PERTURBATIONS OF THE MONOTONE SHIFT

By

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1. Let $H$ be a Hilbert space with orthonormal basis $\{e_i\}^\infty_{i=0}$. If the operator $S$ is defined on $H$ by $Se_i=\alpha_ie_{i+1}$ for $i=0, 1, \cdots$ where $|\alpha_i|\leq|\alpha_{i+1}|\leq M$ for $i=0, 1, 2, \cdots$ then $S$ will be called a monotone shift. For the case $\alpha_i=\alpha_{i+1}$, $i=0, 1, 2, \cdots$ the lattice of invariant subspaces of the shift operator $S$ on $l^2(0, \infty)$ has been characterized by Beurling [1] and in [2] J. Freeman shows that for a large class of compact operators, the perturbed shift $S+C$ is similar to the unperturbed shift $S$. A characterization of the operators which are similar to the weighted shift (i.e., not necessarily monotone) using the geometry of invariant subspaces appear in [5].

In general, if one perturbs an arbitrary operator by a compact operator this may markedly change the fine structure of the spectrum. Using the results of Friedrichs [3], [4] and Freeman [2] we prove that every monotone shift $S$ with $\alpha_0\neq 0$ is similar to $S+P$ where $P$ is a compact operator with strictly lower-triangular matrix.

2. Let $\mathcal{A}$ the algebra of operators on $l^p(0, \infty)(1\leq p\leq \infty)$ with the matrix $A=(a_{nm})_{n.m=0}^\infty$ which satisfies the conditions:

$$\sup_{n} \sum_{m=0}^{\infty} |a_{nm}|=a<\infty \ , \ \sup_{m} \sum_{n=0}^{\infty} |a_{nm}|=b<\infty \ ,$$

By a theorem of M. Riesz the operators $A$ are bounded on $l^p(0, \infty)(1\leq p\leq \infty)$ and

$$\|A\|_p\leq \max \{\|A\|_1, \|A\|_\infty\} \ ,$$

where $\|A\|_1=a$ and $\|A\|_\infty=b$.

It is easy to see that $\mathcal{A}$ is a Banach algebra with the norm

$$\|A\| = \max \{\|A\|_1, \|A\|_\infty\} \ ,$$

and that the class $\mathcal{L}$ of matrices $P$ such

$$|P|=\sum_{n,m=0}^{\infty} |p_{nm}|<\infty \ ,$$
is an ideal in $\mathfrak{A}$ and $|APB| \leq \|A\||P||B\|$ for all $A, B \in \mathfrak{A}, P \in \mathcal{L}$. By $\mathfrak{A}_0$ we denote the subalgebra of the Banach algebra $\mathfrak{A}$ consisting of these matrices $A \in \mathfrak{A}$ which are lower-triangular, i.e., $a_{mn}=0$ unless $n \geq m$ and $\mathcal{L}_0$ is the subspace of $\mathcal{L}$ consisting of such $P \in \mathcal{L}$ with $p_{mn}=0$ unless $n>m$ (strictly lower-triangular).

Let $S_i$ the weighted shift defined by $S_i e_i = \frac{1}{\alpha_i} e_{i+1}$ for $i=0,1,2, \cdots$. The matrices representing $S_i$ and $S_i^*$ are

$$S_i = \begin{pmatrix}
0 & 0 & 0 & \cdots \\
1/\alpha_0 & 0 & 0 & \cdots \\
0 & 1/\alpha_1 & 0 & \cdots \\
& \cdots & \cdots & \cdots
\end{pmatrix}$$

and

$$S_i^* = \begin{pmatrix}
0/\alpha_0 & 0 & \cdots \\
0 & 1/\alpha_1 & \cdots \\
0 & 0 & \cdots \\
& \cdots & \cdots & \cdots
\end{pmatrix},$$

and we have that $S_i S_i^* = E$ and $S_i^* S_i = I$ where $E = \text{diag}(0,1,1, \cdots)$.

