GENERAL POSITIONING IN A MAPPING CYLINDER

By

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For the Basic definitions of P.L. topology the reader is referred to Hudson [4]. Some other definitions follow.

$E^n$ denotes $n$-dimensional Euclidean space. If $a, b \in E^n$, $[a, b]$ denotes the closed line in $E^n$ between $a$ and $b$.

If $K$ is a complex and $L$ is a subcomplex of $K$, then $\text{St}_K(L) = \bigcup \{A \in K: A \cap |L| \neq \emptyset\}$, where $|L| = \bigcup \{x: x \in A \in L\}$. $A < B$ for two simplices $A$ and $B$ means $A$ is a face of $B$.

If $P$ is a polyhedron, $Q$ a subpolyhedron of $P$, and $T$ a triangulation of $P$ in which $Q$ is triangulated, then $T|Q = \{A \in T: A \subset Q\}$.

All manifolds in this paper are compact combinatorial manifolds. If $M$ is a manifold, the interior of $M$ is denoted $\text{Int}(M)$ or $\text{Int}M$.

Let $M$ be an $m$-manifold, $L$ a triangulation of $M$. For each vertex $v_i \in L$, $\text{St}_L(v_i)$ is a combinatorial $m$-ball; let $\theta_i: \text{St}_L(v_i) \to B^m$ be a P.L. homeomorphism from $\text{St}_L(v_i)$ onto the standard $m$-simplex. If $K$ is a complex and $g: |K| \to \text{Int}(M)$, where $g$ is a P.L. mapping, then $g$ is semi-simplicial iff for each $A \in K$, there exists a vertex $v_i$ of $L$ such that $g(\text{St}_K(A)) \subset \text{Int}(\text{St}_L(v_i))$ and $\theta_i g|\text{St}_K(A)$ is a linear mapping of $\text{St}_K(A)$ into $\text{Int}(B^m)$.

Let $g: K \to E^m$ be a semi-simplicial mapping of a complex $K$ into $E^m$, i.e., for $A \in K$, $g|A$ is linear. $g$ is in general position if for any collection of vertices $\{w_0, \ldots, w_r\}$, $r \leq m$, of $K$, $\{g(w_0), \ldots, g(w_r)\}$ spans an $r$-dimensional hyperplane in $E_m$.

If $g$ is in general position and $A_1, A_2 \in K$, then $\dim g(\text{Int}(A_1) \cap \text{Int}(A_2)) \leq \dim A_1 + \dim A_2 - m$.

Let $\{v_0, \ldots, v_r, \ldots, v_t\} \subset E^r$. Let the hyperplane spanned by $\{v_0, \ldots, v_r\}$ be of dimension $p$, $p \leq r$. Let $\{w_0, \ldots, w_p\}$ be $p+1$ linearly independent points in $\{v_0, \ldots, v_r\}$. Then $\{v_0, \ldots, v_r, \ldots, v_t\}$ is in general position with respect to (g.p.w.r.t.) $\{v_0, \ldots, v_r\}$,
If \( \{w_0, \cdots, w_r, v_{i+1}, \cdots, v_t\} \) is in general position in \( E' \). "With respect to" is abbreviated by "w.r.t."

If \( D \) is a hyperplane in \( E' \), \( \{v_1, \cdots, v_i\} \subset \, E' \), and \( \{w_0, \cdots, w_r\} \) is any linearly independent set of points spanning \( D \), then \( \{v_1, \cdots, v_i, D\} \) is in g.p.w.r.t. \( D \), if \( \{v_1, \cdots, v_i, w_0, \cdots, w_r\} \) is in general position in \( E' \).

If \( \{w_1, \cdots, w_s\} \subset \, E' \), its convex closure is denoted by \( \langle w_1, \cdots, w_s \rangle \). If \( A_1, \cdots, A_t \) are convex subsets of \( E' \), the convex closure of \( \{w_1, \cdots, w_s\} \cup A_1 \cup \cdots \cup A_t \) is denoted by \( \langle w_1, \cdots, w_s, A_1, \cdots, A_t \rangle \).

