NOTE ON AN ALMOST SURE INVARIANCE PRINCIPLE FOR SOME EMPIRICAL PROCESSES

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1. Summary. Let \{\xi_i\} be a strictly stationary sequence of random variables which are distributed uniformly over the interval [0, 1] and satisfy the strong mixing (s.m.) condition

\[ \alpha(n) = \sup_{A \in \mathcal{F}_a, B \in \mathcal{F}_b} |P(A \cap B) - P(A)P(B)| \downarrow 0 \]

as \( n \to \infty \), where \( \mathcal{F}_\alpha \) is the \( \sigma \)-algebra generated by \( \xi_a, \cdot \cdot \cdot, \xi_b (\alpha \leq b) \).

Recently, Berkes and Philipp (1977) proved an almost sure invariance principle for some empirical processes by which a functional law of the iterated logarithm for the functions of s.m. sequences, a two-dimensional functional law of the iterated logarithm, etc., are easily obtained. In this note, we shall prove that Theorem 1 in Berkes and Philipp [1] remains true under the less restrictive s.m. condition.

2. The main result. Let \( F_N(s) (0 \leq s \leq 1) \) be the empirical distribution function defined by \( \xi_1, \cdot \cdot \cdot, \xi_N \).

Let

\[ R(s, t) = [t] (F_{\lfloor t \rfloor}(s) - s), \quad 0 \leq s \leq 1, \quad t \geq 0 \]

where \([t]\) denotes the largest integer not exceeding \( t \). Write

\[ g_i(\alpha) = I_{(s, t)}(\alpha) - t \]

where \( I_{(s, t)}(\cdot) \) denotes the indicator function of the interval \([s, t)\) and for fixed \( s \) and \( t \) with \( 0 \leq s < t \leq 1 \), put

\[ x_n(s, t) = g_i(\xi_n) - g_s(\xi_n) \]

Then, we can rewrite \( R(s, t) \) as

\[ R(s, t) = \sum_{j=1}^{[t]} x_j(0, s) \]

Consider the covariance function
(2.5) \( \Gamma(s, t) = E \sum_{n=1}^{\infty} E g_{\xi_1} \xi_1 \sum_{n=1}^{\infty} E g_{\xi_1} \xi_1 + \sum_{n=1}^{\infty} E g_{\xi_1} \xi_1 \) \( 0 \leq s, t \leq 1 \).

(It is known that under the conditions of Theorem (below) the two series in (2.5) converge absolutely for \( 0 \leq s, t \leq 1 \).

Let

(2.6) \( \sigma^2(s, t) = \Gamma(s, s) + \Gamma(t, t) - 2 \Gamma(s, t) \).

It is clear that if \( \Gamma(s, t) \) is positive definite, then \( \sigma^2(s, t) > 0 \) for \( 0 \leq s < t \leq 1 \). Further, let \( \{K(s, t), 0 \leq s \leq 1, t \geq 0\} \) be a Kiefer process, i.e., a separable Gaussian process \( K(s, t) \) on \([0, 1] \times [0, \infty)\) such that \( K(0, t) = K(1, t) = K(s, 0) \) for all \( 0 \leq s \leq 1, t \geq 0 \), (2.7) \( EK(s, t) = 0 \) and

(2.8) \( EK(s, t)K(s', t') = \min(t, t') \Gamma(s, s') \).

We prove the following

**Theorem.** Let \( \{\xi_i\} \) be random variables defined above. Suppose that \( \alpha(n) = O(n^{-3/a}) \) for some \( 0 < a < 1 \). Suppose that \( \Gamma(s, s') \) is positive definite. Then, without changing the distribution of the empirical process \( R(s, t) \) of \( \{\xi_n\} \) we can redefine \( R \) on a richer probability space on which there exists a Kiefer process with covariance \( \min(t, t') \Gamma(s, s') \) such that

(2.9) \( \sup_{0 \leq s \leq 1} \sup_{0 \leq t \leq T} |R(s, t) - K(s, t)| = O(T^{1/2}(\log T)^{-\lambda}) \) \( a.s. \)

for some \( \lambda > 0 \).