The existence of an invertible operator $X$ such that $S$ is similar to its perturbations with $P \in \mathcal{L}_0$ will be obtained using the formal analogy between the equation $AX = SX - XS = XP$ and the classical differential equations [2], [3], [4].

If we define an operator $\gamma$ on $\mathcal{L}_0$ by

$$\gamma(P) = \sum_{k=0}^{\infty} S_i^{*k} P S^k,$$

then $\gamma(P)$ is a strictly lower-triangular matrix and

$$\|\gamma(P)\| \leq |P|.$$

Indeed, since

$$S_i^{*k} P S^k = \begin{pmatrix}
\frac{1}{\alpha_0} & \frac{1}{\alpha_0} & \frac{1}{\alpha_0} & \cdots & \frac{1}{\alpha_0} & \frac{1}{\alpha_0} & \cdots \\
\frac{1}{\alpha_1} & \frac{1}{\alpha_0} & \frac{1}{\alpha_0} & \cdots & \frac{1}{\alpha_0} & \frac{1}{\alpha_0} & \cdots \\
\frac{1}{\alpha_0} & \frac{1}{\alpha_1} & \frac{1}{\alpha_0} & \cdots & \frac{1}{\alpha_0} & \frac{1}{\alpha_0} & \cdots \\
\frac{1}{\alpha_0} & \frac{1}{\alpha_1} & \frac{1}{\alpha_1} & \cdots & \frac{1}{\alpha_0} & \frac{1}{\alpha_0} & \cdots \\
\frac{1}{\alpha_0} & \frac{1}{\alpha_1} & \frac{1}{\alpha_1} & \cdots & \frac{1}{\alpha_0} & \frac{1}{\alpha_0} & \cdots \\
\frac{1}{\alpha_1} & \frac{1}{\alpha_1} & \frac{1}{\alpha_1} & \cdots & \frac{1}{\alpha_0} & \frac{1}{\alpha_0} & \cdots \\
\frac{1}{\alpha_1} & \frac{1}{\alpha_1} & \frac{1}{\alpha_1} & \cdots & \frac{1}{\alpha_0} & \frac{1}{\alpha_0} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix},$$

and if we consider

$$\sum_{k=0}^{\infty} \frac{1}{\alpha_0} \cdots \frac{1}{\alpha_{i+k-1}} \alpha_j \cdots \alpha_{j+k} \frac{1}{\alpha_{j+k+1}} \frac{1}{\alpha_{j+k+2}},$$

we have

$$\sup_j \sum_{k=0}^{\infty} \frac{1}{\alpha_i} \cdots \frac{1}{\alpha_{i+k-1}} \alpha_j \cdots \alpha_{j+k} \frac{1}{\alpha_{j+k+1}} \frac{1}{\alpha_{j+k+2}} \leq \sup_j \sum_{k=0}^{\infty} \frac{\alpha_j}{\alpha_i} \frac{\alpha_{j+1}}{\alpha_{i+1}} \cdots \frac{\alpha_{j+k}}{\alpha_{i+k}}.$$
\[ \times |p_{k+i,k+j}| \leq \sup_{j} \sum_{k=0}^{\infty} |p_{k+i,k+j}| \leq |P| , \]
and similarly
\[ \sup_{j} \sum_{k=0}^{\infty} \frac{1}{a_{i}} \cdots \frac{1}{a_{i+k-1}} \alpha_{j} \cdots \alpha_{j+k-1} \rho_{k+i,k+j} \leq |P| . \]

Then, if we denote by
\[ [\tilde{\gamma}(P)]_{ij} = \sum_{k=0}^{\infty} \frac{1}{a_{i}} \cdots \frac{1}{a_{i+k-1}} \alpha_{j} \cdots \alpha_{j+k-1} \rho_{k+i,k+j} , \]
we obtain that \( \|\tilde{\gamma}(P)\|_{1} \leq |P| \) and \( \|\tilde{\gamma}(P)\|_{\infty} \leq |P| \). Therefore \( \|\tilde{\gamma}(P)\| \leq |P| . \)