If \( P, Q \) are polyhedra and \( f: P \to Q \) is a P.L. mapping, then a point \( x \in P \) such that \( f^{-1}f(x) \neq x \) is a singular point of \( f \); the closure of the set of singular points of \( f \) is the singular set of \( f \) and is denoted \( S_f \).

Let \( K, L \) be triangulations of \( P, Q \) such that \( f: K \to L \) is simplicial. Let \( L_b \) be the barycentric subdivision of \( L \) and \( K_f \) a subdivision of \( K \) such that \( K_f \) is isomorphic to \( K_b \), and \( f: K_f \to L_b \) is simplicial. The mapping cylinder of \( f, C_f \) is formed as follows. For \( b \) a vertex in \( K_f \), let \( B(b) \) be the simplex of \( K \) such that \( b \in \text{Int}(B(b)) \); similarly if \( a \) is a vertex in \( L_b \), let \( A(a) \) be the simplex of \( L_b \) such that \( a \in \text{Int}(A(a)) \). The vertices of \( C_f \) are those of \( L_b \) plus those of \( K_f \). \( \{b_0, b_1, \cdots, b_i, a_{i+1}, \cdots, a_j\} \) is a simplex of \( C_f \) iff \( B(b_0)>B(b_1)>\cdots>B(b_i) \) and \( f(B(b_i)) \supseteq A(a_{i+1})>\cdots>A(a_j) \). By an obvious identification \( L_b, K_f \subset C_f \), so that \( Q, P \subset \{C_f \} \).

Define the onto simplicial mapping \( p_f: C_f \to L_b \) by \( p_f(b)=f(b) \) for \( b \) a vertex of \( K_f \), and \( p_f(a)=a \) for \( a \) a vertex of \( L_b \).

Given any triangulation \( T \) of \( \subset C_f \), there exists a refinement \( T' \) of \( T \) such that \( p_f: T' \to T \) is simplicial. \( T' \) is called a cylindrical subdivision of \( T \); \( p_f(\subset C_f) \) is the base of \( C_f \). For \( x \in \text{base of} \, C_f \), call \( p_f^{-1}(x) \) the fibre over \( x \). If \( A \in T' \), then \( p_f^{-1}(x) \cap A \) is the fibre over \( x \) in \( A \).

Let \( II \) be a P.L. mapping of \( a \) polyhedron \( F \) onto a polyhedron \( G \) such that for each \( x \in G \), \( II^{-1}(x) \) is collapsible. The triple \( \{F, G, II\} \) is called a semi-forest. If for each \( x \in G \), diameter \( II^{-1}(x)<\epsilon \), then \( \{F, G, II\} \) is called an \( \epsilon \)-semi-forest. These notions are due to Homma [3].

2. General Positioning in a Mapping Cylinder. Consider the P.L. mapping \( f: M \to \text{Int}(N) \) of a combinatorial closed \( m \)-manifold into the interior of a combinatorial \( n \)-manifold, where \( m \leq 3n/4-5/4 \), and \( n \geq 4 \). Assume \( M \) has a triangulation \( L \) and \( N \) has a triangulation \( K \), and that \( f \) is in general position and is semi-simplicial with respect to these two triangulations. Let \( P=S_f \), the singular set of \( f \), and \( Q=f(P) \). Let \( |C_f| \) be the mapping cylinder associated with \( f: P \to Q \).

The mapping \( g \). There exists a semi-simplicial mapping \( g: C_f \to M \), where \( C_f \)
is a cylindrical triangulation, satisfying:

(i) if $A, B \in C_f$ with $g(A) \cap g(B) \neq \emptyset$, then there exists a vertex $v \in L$ with $g(A) \cup g(B) \subseteq \text{Int}(St_L(v))$;

(ii) $g$ is in general position;

(iii) $g(x) = x$, for each $x \in P$.

