A NOTE ON THE BIDUAL OF A $C^*$-CROSSED PRODUCT

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ABSTRACT. Let $G$ be a locally compact group. We show that, for any $C^*$-dynamical system $(A, G, \alpha)$, the bidual $(G \times A)''$ of the $C^*$-crossed product $G \times A$ is canonically isomorphic to the von Neumann algebra generated by the regular representation of $G \times A$ if, and only if, $G$ is amenable and the group $C^*$-algebra $C^*(G)$ is scattered.

1. Introduction

Let $(A, G, \alpha)$ be a $C^*$-dynamical system. In [4], we proved that if $G$ is a discrete group which acts freely on $A$ in a strong sense, then the bidual $(G \times A)''$ of the $C^*$-crossed product $G \times A$ is *-isomorphic to the $W^*$-crossed product $G \times A''$ of the $W^*$-dynamical system $(A'', G, \alpha'')$ where $\alpha''$ is the bitransposed action of $\alpha$ on the bidual $A''$ of $A$. As $G \times A''$ is just the von Neumann algebra $M$ generated by the regular representation of $G \times A$ when $G$ is discrete, it is natural to ask under what circumstances such an isomorphism between $(G \times A)''$ and $M$ still persists if $G$ is nondiscrete. The purpose of this note is to show that, given any locally compact group $G$, the bidual $(G \times A)''$ is canonically isomorphic to $M$ for any $C^*$-dynamical system $(A, G, \alpha)$ if, and only if, $G$ is amenable and the group $C^*$-algebra $C^*(G)$ is scattered which in turn, is equivalent to the condition that the Fourier algebra $A(G)$ of $G$ coincides with the Fourier-Stieltjes algebra $B(G)$.

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As usual, we will identify the bidual $A''$ of a $C^*$-algebra $A$ with its universal enveloping von Neumann algebra. A $C^*$-algebra $A$ is called scattered [8] if every (nondegenerate) representation of $A$ is a direct sum of its irreducible subrepresentations or equivalently, if its bidual $A''$ is a direct sum of type I factors. A separable $C^*$-algebra $A$ is scattered if, and only if, its spectrum $\hat{A}$ is countable. If $G$ is a compact group, then every continuous unitary representation of $G$ is a direct sum of
irreducible ones, so the group $C^*$-algebra $C^*(G)$ of $G$ is scattered. On the other hand, the group $C^*$-algebra of the integers $\mathbb{Z}$ is not scattered. Fell (cf. [2; p. 142]) has given an example of a second countable noncompact amenable group with countable dual space and so its group $C^*$-algebra is scattered. It seems an interesting question to find intrinsic characterizations of $G$ for $C^*(G)$ to be scattered. However, we will prove that the group $C^*$-algebra $C^*(G)$ of a discrete group $G$ is scattered (if and) only if $G$ is finite.

2. Fourier algebra

Let $G$ be a locally compact group and let $\lambda_0: G \rightarrow B(L_2(G))$ be the left regular representation which extends to a representation of the group $C^*$-algebra $C^*(G)$ of $G$. In the sequel, we will use the following commutative diagram:

\[
\begin{array}{ccc}
G^*(G) & \overset{\lambda_0}{\longrightarrow} & B(L_2(G)) \\
\downarrow & & \downarrow \tau_0 \\
C^*(G)^{''} & & 
\end{array}
\]

where $\tau_0$ is the extension of $\lambda_0$ on the bidual $G^*(G)^{''}$ and $\lambda_0(C^*(G))$ is the reduced group $C^*$-algebra of $G$. The weak closure $M(G)$ of $\lambda_0(C^*(G))$ in $B(L_2(G))$ is the group von Neumann algebra of $G$. It is well-known that the Banach dual $C^*(G)'$ of $C^*(G)$ is linearly isomorphic to the (complex) linear span $B(G)$ of all continuous positive definite functions on $G$ (cf. [10; 7.1.8, 7.1.10]) and if $B(G)$ is equipped with the pointwise multiplication and the norm inherited from $C^*(G)'$, then it becomes a Banach algebra and is called the Fourier-Stieltjes algebra of $G$. Moreover, the closed subalgebra $A(G)$ of $B(G)$ generated by the positive definite functions with compact supports in $G$ can be identified with the predual $M(G)_{\ast}$ of $M(G)$ and is known as the Fourier algebra of $G$. When $G$ is abelian, $A(G)$ is the image of $L_1(\hat{G})$ under the Fourier transform, where $\hat{G}$ is the Pontryagin dual of $G$. We refer to [1, 6] for other properties of $A(G)$. Recently, De Cannière and Rousseau [5] proved that $A(G)$ is the smallest (nonzero) closed order and algebra ideal of $B(G)$.

