A NOTE ON FLUCTUATIONS OF RANDOM WALKS
WITHOUT THE FIRST MOMENT

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1. Let $X_1, X_2, \cdots$ be a sequence of independent identically distributed real random variables with the common distribution function $F$. Let $S_0=0$, $S_n=X_1 + \cdots + X_n$. Let $F^+$ and $F^-$ denote the distribution function of $X_1^+$ and $X_1^-$ respectively, and let $U^- = \sum_{n=0}^{\infty} (F^-)^{n*}$, where $F^{n*}$ is the $n$-fold convolution of a distribution function $F$ with itself. The purpose of this note is to prove the following theorem.

**Theorem.** Assume that $E|X_1| = \infty$. Then

$$P\{S_n > 0 \text{ i.o.}\} = 0 \text{ or } 1,$$

according as

$$\int_{-0}^{+\infty} U^-(x) dF^+(x) < \infty \text{ or } = \infty.$$

This theorem can be used to obtain the following result due to Williamson[5].

**Corollary (Williamson).** If $1-F^-(x) = L(x)x^{-\alpha}$, $x > 0$, where $0 < \alpha < 1$ and $L$ varies slowly at $+\infty$, then

$$P\{S_n > 0 \text{ i.o.}\} = 0 \text{ or } 1,$$

according as

$$\int_{-0}^{+\infty} [1-F^-(x)]^{-1} dF^+(x) < \infty \text{ or } = \infty.$$

We shall give an example which shows that the assumption on the variation of $F^-$ in the corollary can not be dispensed with.

2. We begin with the proof of the following lemma.

**Lemma.** Assume $F(-0)=0$. If $f$ is a non-negative monotone decreasing function on $[0, +\infty)$, then

$$\sum_{n=0}^{\infty} f(S_n) = \infty \text{ or } < \infty \text{ w.p.1},$$
according as
\[ \int_{-\infty}^{+\infty} f(x) dU(x) = \infty \quad \text{or} \quad < \infty, \]
where \( U = \sum_{0}^{\infty} F^{n*}. \)

**Proof.** Let \( Y_{n} = \sum_{k=0}^{n} f(S_{k}). \) Then, since \( f(S_{k}) \leq f(S_{k}-S_{j}) \) w.p.l if \( j<k, \) we have

\[
EY_{n} = \sum_{j=0}^{n} Ef(S_{j}) = \sum_{j=0}^{n} Ef(S_{j})^{2} + 2 \sum_{j<k} Ef(S_{j}) f(S_{k}) \leq \sum_{j=0}^{n} Ef(S_{j})^{2} + 2 \sum_{j<k} Ef(S_{j}) Ef(S_{k}-S_{j}) \leq 2 \left( \sum_{j=0}^{n} Ef(S_{j}) \right)^{2} = 2 (EY_{n})^{2}.
\]

By Kochen and Stone's generalization of Borel-Cantelli lemma [4], we have

\[
P\left\{ \lim_{n \to \infty} \sup Y_{n}/EY_{n} \geq 1 \right\} > 0.
\]

It follows from the Hewitt-Savage zero-one law that if

\[
\lim_{n \to \infty} EY_{n} = \sum_{n=0}^{\infty} Ef(S_{n}) = \int_{-\infty}^{+\infty} f(x) dU(x) = \infty,
\]

then \( \lim_{n \to \infty} Y_{n} = \infty, \) w.p.l.

This proves the lemma.

The proof of our theorem is an application of a result due to Kesten [3]. The following statement is a slight modification of his theorem and easily derived from Theorem 5 of [3].

**Theorem (Kesten).** If \( EX_{1}^{+} = \infty, \) then

\[
\lim_{n \to \infty} \sup X_{n}^{+}/\sum_{i=1}^{n} X_{i}^{-} = +\infty \quad \text{w.p.l.}, \quad \text{or} \quad =0 \quad \text{w.p.l.}
\]

and

\[
P\{S_{n}>0 \text{i.o.}\} = 1 \quad \text{or} \quad 0,
\]

according as

\[
\lim_{n \to \infty} \sup X_{n}^{+}/\sum_{i=1}^{n} X_{i}^{-} = +\infty \quad \text{or} \quad 0 \quad \text{w.p.l.}
\]

**Proof of Theorem.** If \( EX_{1}^{-} < \infty, \) then \( \lim_{x \to +\infty} U^{-}(x)/x = \left( \int_{-\infty}^{+\infty} x dF^{-}(x) \right)^{-1} > 0, \) and \( \lim_{n \to \infty} S_{n}/n = +\infty. \) If \( EX_{1}^{+} < \infty, \) \( EX_{1}^{-} = \infty, \) then \( \lim_{n \to \infty} S_{n}/n = -\infty \) and \( \int_{-\infty}^{+\infty} U^{-}(x) dF^{+}(x) = \int_{-\infty}^{+\infty} [1-F^{+}(x)] dU^{-}(x) < \infty. \) Hence if \( EX_{1}^{+} < \infty \) or \( EX_{1}^{-} < \infty, \) then the theorem is trivial, and therefore we may assume that \( EX_{1}^{+} = EX_{1}^{-} = +\infty. \) Let \( 1 \leq n_{1} < n_{2} < \cdots \) be the successive indices \( n \) with \( X_{n} > 0. \) The random variables
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\[ V_l = \sum_{n \leq l - 1 < i < n_l} X_i \] and \[ W_l = X_{n_l} \]

are all independent, all \( V_l \) have the same distribution function \( G = \rho \sum_{n=0}^{\infty} (1-\rho)^n (F^-)^n \)
and all \( W_l \) have the same distribution function \( H = 1 - \rho (1-F^+) \). It is obvious that

\[ \limsup \frac{X_n}{\sum_{\ell=1}^{n} X_{\ell}} = \limsup \frac{X_{n_k}}{\sum_{\ell=1}^{n_k} X_{\ell}} = \limsup \frac{W_n}{\sum_{l=1}^{n} V_l} \]