Now if
\[ \tilde{\gamma}_{n}(P) = \sum_{k=0}^{n} S_{1}^{*k} PS_{1}^{*k} , \]
is the sequence of partial sums of series (1) we have
\[ \gamma(P) - \tilde{\gamma}_{n-1}(P) = \gamma(S_{1}^{*n} PS_{1}^{*n}) , \]
and hence
\[ \|\gamma(P) - \tilde{\gamma}_{n-1}(P)\| = \|\gamma(S_{1}^{*n} PS_{1}^{*n})\| \leq \sum_{n=0}^{\infty} \sum_{j=n}^{\infty} |p_{ij}| \rightarrow 0 \quad \text{for} \quad n \rightarrow \infty \]

Let the operator
\[ \tilde{\gamma}_{P}: Q \rightarrow \tilde{\gamma}(Q) P . \]
If \( Q, P \in \mathcal{L}_{0} \), we have that
\[ |\tilde{\gamma}(Q) P| \leq \|\tilde{\gamma}(Q)\| |P| = |Q||P| . \]

For the iterates of \( \tilde{\gamma}_{P} \) we give

**Lemma 2.1.** If \( P, Q \in \mathcal{L}_{0} \) then \( \tilde{\gamma}_{P}(Q) \in \mathcal{L}_{0} \) and
\[ \|\tilde{\gamma}_{P}(Q)\| \leq \frac{|P|^{n}}{n!} |Q| . \]

**Proof.** It is clear that \( \tilde{\gamma}_{P}(Q) \in \mathcal{L}_{0} \) and that
\[ [\tilde{\gamma}_{P}(Q)]_{ij} = \sum_{j<m_{1} \leq i} \sum_{k=0}^{\infty} \frac{1}{a_{i}} \cdots \frac{1}{a_{i+k-1}} \alpha_{j+1} \cdots \alpha_{j+k} \rho_{j+1+k} . \]
and for its iterates we have
\[ [\tilde{\gamma}_{P}^{n}(Q)]_{ij} = \sum_{j<m_{1} \leq \cdots \leq m_{k} \leq i} \sum_{k=0}^{\infty} \frac{1}{a_{i}} \cdots \frac{1}{a_{i+k-1}} \alpha_{m_{1}} \cdots \alpha_{m_{k+1}-1} . \]
Indeed

$$[\gamma^{k+1}_{p}(Q)]_{ij} = \sum_{i=m_{1}+s_{1}}^{\infty} \sum_{j=0}^{\infty} \frac{1}{\alpha_{i}} \cdots \frac{1}{\alpha_{i+s_{1}-1}} a_{m_{2}+s_{2}-1} \cdots a_{m_{k}+s_{k}-1} \times q_{i+s_{1}, m_{2}+s_{2}, \ldots, m_{k}+s_{k}} p_{m_{2}+s_{2}, m_{1}+s_{1}, \ldots, p_{m_{1}, j}}.$$  

and if denotes $m=m_{1}, m_{1}-s=m_{2}, \ldots, m_{k}-s=m_{k+1}, s=s_{1}, s_{1}+s=s_{2}, \ldots, s_{k}+s=s_{k+1}$, 

we obtain

$$[\gamma^{k+1}_{p}(Q)]_{ij} = \sum_{i<s_{1}<m_{2}<\ldots<m_{k+1}<s_{n+1}} \sum_{j=0}^{\infty} \frac{1}{\alpha_{i}} \cdots \frac{1}{\alpha_{i+s_{1}-1}} a_{m_{2}+s_{2}-1} \cdots a_{m_{k}+s_{k}-1} \times q_{i+s_{1}, m_{2}+s_{2}, \ldots, m_{k}+s_{k}} p_{m_{2}+s_{2}, m_{1}+s_{1}, \ldots, p_{m_{1}, j}}.$$  

