ON INCREASING CONVEX FUNCTION OF \( \log \sigma \)

By

SATENDRA KUMAR VAISH

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1. Introduction

Consider a Dirichlet series \( f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n), \) \((s = \sigma + it, \lambda_n \geq 0, \lambda_n < \lambda_{n+1} \rightarrow \infty \) with \( n \)), which we shall assume to be absolutely convergent everywhere in the complex plane \( \mathbb{C} \) and is bounded in any left strip and hence it defines an entire function. The logarithmic mean of \( f(s) \) is defined as

\[
L(\sigma) = \lim_{T \to \infty} \left\{ \frac{1}{2T} \int_{-T}^{T} \log |f(\sigma + it)| \, dt \right\}.
\]

For any \( \delta > 0 \), we define [2, p. 231] the generalized logarithmic mean of \( f(s) \) as

\[
L_{\delta^*}(\sigma) = \lim_{T \to \infty} \left\{ \frac{\sigma^{-\delta-1}}{2T} \int_{0}^{\sigma} \int_{-T}^{T} x^\delta \log |f(x + it)| \, dx \, dt \right\}.
\]

Since \( \log L_{\delta^*}(\sigma) \) is an increasing convex function of \( \log \sigma \) [2, p. 232], we may represent it in terms of an integral given by

\[
\log L_{\delta^*}(\sigma) = \log L_{\delta^*}(\sigma_0) + \int_{\sigma}^{\sigma_0} \frac{U(x)}{x} \, dx,
\]

where \( U(x) \) is a positive real valued indefinitely increasing function of \( x \).

In this paper we are mainly interested in studying certain growth relations of \( U(\sigma) \) and the generalized logarithmic mean function \( L_{\delta^*}(\sigma) \) relative to each other.

2. Main Results

Theorem 1. For \( m > 0 \), let

\[
I_1 = \int_{\sigma_0}^{\infty} \frac{\log L_{\delta^*}(\sigma)}{\sigma^{m+1}} \, d\sigma,
\]

\[
I_2 = \int_{\sigma_0}^{\infty} \frac{U(\sigma)}{\sigma^{m+1}} \, d\sigma.
\]

Then \( I_1 \) and \( I_2 \) converge or diverge together.

Proof. From (1.2), we have
\[
\int_{\sigma_{0}}^{u} \frac{d\sigma}{\sigma^{m+1}} \int_{\sigma_{0}}^{\sigma} \frac{U(x)}{x} \, dx = \int_{\vee}^{u_{0}} \left\{ \log L_{\delta^{*}}(\sigma) - \log L_{\delta^{*}}(\sigma_{0}) \right\} \frac{d\sigma}{\sigma^{m+1}}
\]

Also,

\[
\int_{\sigma_{0}}^{u} \frac{d\sigma}{\sigma^{m+1}} \int_{\sigma_{0}}^{\sigma} \frac{U(x)}{x} \, dx = \int_{\sigma_{0}}^{u} \frac{\log L_{\delta^{*}}(\sigma)}{\sigma^{m+1}} \, d\sigma \quad \text{and} \quad \frac{\log L_{\delta^{*}}(\sigma)}{\sigma^{m}} \rightarrow 0, \quad \text{as} \quad \sigma \rightarrow \infty.
\]

Hence, from (2.3), we find that \( I_{2} \) is also convergent.

Now, if \( I_{1} \) is convergent, then, from (2.3), we get

\[
(2.4) \quad m \int_{\sigma_{0}}^{u} \frac{\log L_{\delta^{*}}(x)}{x^{m+1}} \, dx + \frac{\log L_{\delta^{*}}(\sigma)}{\sigma^{m}} < k
\]

for some \( k > 0 \). But

\[
\int_{\sigma_{0}}^{u} \frac{\log L_{\delta^{*}}(x)}{x^{m+1}} \, dx \quad \text{and} \quad \int_{\sigma_{0}}^{u} \frac{\log L_{\delta^{*}}(\sigma)}{\sigma^{m}} \left( \frac{1}{\sigma_{0}^{m}} - \frac{1}{\sigma^{m}} \right) > 0,
\]

so, both terms on the left hand side of (2.4) are positive. Hence \( I_{1} \) is also convergent. Thus \( I_{1} \) converges if, and only if, \( I_{2} \) converges. Appealing to Modus
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Tollendo Tollen's [1, p. 32] the divergence part of this theorem follows from its convergence part.