It is easy to show that:

**Lemma 1.** dim $C_f \leq n/2 - 2a + 1$, where $a \geq 5/4$.

**Corollary 1.1.** dim $S_g \leq n/4 - 3a + 2$, where $a \geq 5/4$.

**Lemma 2.** Let $A \cup B$ be a complex consisting of two principal simplices $A$ and $B$. Let $A_1$ be a 1-dimensional face of $A$, with dim $(A_1 \cap B) \leq 0$. Let $\phi: [A \cup B] \rightarrow E^n$ be a mapping which is linear on $A$ and $B$, and is in general position, where dim $A \leq m$, dim $B \leq m - 1$. Then given $\epsilon > 0$, there exists a simplicial mapping $\phi': A \cup B \rightarrow E^n$, in general position, where $d(\phi, \phi') < \epsilon$, such that, if $x, y \in E^n$ and the line segment between $x$ and $y$, $[x, y]$, is parallel to $\phi'(A_1)$, then not both $x$ and $y$ lie in $\phi'(B)$.

The proof is straightforward.

By **Lemma 2**, $g$ can be approximated by a semi-simplicial mapping $g' : C_f \rightarrow M$, satisfying (i), (ii), (iii), and (iv'), where

(iv') if $A, B \in C_f$, where $B$ is a 1-dimensional simplex, $p_f(B) = \text{point}$, and dim $(A \cap B) \leq 0$, then any line parallel to $g'(B)$ intersects $g'(A)$ in at most one point.

Note that in order to construct $g'$, dim $C_f$ must be $\leq m - 1$; this follows from Lemma 1.

Assume $g$ satisfies (iv').

**Lemma 3.** $g : |C_f| \rightarrow M$ has the following two properties, the sum of which are called (iv).

(iv$_a$) For any $A, B \in C_f$, if $p_f(A \cap B)$ is a homeomorphism and if there is $x \in A$ such that $g(x) \in (g(A) \cap g(B)) - g(A \cap B)$ then there is no $y \in A$, $y \neq x$, with $g(y) \in (g(A) \cap g(B)) - g(A \cap B)$ and $p_f(x) = p_f(y)$.

(iv$_b$) For any $A, B \in C_f$, if there exists a 1-dimensional face $X$ of $A \cap B$ with $p_f(X) = \text{point}$, and if $x, y \in A - (A \cap B)$; $x', y' \in B - (A \cap B)$; with $g(x) = g(x')$, $g(y) = g(y')$, and $p_f(x) = p_f(y)$ then $p_f(x') = p_f(y')$.

**Proof.** (iv$_a$) follows from (iv'). To prove (iv$_b$) note that if $A \in C_f$ and $A_1 < A$, dim $(A_1) = 1$, with $p_f(A_1) = \text{point}$ in base of $C_f$, then for any line segment $e \subseteq A$,
where $e$ is parallel to $A_1$, $p_f(e)=$ point in base of $C_f$. Now $[x, y]$ is parallel to $X$ in $A$, so $[g(x), g(y)]$ is parallel to $g(X)$. Since $[g(x), g(y)] = [g(x'), g(y')]$, $[g(x'), g(y')]$ is also parallel to $g(X)$, hence $[x', y']$ is parallel to $X$. Therefore $p_f(x') = p_f(y')$.

Thus if $A, B \in C_f$, and a fibre, $g(e)$, in $g(A)$ meets $g(B)$, then $g(e) \cap g(B)$ is contained in a fibre in $g(B)$ and consists of a point or a closed interval.

If $H \subset E^n$, then denote the hyperplane spanned by $H$, by $D(H)$.

**Lemma 4.** Let $G: |K| \rightarrow E^n$ be a semi-simplicial mapping, in general position, from a complex $K$ into $E^n$, where $m=3n/4-a$; dim $K \leq n/2-2a+1$, where $a \geq 5/4$. Then $G$ can be approximated by a semi-simplicial mapping $G': |K| \rightarrow E^n$, which is in general position and is at most 2-to-1.

**Proof.** It is well known that the vertices of the image can be moved slightly and the resulting map will still be in general position. Let $A, B, C \in K$. dim $(G(A) \cap G(B)) - G(A \cap B) \leq n/4-3a+1$. Thus dim $D(G(A) \cap G(B)) \leq n/4-3a+1$. There are 2 cases.

(I) $C \cap (A \cup B) = \phi$. Then the image of $C$ can be adjusted without moving the vertices of $A \cup B$. dim $D(G(A) \cap G(B)) + \dim C - m < -1$.

(II) $C \cap (A \cup B) \neq \phi$. It can be assumed that all the vertices of $C$ are in $A \cup B$.