But

$$\left|\frac{1}{\alpha_{i}} \cdot \alpha_{m_{1}}\right| \leq 1, \quad \left|\frac{1}{\alpha_{i+s_{1}-1}} \cdot \alpha_{m_{2}+s_{2}-1}\right| \leq 1, \ldots,$$

and therefore

$$|\gamma^{t}_{p}(Q)| \leq 0 \leq j \leq m_{1} \leq \ldots \leq m_{n} \leq i \leq \infty \sum_{0 \leq s_{1} \leq \ldots \leq s_{n} \leq \infty} \sum_{0 \leq f \leq \sum_{0 \leq s_{1} \leq \ldots \leq s_{n} \leq \infty}} \times |q_{i+s_{n}, m_{n}+s_{n}} p_{m_{n}+s_{n}-1, m_{n-1}+s_{n-1}} \cdots p_{m_{2}+s_{2}, m_{1}+s_{1}, p_{m_{1}, j}}|$$

Since $P$ is strictly lower-triangular the nonzero terms of the above sum are products of the form
(3) \[ |p_{u_{n}v_{n}} \cdots p_{u_{2}v_{2}}p_{u_{1}v_{1}}| \]

with \( u_{n} > v_{n} \geq \cdots \geq u_{2} > v_{2} \geq u_{1} > v_{1} \) which contains any given entry \( p_{mn} \) of \( P \) at most once and each product (3) occurs at most once in the above sum, since the indices \( j, m_{1}, \ldots, m_{n} \) and \( s_{1}, \ldots, s_{n-1} \) are determined recursively by \( u_{1}, \ldots, u_{n} \) and \( v_{1}, \ldots, v_{n} \); \( j=v_{1}, m_{1}=u_{1}, m_{1}+s_{1}=v_{2}, m_{2}+s_{1}=u_{2} \), etc. The lemma follows from the fact that each product (3) occurs exactly \( n! \) times when the product \(|P|^n = (\sum |p_{ij}|)^n \) is expanded.

In a similar way as in [2] we define an indefinite integral \( \Gamma(P) \) of integrable operators \( P \) which will be used to find an invertible operator \( X \) such that \( S=X(S+P)X^{-1} \).

Let \[ \Gamma(P)=\sum_{k=0}^{\infty}S_{1}^{*k+1}PS^{k} \]

where \( P \in \mathcal{L}_{0} \).

**Proposition 2.1.**

a) \( \Gamma \) maps \( \mathcal{L}_{0} \) in \( \mathcal{A}_{0} \) and for \( P \in \mathcal{L}_{0} \), \( \|\Gamma(P)\| \leq \frac{|P|}{|a_{0}|} \).

b) \( \Delta \Gamma(P)=P. \)

c) the equation

\[ X=I+\Gamma(XP) \]

is uniquely solvable for \( X \in \mathcal{A}_{0} \) and the solution is given by the Peano series

\[ X=I+\Gamma(P)+\Gamma[\Gamma(P)P]+\Gamma[\Gamma[\Gamma(P)P]P]+\cdots, \]

which converges absolutely in \( \mathcal{A}_{0} \).

d) \( \Delta X=XP. \)

**Proof.**

a) It is easy to see that

\[ \Gamma(P)=S^{*}\gamma(P), \]

and since for \( P \in \mathcal{L}_{0} \), \( \gamma(P) \) is a strictly lower-triangular matrix and \( \|\gamma(P)\| \leq |P| \)

we obtain that

\[ \|\Gamma(P)\| \leq \frac{|P|}{|a_{0}|}, \]

where \( \frac{1}{|a_{0}|} = \|S^{*}\|. \)

b) Since \( SS^{*}=E \) where \( E=\text{diag}(0,1,1,1,\cdots) \) we have

\[ \Delta \Gamma(P)=S\Gamma(P)-\Gamma(P)S=SS^{*}\gamma(P)-S^{*}\gamma(P)S=SS^{*}\gamma(P)-[\gamma(P)-P] \]

