THREE-POINT BOUNDARY VALUE PROBLEMS—EXISTENCE AND UNIQUENESS

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Abstract. This paper gives a criterion for the existence and uniqueness of solutions to three-point boundary value problems associated with the third order non-linear differential equations. A matching technique is developed to match solutions of two, two-point boundary value problems which yield a unique solution of a certain class of three-point boundary value problems.

1. Introduction.

The study of three-point boundary value problems is an interesting area of current research and a great deal of work has been done by many authors in the recent years [1, 2, 3, 4, 5]. This paper gives a guarantee for the existence and uniqueness of solutions of three-point boundary value problems associated with the differential equation

\[ y''' = f(x, y, y', y'') \]

where \( f(x, y, y', y'') \) is assumed to be continuous on a subset of \( \mathbb{R}^4 \), solutions to initial value problems associated with \( (1.1) \) exist, are unique, and extend throughout a fixed subinterval of \( R \). In this paper a matching technique is developed to match solutions of two, two-point boundary value problems which yields a unique solution of three-point boundary value problems.

In Section 2 a monotonicity restriction on \( f \) ensures that the following boundary value problems:

\[ y''' = f(x, y, y', y'') \]
\[ y(x_i) = y_i, \quad y(x_i) = y_i, \quad y^{(i)}(x_i) = m \quad (i=1, 2) \]

have solutions and with added hypothesis a unique solution of the following three-point boundary value problem:

is constructed. This is accomplished by matching solutions of (1.2) and (1.3).

We need the following conditions for our latter discussion

(A) \( f: [x_1, x_2] \times \mathbb{R}^3 \rightarrow \mathbb{R} \) and \( g: [x_1, x_2] \times \mathbb{R}^4 \rightarrow \mathbb{R} \) are continuous functions.

(B) for all \( w_1, w_2 \in \mathbb{R} \) \( f(x, y_1, z_1, w_1) - f(x, y_2, z_2, w_2) \geq g(x, y_1 - y_2, z_1 - z_2, w_1 - w_2) \) for all \( x \in (x_1, x_2) \) if \( y_1 \leq y_2, \ z_1 \leq z_2 \) and \( f(x, y_1, z_1, w_1) - f(x, y_2, z_2, w_2) > g(x, y_1 - y_2, z_1 - z_2, w_1 - w_2) \) for all \( x \in (x_2, x_3) \) if \( y_1 \geq y_2, \ z_1 \geq z_2 \) where \( g(x, u, v, w) \) satisfies.

(C) the initial value problems \( z'''' = g(x, z, z', z''), \ z(0) = \delta, \ z'(0) = 0, \ z''(0) = \delta \) or \( z(0) = 0, \ z'(0) = 0, \ z''(0) = 0 \) where \( c \leq x \), has a solution defined for all \( x \geq c \) (\( \delta \) arbitrary).

(D) there exists a number \( h > 0 \) such that no non-trivial solution \( z(x) \) of (C) may satisfy \( z(x) = z(d) = 0 \) with \( 0 < |d-c| < h \).

(E) for any \( p, \ g(x, z_1, v_1, p) \geq g(x, z_2, v_2, p) \) either \( z_1 \leq z_2, \ v_1 \geq v_2, \ x \in [x_1, x_2] \) or \( g(x, z_1, v_1, p) \geq g(x, z_2, v_2, p) \) if \( z_1 \geq z_2, \ v_1 \geq v_2, \ x \in [x_2, x_3] \). Note that the functions defined by \( f(x, y, y', y'') = xy \) and \( g(x, z, z', z'') = -1 \) satisfies all the above conditions at 0 on any interval \((a, b)\) where \( a < 0 < b \).

The major advantage of this study is to find existence and uniqueness of solutions of more general class of the following boundary value problems, which are not covered in [1] and also includes some of the cases of [1] when \( g = 0 \).

2. Existence and uniqueness of solutions to three-point boundary value problems.

In this section we intend to find some criteria under which solutions of (1.1) which satisfy boundary conditions at two-points may be matched to obtain a unique solution of three-point boundary value problems. We assume in the sequel that initial value problems of the type (C) exists and are unique in the interval \([x_1, x_3]\). Theorem 2.1 displays the idea of matching of solutions of three-point boundary value problems associated with (1.4). We now prove the following lemmas which will be used in our subsequent discussion.