3. **Proof.** To prove Theorem, we need some lemmas. In what follows, we denote by the letter \( C \), with or without subscript, various absolute constants.

**Lemma 1.** Let \( X \) be a random variable with finite first moment. Let \( \varphi(t) \) be the characteristic function of \( X \). Further, let \( Z \) be the standardized normal random variable. If there exist two numbers \( L \) and \( T(>1) \) such that for all \( t(|t| \leq T) \)

(3.1) \( |\varphi(t) - e^{-t^2/2}| \leq L \),

then for all \( M(>1) \)

(3.2) \( \sup_u |P(X < u) - \Phi(u)| \leq C \left[ M^{-1}[E|X| + E|Z|] + L \log MT + \frac{1}{T} \right] \).

where

\( \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt \).
Proof. Let $M(>1)$ be an arbitrary number. Since $E|X|<\infty$, so for all $t$ ($|t|\leq M^{-1}$)

$$|\varphi(t)-e^{-t^{2}/2}| \leq E|e^{itX-Z)\ell}-1| \leq C|t|(E|X|+E|Z|)$$

Hence, we have

$$\int_{|t|\leq M^{-1}} \frac{\varphi(t)-e^{-t^{2}/2}}{t} dt = \int_{|t|\leq M^{-1}} \frac{\varphi(t)-e^{-t^{2}/2}}{t} dt \leq C \left[ (E|X|+E|Z|) \int_{|t|\leq M^{-1}} \frac{1}{t} dt + L \int_{M^{-1}<|t|\leq T} \frac{1}{t} dt \right]$$

Now, (3.2) follows from Theorem 2 in [2, Chap. 5, §1], which completes the proof.

We put $l=t-s$ for any pair $(s, t)$ ($0 \leq s < t \leq 1$).

Lemma 2. Suppose that the conditions of Theorem are satisfied. Then there exist positive numbers $\gamma$, $\rho$, $\mu$ and $C_0$ such that $1/2<\rho<\gamma$ and

$$P(\sum_{j=H+1}^{H+N} x_j(s, t) \geq 3A l^{(\gamma-\rho)/2}(2N \log \log N)^{1/2})$$

$$\leq C \{ \exp \left( -A^2 C_0^{-2} l^{\rho} \log \log N \right) + A^{-2} l^{\rho} N^{-1/4} \}$$

uniformly for all pairs $(s, t)$ ($l \geq N^{-1/2-\mu}$) and for all $H\geq 0$, $A>0$ as $N \to \infty$.

Proof. Firstly, we note that if $a(n)=O(n^{-3/a})$ then we can easily find positive numbers $C_0$ and $\gamma\left(\geq 5/9\right)$ such that for $s, t$ ($0 \leq s, t \leq 1$) and for all $n$ sufficiently large

$$\sigma_n^2(s, t) = \frac{1}{n} E\left|\sum_{j=1}^{n} x_j(s, t)\right|^2 \leq C_0^2 l^{\gamma\gamma}$$

since $|x_0(s, t)| \leq 1$ and $E|x_0(s, t)| \leq C l$.

Secondly, as $(\xi_i)$ is strictly stationary, we shall prove Lemma 2 in the case $H=0$.

Let

$$\gamma = \min\left(\frac{5}{9}, \frac{7-3a}{8}\right)$$

and choose $\rho$ so that $1/2<\rho<\gamma$. Let $N$ be a sufficiently large number. Let $p=[N^{1/2} \log N]^{-1/2}$ and $k=[N/2p]$. Choose a number $\mu$ so that

$$0<\mu<(1-a)/(3+a)$$

For brevity, we put

$$\chi_N = (2 \log \log N)^{1/2}$$

and $a = a_\rho(s, t) (>0)$. 