Let $C_A = C \cap A$, $C_B = C \cap B$. Assume dim $C_A \geq$ dim $C_B$, then dim $C_B \leq (n/2-2a+1)/2$. Thus dim $D(G(A) \cap G(B)) \leq n/4-3a+1$. Therefore $G'$ can be constructed. Now assume $g$ satisfies (i), (ii), (iii), (iv), and is 2-to-1.

**Notation for the General Positioning in $C_f$.** Let $S_g$ be the singular set of $g: |C_f| \rightarrow M$. Let $C'_f$ be a complex identical to $C_f$; let $id: |C'_f| \rightarrow |C_f|$ be the identity map. Let $x' = id^{-1}(x)$, for $x \in C_f$; and $A' = id^{-1}(A)$, for $A \subset |C_f|$. Let $R_a$ be a polyhedron homeomorphic to $S_g$; let $r: R_a \rightarrow S_g$ be the homeomorphism. Let $\phi: R_a \rightarrow S_g$ be the homeomorphism satisfying $\phi(r^{-1}(x)) = x'$. For $x$ in base of $C_f(C_f')$, denote the fibre over $x$ by $F_x(F'_x)$.

Let $H = \{A \cap S_g; A \in C_f\}$. Let $T_o$ be a triangulation of $S_g$ which defines $H$, is extendible to a subdivision of $C_f$, and such that $g|T_o$ is simplicial into some triangulation of $M$. Let $T'_o$ and $RT_o$ be the corresponding triangulations of $S'_g$ and $R_g$.

Because $g$ is at most 2-to-1 and $g|T_o$ is simplicial into some triangulation of $M$, for $A \in T_0$, either no other points of $S_g$ are mapped by $g$ into $g(\text{Int } A)$, or there exists a unique $A' \in T_o$ such that $g(A) = g(A')$.

Let $T_1$ be the barycentric subdivision of $T_o$, and $T_2$ the barycentric subdivision of $T_1$. Let $T'_o$, $T'_o$, and $RT_1$, $RT_2$ be the corresponding subdivisions of $S'_o$ and $R_g$.

For $K$ a complex, denote by $K^i$ the i-th skeleton of $K$. 


Let $Q_q = \{x \in |C_f|: g^{-1}g(x) \neq x\}$, $L_q = S_q - Q_q$. Let $p = \text{Max} \{i: L_q \cap \text{Int} (A) \neq \emptyset, A \in C_f^i\}$, $q = \text{Max} \{i: Q_q \cap \text{Int} (A) \neq \emptyset, A \in C_f^i\}$.

**Lemma 5.** $q > p$.

**Proof.** If $A \in C_f$ and $\text{Int} A \cap L_q \neq \emptyset$, then there exists $A_1, A_2 \in C_f$ with $A \subset A_1 \cap A_2$ and $g(A_1) \cap g(A_2) \neq g(A_1 \cap A_2)$ iff dim $A_1 + \dim A_2 - m - \dim (A_1 \cap A_2) \geq 1$.

In this case $A_1 \cap A_2 \subset L_q$, and dim $(A_1 \cap A_2) \leq \dim A_1 + \dim A_2 - m - 1 \leq 2(n/2 - 2a + 1)$.

If $A_1 \in C_f$ and $A_1 \cap Q_q \neq \emptyset$, then there exists $A_2 \in C_f$ such that dim $A_1 + \dim A_2 - m \geq 0$, or $\dim A_1 \geq m - \dim A_2$. Now dim $A_1 \geq m - (n/2 - 2a + 1) = (3n/4 - a) - (n/2 - 2a + 1) = n/4 - a + 1$, or $q \geq n/4 - a - 1$. Thus $q - p \geq (n/4 + a - 1) - (n/4 - 3a + 1) = 4a - 2 > 0$ since $a > 5/4$.

**Notation.** Let $S_q^i = S_q \cap |C_f^i|$.

**Lemma 6.** Let $A$ be a simplex, $G: A \rightarrow E^r$ a linear homeomorphism, $A' < A$. Let $K_b$ be the barycentric subdivision of $A$. Let $G': |K_b| \rightarrow E^r$ such that

(i) $B \in K_b$ implies $G'|B$ is linear
(ii) $G'|A' = G|A'$.

