GAP SERIES AND $\alpha$-BLOCH FUNCTIONS

By
SHINJI YAMASHITA

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1. Introduction. Let $f$ be a function holomorphic in $D=\{|z|<1\}$ with the gap series expansion

\[ f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}, \quad z \in D, \]

where for a constant $q>1$ the natural numbers $n_k$, $k \geq 1$, satisfy

\[ n_{k+1}/n_k \geq q, \quad k \geq 1. \]

A function $g$ in $D$ is called $\alpha$-Bloch ($\alpha>0$) if $g$ is holomorphic in $D$ and if

\[ \sup_{z \in D} (1-|z|)^{\alpha} |g'(z)| < \infty; \]

the family of all $\alpha$-Bloch functions is denoted by $B^\alpha$. A Bloch function [6] is precisely a 1-Bloch function. Let $B_0^\alpha(\alpha>0)$ be the family of $g$ holomorphic in $D$ such that

\[ \lim_{|z| \to 1} (1-|z|)^{\alpha} |g'(z)| = 0. \]

It is easy to observe that $B_0^\alpha \subset B^\alpha$.

Our main result in the present paper is

Theorem 1. Let $f$ be a holomorphic function in $D$ with (1.1) and (1.2). Then for $\alpha>0$, the following two propositions hold.

(I) $f \in B^\alpha$ if and only if

\[ \lim_{k \to \infty} \sup_{k} |a_k| n_k^{1-\alpha} < \infty. \]

(II) $f \in B_0^\alpha$ if and only if

\[ \lim_{k \to \infty} |a_k| n_k^{1-\alpha} = 0. \]

Theorem 1(I) in the case $\alpha=1$ is known; for the proof one should combine [5, Theorem 1] with [5, Theorem 2 (iii)]; note that the latter half of [8, Theorem] is identical with [5, Theorem 2 (iii)].
In Sections 3 and 4 we shall propose some applications of Theorem 1 (II) in the case $\alpha=1$.

The present work arises from the communications with Professor Peter A. Lappan to whom I wish to express my cordial thanks.

2. **Proof of Theorem 1.** We begin with

**Lemma.** Let $g(z) = \sum_{n=0}^{\infty} b_{n} z^{n}$ be holomorphic in $D$. If $g \in B^{\alpha}$ ($g \in B_{0}^{\alpha}$, respectively) for $\alpha>0$, then

\[ \limsup_{n \to \infty} |b_{n}| n^{1-\alpha} < \infty \quad \text{(lim} |b_{n}| n^{1-\alpha} = 0, \text{resp.)} \]

As a special case we obtain [5, Theorem 1] that $\{b_{n}\}$ is bounded if $g \in B^{1}$.

For the proof of **Lemma** we first note that $(1-n^{-1})^{1-n} \to e$ as $n \to \infty$. Assume that $g \in B^{\alpha}$. By the Cauchy formula one obtains for $n \geq 1$,

\[ |b_{n}| = |(2\pi in)^{-1} \int_{0}^{2\pi} g'(re^{i\theta}) r^{1-n} e^{i(1-n)\theta} d\theta| \leq C_{1} n^{-1} (1-r)^{-\alpha} r^{1-n} \]

for all $0<r<1$; hereafter $C_{k}(k=1,\ldots,7)$ denote positive constants. For $n>1$ and for $r=1-n^{-1}$ we thus obtain

\[ |b_{n}| \leq C_{1} n^{\alpha-1} (1-n^{-1})^{1-n} , \]

whence

\[ \limsup_{n \to \infty} |b_{n}| n^{1-\alpha} < \infty . \]

The proof for the case $g \in B_{0}^{\alpha}$ is similar to the above with a few modifications.

In view of **Lemma** the rest we should prove in [Theorem 1] is the "if" parts in (I) and (II). We first consider the case (I). First of all we notice by

\[ \frac{1}{(1-|z|)^{1+\alpha}} = \sum_{n=0}^{\infty} A_{n} |z|^{n} , \quad A_{n} \sim \Gamma(1+\alpha)^{-1} n^{\alpha} , \]

that

\[ \sum_{n=0}^{\infty} (n+1)^{\alpha} |z|^{n} \leq \frac{C_{2}}{(1-|z|)^{1+\alpha}} , \quad z \in D. \]

(2.1)

It then follows from (1.3) that

\[ |zf'(z)| = |\sum_{k=1}^{\infty} a_{k} n_{k} z^{n_{k}}| \leq C_{3} \sum_{k=1}^{\infty} n_{k} |z|^{n_{k}} , \]

whence, on making use of the Cauchy product, one obtains
\[
\frac{|zf'(z)|}{1-|z|} \leq C_3 \sum_{n=1}^\infty \left( \sum_{n_k \leq n} n_k^\alpha \right) |z|^n.
\]

Let \( K = \max \{ k ; n_k \leq n \} \). Then,
\[
\frac{|zf'(z)|}{1-|z|} \leq C_3 \sum_{n=1}^\infty \left( \sum_{n_k \leq n} n_k^\alpha \right) |z|^n.
\]

Therefore,
\[
\frac{|zf'(z)|}{1-|z|} \leq C_5 \sum_{n=1}^\infty n^n |z|^n = C_5 |z| \sum_{n=0}^\infty (n+1)^\alpha |z|^n \leq \frac{C_6 |z|}{(1-|z|)^{1+\alpha}}
\]
for \( z \in D \),

by (2.1), whence \( f \in B^\alpha \). We prove next the "if" part of (II). Given \( \epsilon > 0 \) we may find \( k_0 \geq 2 \) such that
\[
|a_k| n_k^{-\alpha} < \epsilon \quad \text{for all} \quad k \geq k_0.
\]