For any pair \((s, t)\) such that \(l \geqq N^{-1/8-\mu}\), put
\[
y_j = p^{-1/2} \sigma^{-1} \sum_{j=1}^{p} x_{2tj-1} x_{p+j}(s, t) \quad (j=1, \cdots, k)
\]
and
\[
y_{k+1}^* = p^{-1/2} \sum_{i=2kp+1}^{N} x_i(s, t) .
\]

As \(\{x_i(s, t)\}\) is strictly stationary, so
\[
(3.9) \quad \text{LHS of } (3.4) \leqq 2P(|\sum_{j=1}^{k} y_j| \geqq Al^{t}) \sigma k^{1/2} \chi_{N})
\]
\[
+ P(|y_{k+1}^*| \geqq Al^{t}k^{1/2} \chi_{N}) = 2I_1 + I_2 , \quad \text{(say)} .
\]

It follows from (3.5) that
\[
(3.10) \quad I_2 \leqq A^{-2} l^{-1/2} k^{-1/2} \chi_{N}^{-1} E[y_{k+1}^*]
\]
\[
\leqq CA^{-2} l^{-1/2} k^{-1/2} \chi_{N}^{-1} p^{-1}(N-2kp) \leqq CA^{-2} N^{-1/4} l^{r} .
\]

Now, we proceed to estimate \(I_1\). From (3.7), (3.9) and Lemma 1 in Yoshihara [3] we have
\[
E|y_1|^4 \leqq C \sigma^{-4} \{l^{4/3} + l^{1-\alpha} p^{-1}(\log P)\} \leqq C \sigma^{-4} l^{2\gamma} ,
\]
and so from Schwartz's inequality and the fact \(E|y_1|^6 = 1\) we have
\[
(3.11) \quad E|y_1|^2 \leqq (E|y_1|^4)^{1/3} (E|y_1|^6)^{1/3} \leqq C \sigma^{-2} l^{r} ,
\]
Hence, by Lemma 1, [2, Chap. 5, §2] and (3.11) we have that for all \(t (|t| \leqq (1/4)T_N)\)
\[
|E(\exp(ik^{-1/2} t \sum_{j=1}^{k} y_j)) - e^{-t/2}| \leqq C(k\alpha(p) + T_N^{-1})
\]
where
\[
T_N = k^{1/2} (E|y_1|^2)^{1/2} (E|y_1|^6)^{-1} \geqq Ck^{1/3} \sigma^2 l^{-\gamma} .
\]
Since for all \(N\) sufficiently large
\[
E|k^{-1/2} \sum_{j=1}^{k} y_j| \leqq k^{1/2} E|y_1| \leqq k^{1/2} (E|y_1|^2)^{1/2} \leqq k^{1/3}
\]
so using Lemma 1 (with \(M = N^3\)), we have
\[
\sup_z |P(k^{-1/2} \sum_{j=1}^{k} y_j < z) - \Phi(z)| \leqq CN^{-1/4} \sigma^{-2} l^{r} .
\]

Hence, from the non-uniform estimate of the central limit theorem and (3.5)
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(3.12) \[ I_1 \leq C \left[ 1 - \Phi(Al^{1/2}/2^{-1}cN) \right] + \frac{N^{-1/4}l^{-2}l''}{1 + A^2l^{-2}/2^2} \]
\[ \leq C \left[ \exp \left( -A^2C_0^{-2}l^{\rho} \log \log N \right) + A^{-2}N^{-1/4}l'' \right]. \]
Combining (3.9), (3.10) and (3.12), we have (3.4) and the proof is completed.

The following two lemmas correspond Lemmas 5.1 and 5.2 in Berkes and Philipp [1].

Lemma 3. If (3.4) holds, then as \( k \to \infty \)

(3.13) \[ P( \max_{1 \leq j \leq r} \sup_{s_j \leq t \leq s_{j+1}} |R(s, t) - R(s_j, t_k)| \geq t_k^{1/2} (\log t_k)^{-4\epsilon} ) \leq C \exp (-k^{4\epsilon}) \]
where \( r=r_k=[\log k/\log 4], \ t_k=[\exp(k^{1-\epsilon})] \) and \( \epsilon=(\gamma-\rho)/16 \).