Let $A \geq A_0 > A_1 > \cdots > A_t > A'$ with $v_i$ the barycenter of $A_i$. The $G'$ is a linear homomorphism on $\langle v_0, v_1, \ldots, v_t \rangle \ast A'$.

The proof is straightforward.

**Lemma 7.** Let $A \in C_f^i$, $B \in T_1$ or $T_2$ such that $B \cap \text{Int} (A) \neq \emptyset$. Then there exists $A_1 < A_2 < \cdots < A_t = A$ such that $B \cap \text{Int} A_j \neq \emptyset$; and if $A' < A$ with $A' \neq A_j$ for $j = 1, \ldots, t$, then $B \cap \text{Int} A' = \emptyset$.

The proof follows from the way in which a barycentric subdivision divides a complex.

**Corollary 7.1.** Let $A \in C_f^i$, $B \in T_1$ or $T_2$ such that $B \cap \text{Int} A \neq \emptyset$. Let $j < i$, then $B$ meets at most one $j$-dimensional face of $A$.

Let $A \subset |C_f|$. The shadow of $A$ in $|C_f|$ is the set $\{x \in |C_f|: \mathbf{p}_f(x) \in \mathbf{p}_f(A)\}$, and is denoted by Shad $(A)$. If $A \subset C_f^i$, Shad $(A)$ is similarly defined in $C_f^i$.

For each vertex $v_j \in T_2$, where $v_j \in \text{Int} (A_j)$, $A_j \subset C_f^i$, there exists a closed ball $U_j$ centered at $v_j$, $U_j \subset \text{Int} (A_j)$, and this collection of $U_j$'s satisfies: if $F: S_f^i \rightarrow |C_f^i|$ such that $F(v_j) \in U_j$, and $F$ is the linear extension of this vertex map, then there exists an ambient isotopy $H_t: |C_f^i| \rightarrow |C_f^i|$, such that $H_0(x) = F(x)$ for all $x \in S_f^i$, and $H_t(x) = x$ for all $x \in |C_f^i|$. Furthermore $H_t(A) = A$ for all $t \in [0, 1]$ and $A \subset C_f^i$. The notation for $v_j$ and $U_j$ will be used in the following.
Proposition 8. There exists $\phi': R^i_\epsilon \rightarrow |C'_f|$, a P.L. homeomorphism of $R^i_\epsilon$ into $|C'_f|$, ambient isotopic to $\phi$ in $C'_f$ by an isotopy which moves no point more than $\epsilon$, for any given $\epsilon > 0$, and satisfying:

1. $\phi'|R^i_\epsilon = \phi|R^i_\epsilon$ for $0 \leq i \leq q-1$.
2. For each vertex $v_j \in R^i_\epsilon$, $\phi'(v_j) \in U_j$.
3. If $B \in RT^i_\epsilon - RT^{i-1}_\epsilon$, and $C \in RT^i_\epsilon$, if $v_1, \ldots, v_i$ are the vertices of $B$ in $RT^i_\epsilon - RT^{i-1}_\epsilon$, then $\phi(v_j)$, $\phi(v_i)$, $\phi(B)$ is in g.p.w.r.t. $\langle \text{Shad}(\phi(C)) \cap A, \phi(B') \rangle$.

The proof of this follows a lengthy construction.

If $B, C \in T^i_\epsilon$ and $v_1, \ldots, v_i$ are the vertices of $B$ which are not in $C$, and $v_i \in S^i_\epsilon$ for $1 \leq i \leq t$, then $v_1, \ldots, v_i$ are the free vertices of $B$ w.r.t. $C$.

Construction of the $\phi_j$. For $0 \leq i \leq q-1$, define $\phi: R^i_\epsilon \rightarrow |C'_f|$ by $\phi_i = \phi|R^i_\epsilon$. For $q \leq i \leq \dim C_f$, $\phi_i$ will be constructed inductively. The final $\phi_i$ will be $\phi'$.

