathcal{B}_{1}^{\infty})$ so that $P_{\omega}:=\Pi_{n=1}^{\infty}F_{n}^{\omega}$. A monotone class argument shows that $P_{\omega}(B), B \in \mathcal{B}_{1}^{\infty}$, is $\mathcal{A}^{\infty}_{1}$-measurable as a function of $\omega$. So we can define a new probability measure $P$ such that $P(B)=\int P_{\omega}(B)\nu(d\omega)$. Define the process $\{X_{n}\}$ on $(\mathcal{R}_{1}^{\infty}, \mathcal{B}_{1}^{\infty})$ such that $X_{n}(x_{1}, x_{2}, \cdots)=x_{n}$ and set $S_{n}=X_{1}+X_{2}+\cdots+X_{n}$. By the definition of $P_{\omega}$, $\{X_{n}\}$ are independent with respect to $P_{\omega}$ and we also note that $\{X_{n}\}$ is a sequence of

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of independent and identically distributed random variables when \( \mathcal{G} \) has just one element. The purpose of this paper is to study LIL for partial sums of \( X_n \) with respect to \( P_\omega \) and as an application we apply this result to interchangeable processes. The following propositions are important basic tools throughout this paper.

**Proposition 1.** If \( \nu \) is stationary, then \( \{X_n\} \) is a stationary process with respect to \( P \).

**Proof.** Let \( f(\omega)=P_\omega(B), B \in \mathcal{B}_1^\infty \), then \( f \) is a measurable function of \( \omega \). Then

\[
P(B)=\int P_\omega(B)\nu(d\omega)=\int f(\omega)\nu(d\omega)=\int f(\theta(\omega))\nu(d\omega)
=\int P_\theta(\omega)(B)\nu(d\omega)=\int P_\omega(\sigma^{-1}(B))\nu(d\omega)=\int P_\omega(\sigma^{-1}(B)).
\]

**Proposition 2.** If \( \nu \) is ergodic, then \( \{X_n\} \) is ergodic with respect to \( P \).

**Proof.** Let \( C \in \mathcal{B}_1^\infty \) be an invariant set, i.e. \( \sigma^{-1}(C)=C \) and let \( f(\omega)=P_\omega(C) \), then

\[
f(\omega)=P_\omega(C)=P_\omega(\sigma^{-1}(C))=P_\theta(\omega)(C)=f(\theta(\omega)).
\]

This implies \( f \) is an invariant random variable, hence it is a.s. constant, since \( \nu \) is ergodic. By Proposition 6.32 [1] and Kolmogorov zero-one law, \( f(\omega)=0 \) or 1, then \( P_\omega(C)=0 \) \( \nu \)-a.e. \( \omega \) or 1 \( \nu \)-a.e. \( \omega \). Hence \( P(C)=0 \) or 1, therefore \( \{X_n\} \) is ergodic with respect to \( P \).

**Proposition 3.** Let \( A \subset \mathcal{B}_1^\infty \) be measurable. Then \( P_\omega(A)=1 \) for \( \nu \)-a.e. \( \omega \) if and only if \( P(A)=1 \).

**Proof.** The proof follows directly from the definition of \( P \).

2. Results and Proofs

As a generalization of the Hartman-Wintner theorem, we first prove the following theorem

**Theorem 1.** Let \( \mathcal{G}=\{F|\int xdF(x)=0\} \) and let \( \nu \) be stationary and ergodic with \( \int x^2dF(x)\pi(dF)=1 \). Then we have

\[
P_\omega\left\{\limsup\frac{S_n}{(2n \log \log n)^{1/2}}=1\right\}=1, \ \nu\text{-a.e. } \omega.
\]

To prove above theorem we need the following lemma.
Lemma 1. Let $\mathcal{F} = \{ F | \int x dF(x) = 0 \}$ and $\nu$ stationary with $\int |x| dF(x) \pi(dF) < \infty$. Then $\{X_n\}$ with respect to $P$ satisfies $E[X_1|X_1, X_2, \cdots, X_{i-1}] = 0$ a.s. for all $i \geq 2$.