**Lemma 2.1.** Assume that conditions (A), (B), (C), (D) and (E) hold. Then if \( x_2-x_1 \leq h \) (or \( x_3-x_2 \leq h \)) and for each \( y_1, \ y_2, \ y_3, m \in \mathbb{R} \), there exists at most one solution of either of the following boundary value problems

\[
\begin{align*}
y''' &= f(x, y, y', y'') \\
y(x_1) &= y_1, & \ y(x_2) &= y_2, & \ y'(x_3) &= m
\end{align*}
\]
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\[ y''' = f(x, y, y', y'') \]
\[ y(x_2) = y_2, \quad y'(x_2) = m, \quad y(x_3) = y_3. \]

**Proof.** The proof of the uniqueness of solution of (2.2) will be given. Similar arguments will hold for (2.1). Suppose \( \theta(x) \) and \( \eta(x) \) be solutions of (2.2) and write \( \phi(x) = \theta(x) - \eta(x) \). Without loss of generality we can assume that there exists \( c \) and \( d \) such that \( x_2 \leq c < d \leq x_3 \), \( \phi(c) = \phi'(c) = \phi(d) = 0 \) and \( \phi(x) > 0 \) for all \( x \in (c, d) \). Since \( \phi(c) = 0 = \phi(d) \), it follows that there exists an \( r \in (c, d) \) such that \( \phi'(r) = 0 \) and \( \phi'(x) > 0 \) for all \( x \in (c, r) \). Consider the solution of (C) with initial conditions \( z(c) = 0, z'(c) = 0 \) and \( z''(c) = \phi''(c) \). Let \( \phi(x) = \phi(x) - Z(x) \). Clearly \( \phi(c) = 0 = \phi'(c) = \phi''(c) \), \( \phi'(d) = -Z(d) < 0 \) and
\[
\psi(t) = \phi(t) - Z(t) = \phi(t) - \phi(t) + Z(t) = Z(t) - \phi(t) = Z(c) - \phi(c) = 0.
\]
Hence it follows that there exists an \( r \in (c, d) \) such that \( \phi(r) = 0 \) and \( \phi(t) > 0 \) for all \( t \in (c, r) \). Since \( \phi(c) = 0, \phi(r) = 0 \) it follows that there exists a \( t_0 \in (c, r) \) such that \( \phi'(t) > 0 \) \( \forall t \in (c, t_0) \) and \( \phi'(t_0) = 0 \). This together with \( \phi'(c) = 0 \) gives that there exists a \( t_1 \in (c, t_0) \) such that \( \phi''(t_1) = 0 \) and \( \phi'''(t_1) \leq 0 \) and \( \phi''(t) > 0 \) for all \( t \in (c, t_1) \). But
\[
\phi'''(t_1) = \phi'''(t_1) - Z'''(t_1)
= f(t_1, \theta(t_1), \theta'(t_1), \theta''(t_1)) - f(t_1, \eta(t_1), \eta'(t_1), \eta''(t_1)) - Z'''(t_1)
> g(t_1, \theta(t_1), \theta'(t_1), \theta''(t_1)) - g(t_1, \eta(t_1), \eta'(t_1), \eta''(t_1)) - Z'''(t_1)
= g(c, 0, 0, 0, Z''(c)) - g(c, 0, 0, Z''(c)) - Z'''(c)
= 0.
\]

**Lemma 2.2.** Assume that conditions (A), (B), (C), (D) and (E) hold. Then if \( x_3 - x_2 \leq h \) and for each \( y_1, y_2, y_3, m \in R \) there exists almost one solution of each of the following boundary value problems:
\[
y''' = f(x, y, y', y'')
y(x_2) = y_2, \quad y'(x_2) = m, \quad y(x_3) = y_3.
\]
or
\[
y(x_1) = y_1, \quad y(x_2) = y_2, \quad y''(x_2) = m.
\]

**Proof.** The proof is analogous as in **Lemma 2.1.**
Theorem 2.1. Let \( f(x, y, z, w) \) satisfy conditions (A), (B), (C), (D) and (E) and suppose that for each \( m \in R \) there exist solutions of (1.2), and (1.3), \( (i=1, 2) \). Then there exists a unique solution of (1.4).

Proof. From Lemmas 2.1 and 2.2 the solutions of (1.2), and (1.3), \( (i=1, 2) \) whenever exists are unique. Let \( \theta \) be a solution of (1.2), with second derivative \( m \) at \( x=x_2 \).