For $q \leq i \leq \dim C_f$, $\phi_i$ will satisfy:

1. $\phi_i|S^i_\epsilon = \phi_{i-1}$.
2. For each vertex $v_j \in RT^i_\epsilon$, $\phi_i(v_j) \in U_j$.
3. If $B \in RT^i_\epsilon - RT^{i-1}_\epsilon$, and $C \in RT^i_\epsilon$, if $v_1, \ldots, v_i$ are the vertices of $B$ in $RT^i_\epsilon - RT^{i-1}_\epsilon$, such that $v_1, \ldots, v_i \cap C = \phi$, if $A \in C'_f$ such that $\phi(B) \subset A$, and if $B'$ is the face of $B$ opposite $\langle v_1, \ldots, v_i \rangle$, then $\phi_i(v_i) \in \langle \text{Shad}(\phi_i(C)) \cap A, \phi_i(B') \rangle$ is in g.p.w.r.t. $\langle \text{Shad}(\phi_i(C)) \cap A, \phi_i(B') \rangle$.

The main construction. For each $i$, $q \leq i \leq \dim C_f$, we prove:

Proposition 8. Given $\phi_j: R^i_\epsilon \rightarrow |C'_f|$, a P.L. homeomorphism, ambient isotopic to $\phi|R^i_\epsilon$ in $|C'_f|$, and satisfying (1), (2), and (3), where $0 \leq j \leq i-1$, and given $\epsilon > 0$, such that $d(\phi_j, \phi|R^i_\epsilon) < \epsilon$, then there exists $\phi_i: R^i_\epsilon \rightarrow |C'_f|$ satisfying (1), (2), (3), and $d(\phi_i, \phi|R^i_\epsilon) < \epsilon$.

Proof. Let $\{v_k\}_k$ be the vertices of $RT^i_\epsilon - RT^{i-1}_\epsilon$. Let $W_k = U_k - N(k, \epsilon)$ where $N(k, \epsilon)$ is the closure of the $\epsilon$-ball centered at $\phi(v_k)$ for $1 \leq k \leq J$.

Consider the set of ordered pairs of simplices $\{(B_1, C_1), \ldots, B_R, C_R)\}$, where $B_k \in RT^i_\epsilon - RT^{i-1}_\epsilon$, $C_k \in RT^i_\epsilon$, and where $B_k$ has free vertices w.r.t. $C_k$ which are contained in $RT^i_\epsilon - RT^{i-1}_\epsilon$. For each such pair $(B_\epsilon, C_\epsilon)$ $r_\epsilon: R^i_\epsilon \rightarrow |C'_f|$ is constructed which satisfies:

1. $r_\epsilon$ is simplicial on $RT^i_\epsilon$.
2. For each vertex $v \in RT^i_\epsilon$, $r_\epsilon(v) = r_{\epsilon-1}(v)$, except when $v$ is a free vertex of $B_\epsilon$ w.r.t. $C_\epsilon$ in $RT^i_\epsilon - RT^{i-1}_\epsilon$. For such a free vertex, $v_\epsilon$, $r_\epsilon(v_\epsilon) \in B_\epsilon$. Define $r_\epsilon$ to be equal to $\phi_{i-1}$ on $R^i_\epsilon - R^{i-1}_\epsilon$, and to be equal to $\phi$ on the vertices of $RT^i_\epsilon - RT^{i-1}_\epsilon$. 

(3) If \( v_1, \ldots, v_t \) are the free vertices of \( B_k \) w.r.t. \( C_k \) in \( RT^{i_2}_k - RT_k^{i_1} \) and \( \phi(B_k) \subset A \) where \( A \in (C_{j})^i \), then \( \{r_s(v_1), \ldots, r_s(v_t)\} \) is in g.p.w.r.t. \( \langle \phi(B) \cap A, r_s(A') \rangle \) where \( B' \) is the face of opposite \( \langle v_1, \ldots, v_t \rangle \), and \( 1 \leq k \leq s \).

Let \( \phi = r_r \).

The construction of \( r_1 \) follows, the construction of \( r_j \), where \( 2 \leq j \leq s \) is similar.

Let \( v_1, \ldots, v_t \) be the free vertices of \( B_1 \) w.r.t. \( C_1 \) which are in \( RT^{i_2}_1 - RT_k^{i_1} \). Let \( \phi(C_1) \subset A \) where \( A \in (C_{j})^i \); thus \( \{\phi(v_1), \ldots, \phi(v_t)\} \subset \text{Int } A \). Let \( B' \) be the face of \( B_1 \) opposite \( \langle v_1, \ldots, v_t \rangle \).