**Theorem 2.** Let

$$\lim_{\sigma \to \infty} \sup \frac{\log U(\sigma)}{F(\sigma)} = P, \quad 0 \leq Q \leq P \leq \infty,$$

where $F(\sigma)$ is a logarithmico exponential function of $\sigma$, such that, as $\sigma \to \infty$, $F(\sigma) \approx F(\sigma)$ ($k$ is a constant $>0$) and $\log \log \sigma = o(F(\sigma))$. Then

$$\lim \inf_{\sigma \to \infty} \frac{\sigma F'(\sigma) \log L_k(\sigma)}{U(\sigma)} = 0 \leq Q \leq P \leq \lim \sup_{\sigma \to \infty} \frac{\sigma F'(\sigma) \log L_k(\sigma)}{U(\sigma)}.$$

In order to prove this theorem we need the following lemma:

**Lemma 1.** Let

$$\phi(x) = A + \int_{x_0}^{x} \frac{g(t)}{t} dt,$$

where $g(x)$ is a positive non-decreasing function of $x$ for $x \geq x_0$ and $A$ is a constant $>0$. If

$$\lim_{x \to \infty} \sup \frac{\log g(x)}{F(x)} = N \leq M \leq \lim_{x \to \infty} \inf \frac{\log g(x)}{F(x)},$$

Then

$$\lim_{x \to \infty} \inf \frac{g(x)}{x \phi(x)} F(x) \leq N \leq M \leq \lim_{x \to \infty} \sup \frac{g(x)}{x \phi(x)} F'(x).$$

**Proof.** We have

$$\phi(x) = A + \int_{x_0}^{x} \frac{g(t)}{t} dt \leq g(x) \log x + \text{const.}$$

So,

$$\lim \sup_{x \to \infty} \frac{\log \phi(x)}{F(x)} \leq \lim \sup_{x \to \infty} \left\{ \frac{\log g(x)}{F(x)} \cdot \frac{\log \log x + \text{cost.}}{F(x)} \right\} = M.$$

Now,

$$\phi(2x) \geq \int_{x}^{2x} \frac{g(t)}{t} dt \geq g(x) \log 2.$$

Therefore,

$$\lim \sup_{x \to \infty} \frac{\log \phi(2x)}{F(2x)} \geq \lim \sup_{x \to \infty} \left\{ \frac{\log g(x)}{F(x)} \cdot \frac{F(x)}{F(2x)} + \frac{\log 2}{F(2x)} \right\} = M.$$
Hence
\[
\lim_{x \to \infty} \sup_{\infty} \frac{\log \phi(x)}{F(x)} = M.
\]
Similarly,
\[
\lim_{x \to \infty} \inf_{\infty} \frac{\log \phi(x)}{F(x)} = N.
\]
Now, from (2.6), we get, for \(x \geq x_0\),
\[
\phi'(x) = \frac{g(x)}{x \phi(x)}.
\]
Integrating in the Lebesgue sense between \(x_0\) and \(x\), we find
\[
(2.8) \quad \log \phi(x) = \int_{x_0}^{x} \frac{g(t)}{t \phi(t)} dt + \text{const}.
\]
Let,
\[
\lim_{x \to \infty} \sup_{\infty} \frac{g(x)}{x \phi(x) F'(x)} = C, \quad 0 \leq C \leq \infty.
\]
We first suppose that \(0 < D, C < \infty\). Then, for any \(\epsilon > 0\) and sufficiently large \(x\),
\[
(D - \epsilon) F'(x) < \frac{g(x)}{x \phi(x)} < (C + \epsilon) F'(x).
\]
Integrating in the Lebesgue sense, we get
\[
(D - \epsilon)(1 - \alpha(1)) \leq \frac{\log \phi(x)}{F(x)} \leq (C + \epsilon)(1 - \alpha(1)),
\]
or,
\[
(2.9) \quad D \leq N \leq M \leq C,
\]
which also holds, when \(D = 0\) or \(C = \infty\). If \(D = \infty\), then so is \(C\) and
\[
\lim_{x \to \infty} (g(x)/x \phi(x) F'(x)) = \infty.
\]
So, taking an arbitrary large real number in place of \(D - \epsilon\) and proceeding as above, we obtain \(M = N = \infty\). Similarly, if \(C = 0\), it can be shown that \(M = N = 0\). Hence, for \(0 \leq D \leq C \leq \infty\), (2.9) implies (2.7).

**Proof of theorem 2.** Replacing \(\phi(\sigma)\) by \(\log L_{\mu}^*(\sigma)\) and \(g(\sigma)\) by \(U(\sigma)\) in (2.7), we get Theorem 2.

**Theorem 3.** Let
\[
\lim_{\sigma \to \infty} \sup_{\sigma \in [h, H]} \frac{\lambda_{p+1} L_{\mu}^*(\sigma)}{L_{\mu}^*} = H, \quad 0 \leq h \leq H \leq \infty.
\]
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Then

\[ \lim_{\sigma \to \infty} \inf \frac{(l_{1}L_{\delta^{*}}(\sigma))(l_{2}L_{\delta^{*}}(\sigma)) \cdots (l_{p}L_{\delta^{*}}(\sigma))}{U(\sigma)(l_{1}\sigma)(l_{2}\sigma) \cdots (l_{q-1}\sigma)} \leq \frac{1}{H} \leq \frac{1}{h} \leq \lim_{\sigma \to \infty} \sup \frac{(l_{1}L_{\delta^{*}}(\sigma))(l_{2}L_{\delta^{*}}(\sigma)) \cdots (l_{p}L_{\delta^{*}}(\sigma))}{U(\sigma)(l_{1}\sigma)(l_{2}\sigma) \cdots (l_{q-1}\sigma)}, \]

where $l_{k}\sigma$ denotes $k$-th iterate of $\log \sigma$.