Proof. By the assumption, $E|X_1| < \infty$ and hence $E[X_1|X_1, X_2, \cdots, X_{i-1}]$ exists for all $i \geq 2$. Now let $A \in \sigma(X_1, X_2, \cdots, X_{i-1})$ and let $(X_1, X_2, \cdots, X_{i-1}) \in B = A$ for some $i-1$ dimensional cylinder set $B$. Then we have

$$\int_A E[X_1|X_1, X_2, \cdots, X_{i-1}]dP = \int_A X_{i-1}dP = \int \int 1_B(x_1, \cdots, x_{i-1}) x_{i-1} dF_{1}^\omega(x_1) \cdots dF_{i-1}^\omega(x_{i-1}) \nu(d\omega)$$

$$= 0,$$

the last equality holding since $\int x dF(x) = 0$ for all $F \in \mathcal{F}$. This proves the lemma.

Proof of Theorem 1. By Propositions 1, 2 and Lemma 1, $\{X_n\}$ is a stationary and ergodic process with respect to $P$ such that $E[X_1|X_1, X_2, \cdots, X_{i-1}] = 0$ a.s. for all $i \geq 2$ and by assumption $EX_1^2 = \int x^2 dF(x) \pi(dF) = 1$. Now applying Stout’s result [5], we have

$$P\left\{ \lim \sup \frac{S_n}{(2n \log \log n)^{1/2}} = 1 \right\} = 1.$$  

Hence by Proposition 3,

$$P_\omega\left\{ \lim \sup \frac{S_n}{(2n \log \log n)^{1/2}} = 1 \right\} = 1, \ \nu-a.e. \ \omega.$$

What if the ergodicity assumption in above theorem is dropped? In this paper we obtain one possible answer for this question, that is, we need to impose one extra condition on $\mathcal{F}$. As an application we apply this result to interchangeable processes.

Theorem 2. Let $\mathcal{F} = \{ F | \int x dF(x) = 0 \}$ and $\mathcal{F} = \{ F | \int x^2 dF(x) = 1 \}$ and let $\nu$ be stationary. Then we have

$$P_\omega\left\{ \lim \sup \frac{S_n}{(2n \log \log n)^{1/2}} = 1 \right\} = 1, \ \nu-a.e. \ \omega.$$

Proof. By the ergodic decomposition theorem [6, Theorem 5.2.16], there
is a probability measure $\rho \nu$ on $M_{1}^{\theta}(\mathcal{F}_{1}^\infty)$, the space of stationary probability measures on $\mathcal{F}_{1}^\infty$, with the properties that $\rho(\nu EM_{1}^{\theta}(\mathcal{F}_{1}^\infty))=1$, where $EM_{1}^{\theta}(\mathcal{F}_{1}^\infty)$ is the set of ergodic elements of $M_{1}^{\theta}(\mathcal{F}_{1}^\infty)$, and

$$\nu = \int_{M_{1}^{\theta}(\mathcal{F}_{1}^\infty)} R \rho \nu(dR)$$

holds. For every $R \in EM_{1}^{\theta}(\mathcal{F}_{1}^\infty)$, we have that

$$\int \int x^{2}dF_{1}^{\omega}(x)R(d\omega)=1,$$

since

$$\int x^{2}dF_{1}^{\omega}(x)=1 \quad \text{for any } \omega.$$ 

Then by above theorem, we have for any $R \in EM_{1}^{\theta}(\mathcal{F}_{1}^\infty)$

$$\int P_{\omega}\{\lim \sup_{n} \frac{S_{n}}{(2n \log \log n)^{1/2}}=1\} R(d\omega)=1.$$

Now

$$\int P_{\omega}\{\lim \sup_{n} \frac{S_{n}}{(2n \log \log n)^{1/2}}=1\} \nu(d\omega)$$

$$= \int \int P_{\omega}\{\lim \sup_{n} \frac{S_{n}}{(2n \log \log n)^{1/2}}=1\} R(d\omega)\rho \nu(dR)=1,$$

which is equivalent to

$$P_{\omega}\{\lim \sup_{n} \frac{S_{n}}{(2n \log \log n)^{1/2}}=1\} = 1, \quad \nu\text{-a.e. } \omega,$$

which completes the proof.