3. $C^*$-crossed products

We now consider the more general set-up of a $C^*$-dynamical system $(A, G, \alpha)$. We will denote by $K(G, A)$ the linear space of continuous functions from $G$ to $A$ with compact support. Let $\pi_u: A \rightarrow B(H_u)$ be the universal representation of $A$ and let $\tilde{\pi}_u: A \rightarrow B(L_2(G, H_u))$ and $\lambda: G \rightarrow B(L_2(G, H_u))$ be the representations defined by

\[
(\tilde{\pi}_u(a)\xi)(t) = \pi_u(\alpha_{-t}(a))\xi(t)
\]

\[
(\lambda_s\xi)(t) = \xi(s^{-1}t)
\]
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for \( a \in A, s, t \in G \) and \( \xi \in L_2(G, H_u) \). As in [10; 7.7], we define the associated regular representation \( \tilde{\pi}_u \times \lambda : G \times A \rightarrow B(\mathcal{L}_2(G, H_u)) \) by

\[
(((\tilde{\pi}_u \times \lambda)f)\xi) Xt = \int_G \pi_u(\alpha_{t^{-1}}(f(s)))\xi(s^{-1}t)ds
\]

where \( ds \) is the left Haar measure on \( G \), \( f \in K(G, A) \) and \( \xi \in L_2(G, H_u) \). We have the following commutative diagram:

\[
\begin{array}{ccc}
G \times A & \xrightarrow{\tilde{\pi}_u \times \lambda} & B(\mathcal{L}_2(G, H_u)) \\
\pi & \downarrow & \\
(G \times A)^{\prime \prime}
\end{array}
\]

where \( \pi \) is the universal representation of \( G \times A \), \( \tau \) is the extension of \( \tilde{\pi}_u \times \lambda \) on the bidual \( (G \times A)^{\prime \prime} \) and \( \tau((G \times A)^{\prime \prime}) \) is the weak closure of \( (\tilde{\pi}_u \times \lambda)(G \times A) \) in \( B(\mathcal{L}_2(G, H_u)) \).

We will denote this weak closure by \( M(A, G, \alpha) \). We note that the previous diagram is a special case of the above one in which \( A = C \), \( \alpha \) reduces to the trivial action, \( G \times C \) is the group \( C^\ast \)-algebra \( C^\ast(G) \) and \( M(C, G, \alpha) \) the group von Neumann algebra \( M(G) \). We also note that \( (\tilde{\pi}_u \times \lambda)(G \times A) \) is the reduced \( C^\ast \)-crossed product \( G \times A \).

If \( G \) is discrete, then the bitransposed action \( \alpha'': g \in G \rightarrow \alpha'' \in \text{Aut}(A'') \) on the bidual \( A'' \) induces a \( W^\ast \)-dynamical system \( (A'', G, \alpha'') \) and in this case, \( M(A, G, \alpha) \) is just the \( W^\ast \)-crossed product \( G \times A'' \). We have shown in [4] that if \( \alpha \) is a strongly centrally free action, then the map \( \tau \) in the above diagram is faithful and so \( (G \times A)^{\prime \prime} \) can be identified with \( G \times A'' \). We now investigate the faithfulness of \( \tau \) in the nondiscrete situation. We say that \( (G \times A)^{\prime \prime} \) is canonically isomorphic to \( M(A, G, \alpha) \) if \( \tau \) is faithful.

