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SOME SERIES INVOLVING RIEMANN ZETA FUNCTION

By

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1. Introduction

We deal with three series involving Riemann zeta function $\zeta(s)$ in each term. The literature of such series is quite interesting and work has been done mainly by Landau [6], Riesz [8], Hardy and Littlewood [3], and Ramaswami [7] in past. Recently, activities are reported by Verma [12-15], Keshava Menon [4], Chowla and Hawkins [2], Suryanarayana [10] and Verma and Kaur [16, 17].

Landau [6] established

(1.1) $$\zeta(s)=\frac{1}{s-1}+1-\sum_{r=0}^{\infty}\frac{s(s+1)\cdots(s+r)}{(r+2)!}\{\zeta(s+r+1)-1\},$$

while Ramaswami [7] established

(1.2) $$(1-2^{1-s})\zeta(s)=\sum_{r=0}^{\infty}\frac{s(s+1)\cdots(s+r)}{(r+2)!}\frac{\zeta(s+r+1)}{2^{s+r+1}}.$$

Both the results were re-established by Keshava Menon [4]. We prove

(1.3) $$\zeta(s)=\frac{1}{s-1}+\frac{1}{2}+\frac{1}{2}\sum_{r=1}^{\infty}\frac{(-1)^{r+1}rs(s+1)\cdots(s+r)}{(r+2)!}\zeta(s+r+1)\qquad(\mathrm{Re}\, s<1)$$

and

(1.4) $$\zeta(s)=1+\frac{s+3}{(s-1)\cdot 2^{s+1}}+\frac{1}{2}\sum_{r=1}^{\infty}\frac{(-1)^{r+1}rs(s+1)\cdots(s+r)}{(r+2)!}\{\zeta(s+r+1)-1\}.$$

The third result which we give is a direct proof of Suryanarayana's [10] result

(1.5) $$\sum_{r=2}^{\infty}\frac{(-1)^{r}\zeta(s)}{s+1}=1+\frac{\gamma}{2}-\frac{\log 2\pi}{2}$$

using methods of Verma [12], where γ is the Euler's constant given by

$$\gamma=\lim_{n\to\infty}\left\{\sum_{m=1}^{n}\frac{1}{m}-\log n\right\}.$$

2. Proof of (1.3)

We have (Titchmarsh [11])

(2.1) $$\zeta(s)=\frac{1}{s-1}+\frac{1}{2}-s\int_1^\infty\left(x-[x]-\frac{1}{2}\right)\frac{dx}{x^{s+1}}, \qquad (\mathrm{Re}\, s>-1).$$

Now

$$\int_1^\infty\left(x-[x]-\frac{1}{2}\right)\frac{dx}{x^{s+1}}=\sum_{n=1}^\infty\int_n^{n+1}\left(x-n-\frac{1}{2}\right)\frac{dx}{x^{s+1}}$$

$$=\sum_{n=1}^\infty\left[\frac{1}{(s-1)n^{s-1}}\left\{1-\left(1+\frac{1}{n}\right)^{1-s}\right\}-\frac{n+1/2}{sn^s}\left\{1-\left(1+\frac{1}{n}\right)^{-s}\right\}\right]$$

$$=\sum_{n=1}^\infty\left[\frac{1}{(s-1)n^{s-1}}\left\{\frac{s-1}{n}-\frac{(s-1)s}{2!\,n^2}+\cdots+\frac{(-1)^{r+1}(s-1)s\cdots(s+r)}{(r+2)!\,n^{r+2}}+\cdots\right\}\right.$$

$$\left.-\frac{1}{sn^s}\left(n+\frac{1}{2}\right)\left\{\frac{s}{n}-\frac{s(s+1)}{2!\,n^2}+\cdots+\frac{(-1)^r s(s+1)\cdots(s+r)}{(r+1)!\,n^{r+1}}+\cdots\right\}\right]$$

$$=\sum_{n=1}^\infty\left[\frac{1}{n^s}\left\{1-\frac{s}{2!\,n}+\cdots+\frac{(-1)^{r+1}s(s+1)\cdots(s+r)}{(r+2)!\,n^{r+1}}+\cdots\right\}\right.$$

$$\left.-\frac{1}{n^s}\left(1+\frac{1}{2n}\right)\left\{1-\frac{s+1}{2!\,n}+\cdots+\frac{(-1)^r(s+1)\cdots(s+r)}{(r+1)!\,n^r}+\cdots\right\}\right]$$

$$=\sum_{n=1}^\infty\left[\frac{1-1}{n^s}-\frac{1}{n^{s+1}}\left(\frac{s}{2!}-\frac{s+1}{2!}+\frac{1}{2}\right)\right.$$

$$\left.+\sum_{r=1}^\infty\frac{(-1)^{r+1}(s+1)\cdots(s+r)}{(r+2)!\,n^{s+r+1}}\left\{s-(s+r+1)+\frac{r+2}{2}\right\}\right]$$