Proof. We write for \( 0 \leq s < s' \leq 1 \) and integers \( P(\geq 0), \ Q(\geq 1) \)

\[ F(P, Q, s, s')=| \sum_{j=P+1}^{P+Q} x_j(s, s') | . \]
Put \( m=|(1/2+\mu) \log t_k/\log 2| \) and write for \( s_j \leq s < s_{j+1} \)

\[ s=s_j + \sum_{\nu=r+1}^{m} \beta_\nu 2^{-\nu} + \theta 2^{-m} \]
where \( \beta_\nu=0,1 \) and \( 0 \leq \theta \leq 1 \). We define the following events:

\[ E_k(\nu, a) = \{ F(0, t_k, a, (a+1)2^{-\nu}) \geq 2C_02^{-\nu}t_k^{1/2} \chi_{\ell_k} \} \]
\[ E_k = \bigcup_{r<\nu \leq m+1} \bigcup_{0 \leq a < 2^\nu} E_k(\nu, a) . \]
Then, applying the same method in the proof of Lemma 5.1 in [1] and using Lemma 2 we have

\[ F(0, t_k, s_j, s) \leq Ct_k^{1/2}(\log t_k)^{-1-(\gamma-\rho)/4} \quad \text{a.s.} \]
and the proof is completed.

Lemma 4. If (3.4) holds, then as \( k \to \infty \)

\[ P( \max_{t_k \leq t \leq t_{k+1}} \sup_{t \geq t_k} |R(s, t) - R(s, t_k)| \geq t_k^{1/2} (\log t_k)^{-4\epsilon} ) \leq Ck^{-2} . \]

Proof. Put \( p=|(1-\mu) \log t_k/\log 4| \) and \( q=[\log (t_{k+1}-t_k)/\log 2] \). We write each integer \( t \) \( (t_k \leq t \leq t_{k+1}) \) in the form

\[ t=t_k + \sum_{0 \leq \nu \leq q} \tau_\nu 2^\nu = t_k + \sum_{p \leq \nu \leq q} \tau_\nu 2^\nu + \theta t_k^{1-\mu/2} \]
where \( \tau_\nu=0,1 \) and \( 0 \leq \theta \leq 1 \). Also, we write \( s(0 \leq s \leq 1) \) in the form

\[ s= \sum_{\nu=1}^{\infty} \sigma_\nu 2^{-\nu} = \sum_{\nu=m} a_\nu 2^{-\nu} + \theta 2^{-m} . \]
Further, let

\[ H_k(\nu, a, j, h) = \{ F(t_k + h2^{j+1}, 2^l, a, a2^{-\nu}, (a+1)2^{-\nu}) \geq 2C_02^{-l\nu/2}(q-j)^{-\nu/2}2^{\nu/2} \} \]

\[ H_k = \bigcup_{p<j} \bigcup_{0<\nu \leq t_1+j/0} \bigcup_{a>0} \bigcup_{0<h'q-j} H_k(\nu, a, j, h). \]

Then, applying the same method in the proof of Lemma 5.2 in [1] and using Lemma 2, we have the lemma.

From Lemmas 3 and 4, we have the following lemma.

**Lemma 5.** If the conditions of Theorem are satisfied, then

\[
(3.14) \quad \max_{t_k \leq t \leq t_{k+1}} \max_{s_j \leq s} |R(s, t) - R(s_j, t_k)| \leq Ct_k^{1/2}(\log t_k)^{-1} \quad a.s.
\]

where \( t_k \) and \( r_k \) are the ones defined in Lemma 3.

**Proof of Theorem.** Since under the conditions of Theorem the corresponding result to Proposition 3.1 in [1] is proved by the method used there, so using the Berkes and Philipp method in [1] and Lemma 5, we have the theorem.

**References**


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