\[ =\gamma(P)-\gamma(P)+P=P. \]
c) If we define the operator $\Gamma_P : \mathcal{L}_0 \to \mathcal{L}_0$ by $\Gamma_P(Q) = \Gamma(Q)P$, then $\Gamma^k_P(Q) = S^k \Gamma(Q)$ since $\Gamma(Q) = S^k \Gamma(Q) = \Gamma(S^k(Q))$ and by Lemma 2.1 and a) we have

$$\|\Gamma(\Gamma_P(Q))\| \leq \frac{|\Gamma(P)|}{|a_0|^n}.$$ 

Now assuming the existence of a solution $X \in \mathcal{A}_0$ we have by successive substitutions

$$X = I + \Gamma(P) + \Gamma(\Gamma(P)P) + \cdots + \Gamma(\Gamma^k_P(Q)) + \Gamma(\Gamma^k_P(XP)),$$

and the above reason for $Q = XP$ gives the uniqueness of the solution. The absolute convergence of the Peano series follows taking $Q = P$ in the same inequality. It is clear that the series satisfies the integral equation and applying $\Delta$ to both sides it follows $\Delta X = XP$.

In a similar way as in [2] we define the product integral

$$\hat{\Gamma}(I+P) = \prod_{k=0}^\infty (I + S^k_{1+i}PS^k),$$

for $P \in \mathcal{L}_0$ (where $\prod_{k=0}^\infty A_k = A_n A_{n-1} \cdots A_1 A_0$ for $A_k \in \mathcal{A}$). For the same reason as in [2] we conclude that for $P \in \mathcal{L}_0$ the infinite products $\prod_{k=0}^\infty (I + S^k_{1+i}PS^k)$ converges in $\mathcal{A}_0$ and that the integral equation $X = I + \Gamma(XP)$ is solved by $\hat{\Gamma}(I+P)$ (the product $\prod_{k=n_0}^\infty A_k$ is said to be convergent if there exists an $n_0$ such that $A_k$ is nonsingular ($A_k \in \mathcal{A}_0$) for $k \geq n_0$ and $\prod_{k=n_0}^\infty A_k$ converges to a nonsingular element of $\mathcal{A}$ as $n \to \infty$).

The main result of this note is the following.

**Theorem.** If $P = (p_{ij})$ is a strictly lower-triangular matrix with $|P| = \sum_{i,j} |p_{ij}| < \infty$ and $p_{i+1,j} = -a_i$, then $S+P$ and $S$ represent similar operators on $l^p(0, \infty)$ ($1 \leq p < \infty$) and the product integral $\hat{\Gamma}(I+P)$ represents an operator which implements the similarity.

**Proof.** Since the similarity of $S+P$ with $S$ is equivalent to the solvability of $\Delta X = XP$ by an invertible operator $X$ and we proved that for $P \in \mathcal{L}_0$ the matrix equation $\Delta X = XP$ is solved by $X = \hat{\Gamma}(I+P) \in \mathcal{A}_0$. For to prove that $\hat{\Gamma}(I+P)$ is an invertible operator on $l^p(0, \infty)$ we remarks that since the infinite product $\prod_{k=0}^\infty (I + S^k_{1+i}PS^k)$ is convergent, then it can be factored

$$\hat{\Gamma}(I+P) = \prod_{k=n_0+1}^\infty (I + S^k_{1+i}PS^k) \prod_{k=0}^{n_0} (I + S^k_{1+i}PS^k),$$

where $\prod_{k=n_0+1}^\infty (I + S^k_{1+i}PS^k)$ is invertible and his inverse is in $\mathcal{A}_0$. Since by the
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Riesz theorem these matrices all represents bounded operators on $l^p(0, \infty)$ it remains to show that the operator $\prod_{k=0}^{m_0} (I+S_1^{k+1}PS^k)$ is invertible on $l^p(0, \infty)$. This assertion follows from the fact that $-1$ is not an eigenvalue of $S_1^{k+1}PS^k$ and these operators are completely continuous for which the Fredholm alternative implies that $-1$ is a regular point.

REFERENCES


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