(2.2) $$=\frac{1}{2}\sum_{n=1}^\infty\sum_{r=1}^\infty\frac{(-1)^r r(s+1)(s+2)\cdots(s+r)}{(r+2)!\,n^{s+r+1}}, \qquad (\mathrm{Re}\, s<1).$$

Each row of this double series is uniformly convergent and sum by rows is an analytic function, so by Weierstrass theorem on double series of complex function (Knopp [5]), summing by columns, this is

(2.3) $$=\frac{1}{2}\sum_{r=1}^\infty\frac{(-1)^r r(s+1)(s+2)\cdots(s+r)}{(r+2)!}\zeta(s+r+1), \qquad (-1<\mathrm{Re}\, s<1),$$

giving

(2.4) $$\zeta(s)=\frac{1}{s-1}+\frac{1}{2}+\frac{1}{2}\sum_{r=1}^\infty\frac{(-1)^{r+1}rs(s+1)\cdots(s+r)}{(r+2)!}\zeta(s+r+1) \qquad (-1<\mathrm{Re}\, s<1).$$

The left side as well as the right side of (2.4) is analytic for Re $s<1$, so continuing analytically, the required result follows.

3. Proof of (1.4.)

To establish (1.4) we write

$$\int_1^\infty\left(x-[x]-\frac{1}{2}\right)\frac{dx}{x^{s+1}}$$

$$=\left\{\int_1^2+\int_2^\infty\right\}\left(x-[x]-\frac{1}{2}\right)\frac{dx}{x^{s+1}}$$

Proceeding as above we easily get that this

$$=\frac{1}{s-1}\left(1-\frac{1}{2^{s-1}}\right)-\frac{3}{2s}\left(1-\frac{1}{2^s}\right)$$

$$+\frac{1}{2}\sum_{r=1}^\infty\frac{(-1)^r r(s+1)\cdots(s+r)}{(r+2)!}\{\zeta(s+r+1)-1\}$$

giving

$$\zeta(s)=1+\frac{s+3}{(s-1)\cdot 2^{s+1}}+\frac{1}{2}\sum_{r=1}^\infty\frac{(-1)^{r+1}rs(s+1)\cdots(s+r)}{(r+2)!}\{\zeta(s+r+1)-1\}$$

for $\operatorname{Re} s>-1$. The restriction on s can be done away with as before, proving the result.

4. Proof of (1.5)

Suryanarayana [10] has proved (1.5) using Robbins [9] formula. We proceed directly as follows.

$$\sum_{r=2}^\infty\frac{(-1)^r\zeta(r)}{r+1}=\sum_{r=2}^\infty\sum_{m=1}^\infty\frac{(-1)^r}{(r+1)m^r}$$

Changing the order of summation of the above absolutely convergent double series, this

$$=\sum_{m=1}^\infty m\sum_{r=2}^\infty\frac{(-1)^r}{(r+1)m^{r+1}}$$

$$=\lim_{n\to\infty}\sum_{m=1}^n\left\{m\log\left(1+\frac{1}{m}\right)-1+\frac{1}{2m}\right\}$$

$$=\lim_{n\to\infty}\left\{n\log(n+1)-\log n!-n+\frac{\log n}{2}+\frac{1}{2}\left(\sum_{m=1}^n\frac{1}{m}-\log n\right)\right\}.$$

Using Stirling's formula (Bruijn [1]), this is

$$=\lim_{n\to\infty}\left\{n\log(n+1)-\left(n+\frac{1}{2}\right)\log n+n-\frac{\log 2\pi}{2}\right.$$

$$\left.+O\left(\frac{1}{n}\right)-n+\frac{\log n}{2}\right\}+\frac{\gamma}{2}$$

$$=1+\frac{\gamma}{2}-\frac{\log 2\pi}{2}.$$

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