It follows from Kesten's theorem that

(1) \[ P(S_n > 0 \text{ i.o.}) = \begin{cases} 1 & \text{or} \ 0, \\
\end{cases} \]

according as

\[ P(W_n > \sum_{l=1}^{n} V_l \text{ i.o.}) = \begin{cases} 1 & \text{or} \ 0. \\
\end{cases} \]

It is obvious that this is equivalent to

\[ P(W_n > \sum_{l=1}^{n} V_l \text{ i.o.}|V_1, V_2, \cdots) = \begin{cases} 1 & \text{or} \ 0 \ 	ext{w.p.l.} \\
\end{cases} \]

Thus by Borel-Cantelli lemma, (1) holds according as

\[ \sum_{n=1}^{\infty} P(W_n > \sum_{l=1}^{n} V_l |V_1, V_2, \cdots) = \begin{cases} \infty & \text{or} \ < \infty \ 	ext{w.p.l.} \\
\end{cases} \]

It follows from the relation

\[ P(W_n > \sum_{l=1}^{n} V_l |V_1, V_2, \cdots) = 1 - H(\sum_{l=1}^{n} V_l), \ 	ext{w.p.l,} \]

and from Lemma that (1) holds according as

\[ \sum_{n=1}^{\infty} \int_{-\infty}^{+\infty} [1-H(x)]dG^*(x) = \int_{-\infty}^{+\infty} [1-H(x)]d(\sum_{n=0}^{\infty} G^*(x)) = \begin{cases} \infty & \text{or} \ < \infty, \\
\end{cases} \]

or equivalently

\[ \int_{-\infty}^{+\infty} (\sum_{n=1}^{\infty} G^*(x))dH(x) = \begin{cases} \infty & \text{or} \ < \infty. \\
\end{cases} \]

Since \( dH(x) = \rho dF^+(x) \), and \( \sum_{n=1}^{\infty} G^*(x) = 1 + \rho(1-\rho)^{-1}U^-(x) \) for \( x > 0 \), this implies the theorem.

**Proof of Corollary.** If \( 1-F^-(x) = L(x)x^{-\alpha}, x > 0 \), where \( 0 < \alpha < 1 \) and \( L \) varies slowly at \(+\infty\), then it is well-known [2] p. 446 that

\[ \lim_{x \to +\infty} [1-F^-(x)]U^-(x) = (\sin \pi\alpha)/(\pi\alpha). \]
Hence the corollary follows immediately from Theorem.

**Example.** Let the common distribution of random variables $X_1, X_2, \cdots$ be such that $P(X_n=k!)=c(k!)^{-1}$ for $k \geq 1$, and $P(X_n=-k!)=c((k-2)!)^{-1}$ for $k \geq 3$, where $c=(2e-2)^{-1}$. It is easy to verify that

\[
\int_{-\infty}^{\infty} [1-F^{-}(tx)]^{-1}dF^{+}(x)=\infty \quad \text{or} \quad <\infty,
\]

according as $t>1$ or $0<t<1$. Let $t'>1$, $0<t''<1$, and let $X'_n=X_n-(t')^{-1}X_n$, $X''_n=X_n-(t'')^{-1}X_n$, $S'_n=S'_n=0$, $S''_n=X'_1+\cdots+X'_n$, $S''_n=X''_1+\cdots+X''_n$. It follows from Kesten's theorem that $P(S'_n>0 \text{ i.o.})=P(S''_n>0 \text{ i.o.})=0$ or 1. This fact together with (2) shows that in Corollary we cannot remove the assumption on the variation of $1-F^{-}$.

**Remark.** Combined with Theorem 6 of [3] our theorem implies that

\[
\lim_{n} n^{-1} S_n = -\infty \quad \text{w.p.1},
\]

if and only if $E|X_1| = \infty$ and

\[
\int_{-\infty}^{\infty} U^-(x) dF^+(x) < \infty.
\]

A sufficient condition for (3) is that there exist constants $0<\alpha<1$, $C>0$ and $x_0>0$ for which $1-F^-(x) \geq Cx^{-\alpha}$ for $x>x_0$ and $\int_{0}^{\infty} x^\alpha dF^+(x) < \infty$. In fact $1-F^-(x) \geq Cx^{-\alpha}$ implies the existence of a distribution function $G$ and a constant $M>0$ such that $G(0)=0$, $1-G(x)=Cx^{-\alpha}$ for $x>M$ and $F^\leq G$. Since $(G^-)^{n*} \leq C^{n*}$, $n \geq 1$, we have $U^- \leq \sum G^{n*} = O(x^n)$ as $x \to \infty$. Thus $\int_{0}^{\infty} U^-(x) dF^+(x) < \infty$ follows. This result includes that of Derman and Robbins [1], which states that (3) holds if for some constants $0<\alpha<\beta<1$, $C>0$ and $x_0>0$, $F(x) \geq C|x|^{-\alpha}$ for $x<x_0$ and $\int_{0}^{x_0} x^\beta dF(x) < \infty$.

**REFERENCES**


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