Claim. \( \theta'(x_2, m) \) is an increasing function of \( m \) with range \( R \). Write \( \phi=\theta(\cdot, m_1)-\theta(\cdot, m_2) \). If \( m_1<m_2, \phi''(x_2)<0 \). Clearly \( \phi(x_1)=0, \phi(x_2)=0 \). Without loss of generality we can assume that \( \phi'(x_2)>0 \) since \( \phi(x_1)=0, \phi(x_2)=0 \) there exists an \( r \in (x_1, x_2) \exists \phi'(r)=0, \phi'(x)>0 \ \forall x \in [r, x_2] \). Clearly \( \phi(x)<0 \) in \( [r, x_2] \). Consider the solution of initial value problem \( x''=g(x, z, z', z''), \ z(x_1)=0, \ z'(x_2)=0 \). Write \( \phi=\psi-\theta \). Clearly \( \phi(x_1)=0, \phi'(x_2)>0, \phi''(x_2)<0, \phi'(r)\leq 0 \). Hence there exists an \( r_1 \in (r, x_2) \) such that \( \phi''(r_1)=0 \) and \( \phi''(x)<0 \) in \( (r_1, x_2) \). These properties imply that \( \phi''(r_1)\leq 0 \). Now \( \phi'(x)>0 \) in \( [r_1, x_2] \) and \( \phi(x)<0 \) in \( [r_1, x_2] \). Consider

\[
\phi''(x_2)=\phi''(r_1)-Z''(r_1)=f(r, \theta(r_1, m_1), \theta'(r_1, m_1), \theta''(r_1, m_1))
- f(r, \theta(r_1, m_2), \theta'(r_1, m_2), \theta''(r_1, m_2)) - g(r_1, Z(r_1), Z'(r_1), Z''(r_1))
> \theta(r_1, \phi(r_1, m_1), \phi'(r_1, m_1), \phi''(r_1, m_1)) - g(r_1, Z(r_1), Z'(r_1), Z''(r_1))
> 0. \text{ Hence a contradiction.}
\]

Hence \( \phi'(x_2)\leq 0 \).

Let \( m \in R \). Now the problem (1.2) has a unique solution in \( [x_1, x_2] \) and let it be \( \phi_m \). Since \( \phi_m \) and \( \theta(\cdot, \phi_m''(x_2)) \) are solutions of the problem \( y''=f(x, y, y', y''), \ y(x_1)=y_1, \ y(x_2)=y_2, \ y''(x_2)=\phi_m''(x_2) \) in \( [x_1, x_2] \), we must have \( \theta(\cdot, \phi_m''(x_2))=\phi_m \). Hence \( \theta'(x_2, \phi_m''(x_2))=\phi_m'(x_2)=m \). Thus \( R \) is the range of \( \theta'(x_2, \cdot) \). Hence the claim. The remainder of the proof of existence follows as in [Theorem 2.1 in [1]].

Now to establish uniqueness, let \( \phi \) and \( \psi \) be solutions.

Claim. \( \phi''(x_2)=\psi''(x_2) \).

Suppose to the contrary \( \phi''(x_2)\neq\psi''(x_2) \). Without loss of generality assume that \( \phi''(x_2)<\psi''(x_2) \). Let \( \phi''(x_2)=m_1 \) and \( \psi''(x_2)=m_2 \). Then \( \phi \) is a solution of the problem (1.2) with second derivative \( m_1 \) at \( x=x_2 \) and \( \psi \) is a solution of the problem (1.3) with second derivative \( m_2 \) at \( x=x_2 \). Hence we must have

\[
\begin{align*}
\phi(x) &= \theta(x, m_1) \\
\psi(x) &= \theta(x, m_2)
\end{align*}
\]

for all \( x \in [x_1, x_2] \).

Therefore
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Now $m_1 < m_2 \Rightarrow \theta'(x_2, m_1) < \theta'(x_2, m_2)$ i.e.,

$$\phi'(x_2) < \psi'(x_2).$$  \hspace{1cm} (2.3)

Similarly $\phi$ and $\psi$ are solutions of the problem (1.3). Therefore, we must have

$$\phi(x) = \eta(x, m_1) \quad \text{for all} \quad x \in [x_2, x_3].$$

Hence

$$\phi'(x_2) = \eta'(x_2, m_1); \quad \phi'(x_2) = \eta'(x_2, m_2) \quad (i=1,2).$$

Now $m_1 < m_2 \Rightarrow \eta'(x_2, m_1) > \eta'(x_2, m_2)$ i.e.,

$$\phi'(x_2) > \psi'(x_2).$$  \hspace{1cm} (2.4)

(2.3) and (2.4) contradict each other. Hence the claim. Thus uniqueness is established.

The next theorem establishes validity of hypothesis (ii) in Theorem 2.1.

**Theorem 2.2.** Let $f: [x_1, x_3] \times \mathbb{R} \to \mathbb{R}$ with $x_1 < x_2 < x_3$ and suppose there exists a constant $N > 0$ such that $|f(x, y, z, w)| \leq N$ for all $x \in [x_1, x_3], -\infty < y, z, w < \infty$ then there exist solutions of the problems (1.2) and (1.3), $(i=1,2)$.

**Proof.** The proof is analogous as in Theorem 2.3 in [5].

**References**


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