Let \( P_0 = \langle r_s(B'), \phi(B) \cap A \rangle \). \( P_0 \) determines a hyperplane in \( A \), denoted by \( D(P_0) \).

Note \( \dim P_0 < \dim B_1 + \dim (\phi(B) \cap A) + 1 \). In order to show that \( \dim B_1 + \dim (\phi(B) \cap A) - \dim A \leq -1 \) note that \( \dim A = n/2 + 2a + 1 - b \), where \( a \geq 5/4, \ b \geq 0 \). \( \dim B_1 \leq \dim (S_{r} \cap A) \leq \dim A + \dim C_j^m \leq n/4 - 3a + 2 - b \). Thus \( \dim (\phi(B) \cap A) \leq n/4 - 3a + 3 - b \), and \( \dim A_1 + \dim (\phi(B) \cap A) - \dim A \leq -4a + 4 - b \leq -4a + 4 \leq -1 \).

Thus \( D(P_0) \) does not fill up \( W_i \), so there exists \( y_1 \in W_1 - D(P_0) \). Let \( P_1 = \langle y_1, P_0 \rangle \). If \( t > 1 \), there exists \( y_2 \in W_2 - D(P_1) \). By induction \( y_k \in W_k - D(P_{k-1}) \) and \( P_k = \langle y_k, P_{k-1} \rangle \) can be obtained for \( 2 \leq k \leq t \).

Now \{\( y_1, \ldots, y_t, P_0 \} \) is in g.p.w.r.t. \( P_0 \). Let \( r_1(v_k) = y_k \) for \( 1 \leq k \leq t \); \( r_i(v) = r_1(v) \) for each \( v \) in \( RT^{i_2}_t \) for \( v \neq v_k, 1 \leq k \leq t \). Extend \( r_i \) linearly to all of \( R^{i_2}_t \).

The following corollaries now follow.

**Corollary 8.1.** If \( q \leq i \leq \dim C_f \), and \( B, C \in RT^{i_2}_i \), \( \phi'(B) \cap A \neq \phi \) with \( A \in (C_{j})^i \); if \( v_1, \ldots, v_t \) are the free vertices of \( B \) w.r.t. \( C \), then \( x \in \text{Int } (B) \) implies that \( \phi'(x) \in \phi(\phi'(C)) \).

**Corollary 8.2.** If \( B \in RT^{i_2}_k \) and \( C \in RT^{i_1}_k - RT^{i_2}_k \) where \( q \leq i < j \leq \dim C_j \); if \( A \in (C_{j})^i \) with \( \phi'(B) \subset A \); if \( v_1, \ldots, v_t \) are the free vertices of \( B \) w.r.t. \( C \), then \( x \in \text{Int } (B) \) implies that \( \phi'(x) \in \phi(\phi'(C)) \).

**Corollary 8.3.** If \( x \in \text{Int } (A) \), \( y \in \text{Int } (B) \), where \( B, C \in RT_2^i - RT_2^{-1} \); and there exists a free vertex of \( B \) w.r.t. \( C \), then there is no \( z \in \text{base of } C_f \) such that \( \phi'(x) \) and \( \phi'(y) \in F_z \).

Since \( \phi' \) is ambient isotopic to \( \phi \) in \( |C_f| \), there is an ambient isotopy \( H_t: |C_f| \rightarrow |C_f| \) such that \( H_t(x) = x \) for all \( x \in |C_f| \), \( H_t(\phi'(x)) = \phi(x) \) for all \( x \in R_p \), and \( H_t|A \) is a P.L. homeomorphism of \( A \) onto \( A \) for each \( t \in [0, 1] \) and \( A \in C_f \).

Define \( h: |C_f| \rightarrow |C_f| \) by \( h(x) = idH_t(x) \).
Theorem 9. Let $z \in \text{base of } C_f$. Suppose $gh(F_z) \cap gh(|C_f^\prime| - F_z) \neq \phi$, then there exists a unique $z^\prime \in \text{base of } C_f$, $z^\prime \neq z$, such that $gh(F_z) \cap gh(F_{z^\prime}) \neq \phi$. Furthermore $gh(F_z) \cap gh(F_{z^\prime})$ is connected.