We first observe that if \( B \) is any \( C^\ast \)-algebra and if \( \phi : B \rightarrow M \) is a \( * \)-homomorphism into a von Neumann algebra \( M \) with predual \( M_\ast \), then the extension \( \check{\phi} \) of \( \phi \) on the universal envelope \( B'' \) in the diagram

\[
\begin{array}{ccc}
B & \xrightarrow{\phi} & M \\
\downarrow & \nearrow \check{\phi} & \\
B''
\end{array}
\]

is faithful if, and only if, \( \phi'(M_\ast) = B' \) where \( \phi' : M' \rightarrow B' \) is the transpose of \( \phi \). Note that \( \check{\phi} \) is the transpose of the restriction of \( \phi' \) to \( M_\ast \subset M' \) and if \( \phi(B) \) is weakly dense in \( M \), then \( \phi' \) is faithful on \( M_\ast \).

**Proposition 1.** Let \( (A, G, \alpha) \) be a \( C^\ast \)-dynamical system. Then the following
conditions are equivalent:

(i) \((G \times A)^\tau\) is canonically isomorphic to \(M(A, G, \alpha)\);

(ii) For any \(\psi \in (G \times A)^\tau\), there exist sequences \((\xi_n)\) and \((\eta_n)\) in \(L_2(G, H_u)\) with 
\[\sum_n \|\xi_n\|^2 < \infty \text{ and } \sum_n \|\eta_n\|^2 < \infty\] such that
\[\psi(y) = \sum_n \langle (\pi_u \lambda) y \xi_n, \eta_n \rangle \quad (y \in G \times A).\]

**Proof.** Let \(M_\alpha\) be the predual of \(M(A, G, \alpha)\). By the above remark, \(\tau\) is faithful if and only if \((\pi_u \lambda')'(M_\alpha) = (G \times A)'\). But if \(\psi \in (G \times A)'\) is equal to \((\pi_u \lambda)'(\omega)\) for some \(\omega \in M_\alpha\), then there exist sequences \((\xi_n)\) and \((\eta_n)\) in \(L_2(G, H_u)\) with 
\[\sum \|\xi_n\|^2 < \infty,\] 
\[\sum \|\eta_n\|^2 < \infty\] and
\[\omega(\cdot) = \sum \langle \cdot \xi_n, \eta_n \rangle\] such that
\[\psi(y) = (\pi_u \lambda)'(\omega)(y) = \omega((\pi_u \lambda)(y)) = \sum \langle \pi_u \lambda y \xi_n, \eta_n \rangle.\]

**Remark.** The faithfulness of \(\tau\) implies that of \(\pi_u \lambda\) in which case \(G \times A\) is isomorphic to the reduced \(C^*\)-crossed product \(G \times A\). Therefore \(\tau\) need not be faithful in general.

We write \(A(G \times A)\) for \((\pi_u \lambda)'(M_\alpha)\) which is the norm-closed subspace of \((G \times A)'\) consisting of functions of the form 
\[\psi(y) = \sum \langle (\pi_u \lambda) y \xi_n, \eta_n \rangle.\] Thus, \((G \times A)''\) is canonically isomorphic to \(M(A, G, \alpha)\) if, and only if, \(A(G \times A) = (G \times A)'\). Note that if \(A = C\), then \((G \times C) = \lambda_0^*(M(G)_\alpha)\) which is just the Fourier algebra \(A(G)\) and so the last condition is \(A(G) = B(G)\).

As before, we can identify the Banach dual \((G \times A)'\) with the linear span \(B(G \times A)\) of \(A'\)-valued functions \(\Phi : G \rightarrow A'\) which are positive definite with respect to \(\alpha\) (cf. [10; 7.6.10]). Moreover, if \(\Phi \in B(G \times A)\) is positive definite, then \(\|\Phi\| = \|\Phi(e)\|\) where \(e\) is the identity of \(G\). Also, if \(\psi \in A(G)\) is positive definite, then \(\psi \cdot \Phi \in B(G \times A)\) is positive definite with respect to \(\alpha\) and also \(\|\psi \cdot \Phi\| = \|\psi(e)\Phi(e)\| = \|\psi\| \cdot \|\Phi\|\) (cf. [10; 7.6.9]). Hence for \(\phi \in A(G)\) with positive decomposition \(\phi = \phi_1 - \phi_2\) where
\[\phi = \phi_1 + \phi_2\] we have
\[\phi \cdot \Phi = \phi_1 \cdot \Phi - \phi_2 \cdot \Phi \in B(G \times A)\]
and
\[\|\phi \cdot \Phi\| = \|\phi_1 \cdot \Phi - \phi_2 \cdot \Phi\| = \|\phi_1 \cdot \|\Phi\| + \|\phi_2 \cdot \|\Phi\| = \|\phi_1\| \cdot \|\Phi\| + \|\phi_2\| \cdot \|\Phi\| = (\|\phi_1\| + \|\phi_2\|) \cdot \|\Phi\| = \|\phi\| \cdot \|\Phi\|.\]