The proof of this theorem is based on the following lemma:

Lemma 2. Let

\[ G(x) = A + \int_{x_{0}}^{x} \frac{\Psi(t)}{t} dt, \]

where $\Psi(x)$ is a positive and non-decreasing function of $x$ for $x \geq x_{0}$. If

\[ \lim_{x \to \infty} \sup \frac{l_{p}G(x)}{l_{q}x} = S, \quad 0 \leq S \leq T \leq \infty, \]

Then

\[ \lim_{x \to \infty} \inf \frac{G(x)(l_{1}G(x))(l_{2}G(x)) \cdots (l_{p-1}G(x))}{\Psi(x)(l_{1}x)(l_{2}x) \cdots (l_{q-1}x)} \leq \frac{1}{T} \leq \frac{1}{S} \leq \lim_{x \to \infty} \sup \frac{G(x)(l_{1}G(x))(l_{2}G(x)) \cdots (l_{p-1}G(x))}{\Psi(x)(l_{1}x)(l_{2}x) \cdots (l_{q-1}x)}. \]

Proof. Let

\[ \lim_{x \to \infty} \sup \frac{G(x)(l_{1}G(x))(l_{2}G(x)) \cdots (l_{p-1}G(x))}{\Psi(x)(l_{1}x)(l_{2}x) \cdots (l_{q-1}x)} = C, \quad 0 \leq C \leq \infty, \]

and suppose that $d > 0$. Then, for any $\varepsilon > 0$ and $x \geq x_{0}$, we have

\[ G(x)(l_{1}G(x))(l_{2}G(x)) \cdots (l_{p-1}G(x)) > (d - \varepsilon)\Psi(x)(l_{1}x)(l_{2}x) \cdots (l_{q-1}x). \]

Differentiating (2.11), we get

\[ G'(x) = \frac{\Psi(x)}{x}. \]

Therefore,

\[ \frac{G'(x)}{G(x)(l_{1}G(x))(l_{2}G(x)) \cdots (l_{p-1}G(x))} < \frac{\Psi(x)}{(d - \varepsilon)\Psi(x)(l_{1}x)(l_{2}x) \cdots (l_{q-1}x)}. \]

Integrating (2.13) in the Lebesgue sense, between $x_{0}$ and $x$, we obtain

\[ l_{p}G(x) = \int_{x_{0}}^{x} \frac{G'(t)}{G(t)(l_{1}G(t))(l_{2}G(t)) \cdots (l_{p-1}G(t))} dt < \frac{l_{p}x}{d - \varepsilon}. \]
or,
\[
\frac{\log L_{\delta^{*}}(x)}{l_{x}x} < \frac{1}{d-\epsilon}.
\]
So,
\[
(2.14)
\]
\[
d \leq \frac{1}{T},
\]
which also holds when \(d=0\). If \(d=\infty\), the above argument with an arbitrary large real number instead of \(d-\epsilon\) gives \(T=0\). Hence, for \(0 \leq d \leq \infty\), (2.14) gives the left hand side of (2.12). Similarly, the right hand side follows.

**Proof of theorem 3.** Replacing \(G(\sigma)\) and \(\Psi(\sigma)\) by \(\log L_{\delta^{*}}(\sigma)\) and \(U(\sigma)\), respectively, we get the required result.

**Theorem 4.** If \(F(\sigma)\) is a logarithmic exponential function of \(\sigma\), such that, \(F(k\sigma) \approx F(\sigma)\) and \(\log L_{\delta^{*}}(\sigma) \approx F(\sigma)\). Then,
\[
\lim_{\sigma \to \infty} \frac{\log L_{\delta^{*}}(\sigma)}{U(\sigma)} = \infty.
\]

**Proof.** For any \(\epsilon > 0\) and \(\sigma \geq \sigma_{0}\),
\[
\frac{F(\sigma)}{1-\epsilon} > \frac{F(2\sigma)}{1-\epsilon} > \log L_{\delta^{*}}(2\sigma) = \log L_{\delta^{*}}(\sigma) + \int_{\sigma}^{2\sigma} \frac{U(x)}{x} dx \geq \log L_{\delta^{*}}(\sigma) + U(\sigma) \log 2,
\]
and
\[
\frac{F(\sigma)}{1+\epsilon} < \log L_{\delta^{*}}(\sigma).
\]
So,
\[
\{(1-\epsilon)^{-1} - (1+\epsilon)^{-1}\}F(\sigma) > U(\sigma) \log 2.
\]
Thus,
\[
\lim_{\sigma \to \infty} \frac{U(\sigma)}{F(\sigma)} = 0.
\]
Since, \(F(\sigma) \approx \log L_{\delta^{*}}(\sigma)\). Hence
\[
\lim_{\sigma \to \infty} \frac{\log L_{\delta^{*}}(\sigma)}{U(\sigma)} = \infty.
\]
Thus the proof of **Theorem 4** follows.

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Department Mathematics
University of Roorkee
Roorkee-247672 (U. P.)
India