The proof of Theorem 9 follows from Corollary 8.3.

Let $B^\prime = \text{base of } C_f$. For $z \in B^\prime$, define $B^\prime(z) = \{z^\prime \in B^\prime : gh(F_z) \cap gh(F_{z^\prime}) \neq \phi\}$. $B^\prime(z)$ contains at most 2 points. Let $G^\prime = \{z \in B^\prime\}$. Let $G = B^\prime\mid G'$, i.e. $G$ is the decomposition space formed by identifying $z$ with $z^\prime$ in $B^\prime$ if $B^\prime(z) = B^\prime(z^\prime)$. Let $\Theta : B^\prime \rightarrow G$ be defined by $\Theta(z) = B^\prime(z)$. A subset $U$ of $G$ is open if $\Theta^{-1}(U)$ is open in $B^\prime$. $G$ is a polyhedron and $\Theta$ is a P.L. map. Let $\delta_1 = \text{Max}\{\text{diameter } g\Phi y(z) : x \in \text{base of } C_f\}$.

Theorem 10. Given $\delta > 0$, there exists a $(2\delta_1 + \delta)$ semi-forest $\Gamma = \{F, G, \pi\}$ and a P.L. mapping $\Theta$ of $Q$ onto $G$ satisfying:

a) $P \subset F \subset \text{Int} (M)$

b) $\Theta f = \pi|P$

c) $\dim S_{\Theta} < \dim Q$

d) $\dim \pi^{-1}(x) \leq \text{Max dim } f^{-1}(y) + 1$, for any $x \in G$.

Proof. Given $\delta > 0$, there exists $\epsilon > 0$ such that if $\phi'$ is constructed as in the proof of Proposition 8 so that $d(\phi, \phi') < \epsilon$, then $d(gh(x), gid(x)) < \delta$ for any $x \in |C_f^\prime|$. Let $K^\prime$ be a triangulation of $|C_f^\prime|$ which extends $T^\prime$. Let $K(P)^\prime$ denote the subcomplex of $K'$ which triangulates id$^{-1}g^{-1}(P)$.

$h$ can be chosen so that there is an ambient isotopy $H_t : M \rightarrow M$, such that $H_t(x) = x$ for $x \in M$, $H_t(gh(x)) = gid(x)$ for $x \in |K(P)^\prime|$, and $d(x, H_t(x)) < \delta/8$ for any $t \in [0, 1]$, $x \in M$. Thus diameter $(gh(F_z)) < \delta_1 + \delta/4$ for any $z$ in base of $C_f'$; and diameter $(H_t(gh(F_z))) < \delta_1 + \delta/2$. If $H_t(gh(F_z)) \cap H_t(gh(F_{z^\prime})) \neq \phi$ for $z, z^\prime$ in base of $C_f'$, then diameter $(H_t(gh(F_z)) \cup H_t(gh(F_{z^\prime}))) < 2\delta_1 + \delta$.

Let $F = H_t gh(|C_f^\prime|)$. Identify $Q$ with $B^\prime$, thus $\Theta : Q \rightarrow G$. For $x \in F, x \in H_t gh(F_z)$ for some $x \in B^\prime$. Let $\pi(x) = B^\prime(x) \in G$. Since $x \in H_t gh(F_{z^\prime})$ implies $B^\prime(x) = B^\prime(z^\prime)$, $\pi$ is well-defined.

For $x \in G$, $\pi^{-1}(x)$ can be shown to be collapsible by Theorem 9. It is clear that $\dim \pi^{-1}(x) \leq \text{Max dim } f^{-1}(y) + 1$, for any $x \in G$.

To show that $\dim S_{\Theta} < \dim Q$, note that $n \geq 4$ and: $\dim Q = \dim S_f \leq n/2 - 2a$, where $a \geq 5/4$. Thus $\dim Q = n/2 - 2a - b$ where $b \geq 0$. $\dim S_\Theta \leq \dim S_f \leq 2(\dim C_f) - m = n/4 - 3a - 2b + 2$. Therefore $\dim Q - \dim S_{\Theta} > 0$, and the theorem follows.
References


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