**Lemma 2.** Let \(A\) be a \(C^*\)-algebra and suppose that there is a minimal central projection \(p\) in \(A''\). If \(z\) is a central projection in \(A''\) with \(pz = 0\), then \(A' : z\) is not \(w^*\)-dense in \(A'\), where we define as usual \((f \cdot z)(a) = f(az)\) for \(f \in A'\) and \(a \in A\).
Proof. We note that the split (invariant) faces of the state space of a unital C*-algebra B are in natural one-one correspondence with the central projections in B' (cf. [11; III.6]).

Let A, be the C*-algebra obtained by adjunction of an identity to A. Let Q = \{f ∈ A_+: ∥f∥ ≤ 1\} be the quasi-state space of A which is affine w*-homeomorphic to the state space of A, [11; p. 166]. Now A''_1 = A'' ⊕ C and p is a minimal central projection in A''_1. So F = \{f ∈ Q: f(p) = 0\} is a (proper) maximal split face of Q. But the w*-closure \(\overline{F}\) of F is also a split face of Q and \(\overline{F} ≠ Q\). Hence \(F = \overline{F}\) is w*-closed. Now \(V_z = ∪_{λ ≥ 0} λ F\) is a hereditary subspace of A_1 which is w*-closed since F is the intersection of \(V_z\) with the closed unit ball of A' [11; p. 146]. Therefore, by [11; Proposition III.4.13], \(V_z = V ∩ A_1\), for some proper w*-closed (invariant) subspace \(V\) of A'. It follows from \(pz = 0\) that \(A' \cdot z ⊂ V\) and so the w*-closure of \(A' \cdot z\) is properly contained in A'.

Now we prove the main result.

Theorem 3. Let G be a locally compact group. Then the following conditions are equivalent:
(i) G is amenable and \(C^*(G)\) is a scattered C*-algebra;
(ii) \(A(G) = B(G)\);
(iii) For any C*-dynamical system \((A, G, \alpha)\), the bidual \((G × A)'\) is canonically isomorphic to \(M(A, G, \alpha)\).

Proof. (i) ⇒ (ii). We prove that \(λ_0^t(M(G)_e) = C^*(G)'\) since there is a linear isomorphism which identifies \(A(G)\) with \(λ_0^t(M(G)_e)\) and \(B(G)\) with \(C^*(G)\)' as \(G\) is amenable, \(λ_0^t(M(G)_e)\) is w*-dense in \(C^*(G)'\) [10; 7.3.9]. To show that they are actually equal, we prove that \(τ_0\) is faithful. As \(C^*(G)\) is scattered, there is a family \(\{p_j\}\) of minimal central projections in \(C^*(G)''\) with \(∑ p_j = 1\). Let \(z\) be a central projection in \(C^*(G)''\) such that \(ker τ_0 = C^*(G)'(1-z)\). By minimality, we have either \(pz = 0\) or \(pz = p_j\) for each j. Suppose \(z ≠ 1\), then \(pz = 0\) for some j. By Lemma 2, \(C^*(G)' \cdot z\) is not w*-dense in \(C^*(G)'\). If \(ω ∈ M(G)_e\), then \((1-z)(λ_0^t(ω)) = τ_0(1-z)(ω) = 0\). Therefore \(λ_0^t(M(G)_e) ⊂ C^*(G)' \cdot z\) which implies that \(λ_0^t(M(G)_e)\) is not w*-dense in \(C^*(G)'\). This is impossible. Hence \(z = 1\) and \(τ_0\) is faithful. So we have \(A(G) = B(G)\).