Set
\[
P(z) = |z|^{-1} \sum_{k=1}^{k_o-1} |a_k| n_k |z|^{n_k},
\]
so that \( P \) is bounded on \( D \). Then, there exists \( 0 < r < 1 \) such that
\[
(1-|z|)^\alpha P(z) < \epsilon \quad \text{for} \quad r < |z| < 1.
\]

Now,
\[
zf'(z) \leq \sum_{k=1}^\infty |a_k| n_k |z|^{n_k} \leq |z| P(z) + \epsilon \sum_{k=k_0}^\infty n_k^\alpha |z|^{n_k},
\]
so that
\[
\frac{|zf'(z)|}{1-|z|} \leq \frac{|z| P(z)}{1-|z|} + \epsilon \sum_{n=1}^\infty \left( \sum_{n_k \leq n} n_k^\alpha \right) |z|^n.
\]

It then follows from (2.2), together with (2.1), that
\[
\frac{|zf'(z)|}{1-|z|} \leq \frac{|z| P(z)}{1-|z|} + \frac{\epsilon C_7 |z|}{(1-|z|)^{1+\alpha}}.
\]

Combining (2.4) with (2.3) one obtains
\[
(1-|z|)^\alpha |f'(z)| \leq (1+C_7) \epsilon, \quad r < |z| < 1,
\]
which proves that $f \in B_0^\alpha$.

3. **Fatou points.** We begin with a corollary of Theorem 1.

**Corollary.** Let $f$ be holomorphic in $D$ with (1.1) and (1.2). Assume that

$$\sum_{k=1}^{\infty} |a_k|^2 = \infty \quad \text{and} \quad \lim_{k \to \infty} |a_k| = 0.$$  

Then, $f \in B_0^1$ and $f$ has not finite radial limit at almost every point of the circle $\Gamma = \{|z| = 1\}$.

Lappan proved the special case $n_k = q^k$ and $a_k = k^{-1/2}$.

For the proof of the Corollary we first note that (1.4) holds because $|a_k| \to 0$. It follows from the theorems of G. H. Hardy and J. E. Littlewood and of A. Zygmund (see, for instance, [4, Theorem A and Theorem B]) that $f$ has not finite radial limit at almost every point of $\Gamma$.

Let $f$ be meromorphic in $D$, let $F(f)$ be the set of all Fatou points [3, p. 21] of $f$, and let $F^*(f)$ be the set of $\zeta \in F(f)$ where $f$ has a finite angular limit. Then $F(f) - F^*(f)$ is of Lebesgue measure zero.

Consider now $f$ in the Corollary. Then $F'(f) = \Gamma$ and $F^*(f)$ is of measure zero, in other words, the set $F'(f) - F^*(f)$ is metrically very large.

4. **Conformal and semiconformal points.** Let $S$ be the family of all functions holomorphic and univalent in $D$. Then $f \in S$ is called conformal at $\zeta \in \Gamma$ if $f$ has the angular limit $f(\zeta) \neq \infty$ at $\zeta$ and if the function

$$\arg \left((f(\zeta) - f(z))/(\zeta - z)\right)$$

of $z$ has a finite angular limit at $\zeta$ [7, p. 303]. We call $f \in S$ semiconformal at $\zeta \in \Gamma$ if the radial limit $f^*(\zeta) \neq \infty$ (being also the angular limit) exists and if the function

$$(f^*(\zeta) - f(z))/(\zeta - z)f'(z)$$

of $z$ has the angular limit one at $\zeta$; see [2] and [10]. We denote by $\mathcal{C}(f)$ ($\mathcal{C}_s(f)$, resp.) the set of all conformal (semiconformal, resp.) points of $f \in S$. It is known that $\mathcal{C}(f) \subset \mathcal{C}_s(f)$. J. L. Walsh and D. Gaier [9, p. 85] essentially proved that there exists $f \in S$ such that $1 \in \mathcal{C}_s(f) - \mathcal{C}(f)$. A natural question, therefore, arises: How large may be the set $\mathcal{C}_s(f) - \mathcal{C}(f)$ for $f \in S$? We shall show
Theorem 2. There exists $g \in S$ such that $\mathcal{C}_s(g) = \Gamma$ and $\mathcal{C}(g)$ is of measure zero.

Proof. Consider $f \in B_0^1$ of Corollary in Section 3. We may assume, on dividing $f$ by a suitable positive constant, that

$$\sup_{z \in D} (1 - |z|^2) |f'(z)| < 1.$$  

Set

$$g(z) = \int_0^z \exp [i(f(w) - f(0))] dw, \quad z \in D,$$

so that $g(0) = g'(0) - 1 = 0$. First of all it follows from (4.1), together with [1, Corollary 4.1, p. 36] and $|g''/g'| = |f'|$, that $g \in S$. It is known [10, Lemma 1] that $\zeta \in \mathcal{C}_s(g)$ if and only if the angular limit of $(1 - |z|)|g''(z)/g'(z)| = (1 - |z|)|f'(z)|$ at $\zeta$ is zero. Therefore, $\Gamma = \mathcal{C}_s(g)$. On the other hand, it is known [11, Theorem 2, p. 121] that $\zeta \in \mathcal{C}(g)$ if and only if $\arg g' = \text{Re}(f - f(0))$ has a finite angular limit at $\zeta$. It then follows from Plessner's theorem [3, Theorem 8.2, p. 147], applied to $f - f(0)$, that

$$\mathcal{C}(g) - F^*(f)$$

is of measure zero. Since $F^*(f)$ is of measure zero, it follows that $\mathcal{C}(g)$ is of measure zero.

Remark. It is easy to see that $g$ may be extended one-to-one quasiconformally to the whole extended plane.

References


Department of Mathematics
Tokyo Metropolitan University
Fukazawa, Setagaya-ku
Tokyo 158, Japan