Next we consider a special case of the model in introduction. Let $\nu_{\omega}\{F_{i}^{\omega} = F_{j}^{\omega} \text{ for all } i \neq j\}=1$. Then clearly $\nu$ is stationary and hence we see from Theorem 2 and the definition of $P$ that

$$P\{\lim \sup_{n} \frac{S_{n}}{(2n \log \log n)^{1/2}}=1\}=1$$

if

$$\nu_{\omega}\{\int xdF_{i}^{\omega}(x)=0 \text{ and } \int x^{2}dF_{i}^{\omega}(x)=1\}=1.$$ 

We shall show that this condition is also necessary. Since $\{X_{n}\}$ is independent and identically distributed with respect to $P_{\omega}$, $\nu$-a.e. $\omega$, 


$P_{\omega}\{\lim \sup \frac{S_{n}}{(2n \log \log n)^{1/g}} = 1\} = 1$, $\nu$-a.e. $\omega$

implies

$$\int x dF_{1}^{\omega}(x) = 0 \quad \text{and} \quad \int x^{2} dF_{1}^{\omega}(x) = 1 \quad \nu$-a.e. $\omega$.

by Martikainen theorem [4]. Hence

$$\nu\{\omega|\int x dF_{1}^{\omega}(x) = 0 \quad \text{and} \quad \int x^{2} dF_{1}^{\omega}(x) = 1\} = 1.$$

We summarize in

**Lemma 2.** Let $\nu\{\omega|F_{i}^{\omega} = F_{j}^{\omega} \quad \text{for all} \quad i \neq j\} = 1$. Then

$$P\{\lim \sup \frac{S_{n}}{(2n \log \log n)^{1/g}} = 1\} = 1$$

if and only if

$$\nu\{\omega|\int x dF_{1}^{\omega}(x) = 0 \quad \text{and} \quad \int x^{2} dF_{1}^{\omega}(x) = 1\} = 1.$$

A random variables $\{Y_{n}, n \geq 1\}$ on $(\Omega, \mathcal{G}, \tilde{P})$ are called interchangeable if the joint distribution of every finite subset of $k$ of these random variables depends only on $k$ and not the particular subset, $k \geq 1$. According to Theorem 7.3.2 [2] the random variables $\{Y_{n}, n \geq 1\}$ are conditionally i.i.d. given the $\sigma$-field $\mathcal{G}$ of permutable events and according to Corollary 7.3.5 [2] there is a regular conditional distribution, say $P^{\omega}$, for $Y=(Y_{1}, Y_{2}, \cdots)$ given $\mathcal{G}$ such that for each $\omega \in \Omega$ the coordinate random variables $\{X_{n}, n \geq 1\}$ of the probability space $(\mathcal{R}^{1}, \mathcal{B}^{1}, P^{\omega})$ are i.i.d.. Now suppose $EY_{1}=0$ and $EY_{1}^{2}=1$. Then, moreover, we see that

$$Cov(Y_{1}, Y_{2}) = 0 = Cov(Y_{1}^{2}, Y_{2}^{2})$$

is equivalent to

$$E[Y_{1}|\mathcal{G}] = 0 \quad \text{and} \quad E[Y_{1}^{2}|\mathcal{G}] = 1 \quad \text{a.s.} \quad [2, \text{p. 310}].$$

Consequently, for almost all $\omega$

$$\int_{-\infty}^{\infty} \xi_{1} dP^{\omega} = 0, \quad \int_{-\infty}^{\infty} \xi_{1}^{2} dP^{\omega} = 1,$$

and thus, by Lemma 2 we have the following theorem.

**Theorem 3.** Let $\{Y_{n}, n \geq 1\}$ be an interchangeable process with mean zero and variance one. Then the law of the iterated logarithm holds for the process if
and only if

\[ \text{Cov}(Y_1, Y_2) = 0 = \text{Cov}(Y_1^2, Y_2^2). \]

References


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