(ii) ⇒ (iii). Let \((A, G, \alpha)\) be a C*-dynamical system. By [Proposition 1], the faithfulness of \(τ\) is equivalent to \(A(G × A) = (G × A)'\). We prove the latter. It suffices to show that the positive definite \(A'\)-valued functions \(Φ\) in \(B(G × A)\) are contained in \(A(G × A)\). As \(A(G) = B(G)\), the constant 1-function on \(G\) is the norm-limit of a sequence \((φ_n)\) of positive definite functions with compact supports in \(G\) [10; 7.2.5]. By [10; 7.7.6], we have \(φ_n · Φ ∈ A(G × A)\). Now \(∥Φ − φ_n · Φ∥ ≤ ∥1 − φ_n∥ · ∥Φ∥ → 0\) as \(n → ∞\). Hence \(Φ ∈ A(G × A)\). This proves that \(A(G × A) = \)
B(G \times A) and so $\tau$ is an isomorphism from $(G \times A)^{''}$ onto $M(A, G, \alpha)$.

(iii) $\Rightarrow$ (i). First, by considering the trivial $C^*$-dynamical system $(C, G, i)$, $\tau_0: C^*(G)^{''} \to B(L_2(G))$ is faithful and so is the regular representation $\lambda_0: C^*(G) \to B(L_2(G))$. Therefore $G$ is amenable. Let $A$ be any $C^*$-algebra and consider the $C^*$-dynamical system $(A, G, i)$ in which $i$ is the trivial action. The bitranspose $i''$ of $i$ induces the $W^*$-dynamical system $(A'', G, i'')$ and it is not difficult to verify that $M(A, G, i)$ is naturally isomorphic to the $W^*$-tensor product $A'' \hat{\otimes} M(G)$. Also the $C^*$-crossed product $G \times A$ is naturally isomorphic to the projective $C^*$-tensor product $A \hat{\otimes} C^*(G)$ where $C^*(G)^{''}$ is isomorphic to $M(G)$ via $\tau_0$.

We therefore have the following canonical isomorphisms

$$(A \hat{\otimes} C^*(G))^{''} \approx (G \times A)^{''} \approx M(A, G, i) \approx A'' \hat{\otimes} M(G) \approx A'' \hat{\otimes} C^*(G)^{''}$$

As $A$ was arbitrary, by a result of Huruya [7; p. 23], $C^*(G)$ is a scattered $C^*$-algebra. The proof is complete.

We conclude with two relevant results.

**Proposition 4.** Let $G$ be a discrete group. Then the group $C^*$-algebra $C^*(G)$ is scattered if, and only if, $G$ is finite.

**Proof.** We need only prove the sufficiency. As $C^*(G)$ is a type I $C^*$-algebra, by Thoma's characterization of type I groups [12], there is an abelian normal subgroup $\Delta$ of $G$ with finite index. It suffices to show that $\Delta$ is finite. Since $C^*$-subalgebras of a scattered $C^*$-algebra are also scattered [3], the group $C^*$-algebra $C^*(\Delta)$ is also scattered. But $C^*(\Delta)$ is the $C^*$-algebra $C(\hat{\Delta})$ of continuous functions on the dual group $\hat{\Delta}$ which is compact and so has finite Haar measure $\mu$. By scatteredness of $C(\hat{\Delta})$, $\mu$ must be atomic [8] and hence $\hat{\Delta}$ must be finite since $\mu$ is finite and invariant. It follows that $\Delta$ is finite by Pontryagin duality.

If $G$ is discrete, then $A''$ can be embedded into $(G \times A)^{''}$ as in [3, 4] where it has been shown that if the relative commutant of the centre of $A''$ in $(G \times A)^{''}$ is contained in $A''$, then $(G \times A)^{''}$ is canonically isomorphic to the $W^*$-crossed product $G \times A''$. Conversely we have the following result.

**Proposition 5.** Let $(A, G, \alpha)$ be a $C^*$-dynamical system in which $G$ is a discrete group. Then the following two conditions are equivalent:

(i) If $m \in (G \times A)^{''}$ commutes with the centre of $A''$, then $m \in A''$;

(ii) The bitranspose $\alpha''$ acts freely on the centre of $A''$ and $(G \times A)^{''}$ is canonically isomorphic to the $W^*$-crossed product $G \times A''$.

**Proof.** This follows readily from a result of Nakagami and Takesaki [9; p. 102] which states that $\alpha''$ acts freely on the centre of $A''$ if and only if the relative commutant of the centre of $A''$ in $G \times A''$ is contained in $A''$. 


References


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