A THEOREM OF PIECEWISE LINEAR APPROXIMATIONS

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1. In this paper we shall assume that a complex is finite. If $K$ is a complex, $P = \bigcup_{\xi} \xi$ is called a polyhedron and denoted by $|K|$, and $K$ is called a simplical division of $P$. Let $K, H$ be complexes and $f: |K| \rightarrow |H|$ be a continuous mapping such that for any $\xi \in K$ there is a $\sigma \in H$ satisfying i) $f(\xi) = \sigma$ and ii) $f|_{\xi} \rightarrow \sigma$ is linear. Then $f$ is called a simplicial mapping of $K$ into $H$ and also a piecewise linear (p. w. l.) mapping of $|K|$ into $|H|$. If $K$ is a complex and $f: |K| \rightarrow E^n$ is a continuous mapping of $|K|$ into $E^n$ such that for any $\xi \in K$ $f|_{\xi} \rightarrow E^n$ is linear, $f$ is called a semi-simplicial mapping of $K$ into $E^n$ and also a p. w. l. mapping of $|K|$ into $E^n$. A p. w. l. mapping $f: |K| \rightarrow |H|$ (or $E^n$) is said to be non-degenerate, if for any $\xi \in K$ $\dim \xi = \dim f(\xi)$. If $f: P \rightarrow Q$ is a p. w. l. mapping, a point $p \in P$, such that $f^{-1}(f(p))$ is a singular point of $f$ and the closure of the set of singular points of $f$ is denoted by $S_f$. If $K$ is a complex and $\xi$ is a simplex of $K$, we denote by $St_k(\xi)$ the polyhedron which is the union of all simplexes of $K$ having $\xi$ as a face. If $K$ is a complex and for any $\xi \in K$ $St_k(\xi)$ is p. w. l. homeomorphic to an $n$-simplex, the polyhedron $|K|$ is called a (combinatorial) $n$-manifold. If $M$ is an $n$-manifold, we denote by $\bar{M}$ and $\dot{M}$ the interior of $M$ and the boundary of $M$ respectively. Throughout this paper we shall assume that any complex $K$ is contained in some euclidean space $E^k$ and any simplex $\xi$ of $K$ is linearly imbedded in $E^k$. If $\Gamma = \{P_1, \ldots, P_l\}$ is a set of polyhedra such that $P_1 \cup \cdots \cup P_l$ is connected and $P_i \cap P_j$ is a point or $\phi$ for $i \neq j$, $\Gamma$ is called a chain of polyhedra and $l$ is called a length of $\Gamma$. If

$$P_i \cap P_j = \begin{cases} \text{one point} & \text{for } |i-j| = 1 \\ \phi & \text{for } |i-j| > 1, \end{cases}$$

$\Gamma$ is called a simple chair. If

$$P_i \cap P_j = \begin{cases} \text{one point} & \text{for } |i-j| = 1, l-1 \\ \phi & \text{for } 1 < |i-j| < l-1, \end{cases}$$

$\Gamma$ is called a cyclic chain. The main theorem of this paper is following:

**Theorem 1.** Let $M$ be an $n$-manifold. Then there is a positive number $\eta(M)$ such that if $f: P \rightarrow Q$ is a p. w. l. mapping of a polyhedron $P$ onto a polyhedron $Q$, $g: P \rightarrow \bar{M}$ is a p. w. l. mapping and $R$ is a subpolyhedron of $P$ satisfying
$g\mid R$ is a homeomorphism

$n > \dim Q + 2 \max_{q\in Q} \dim f^{-1}(q)$

$\eta(M) > \text{dia} f^{-1}(q)$, for any $q \in Q$.

then for any $\epsilon > 0$ there is a p.w.l. mapping $h : P \to H$ satisfying

1) $d(h, g)(=\sup_{p} d(h(p), g(p)) < \epsilon$

2) $h\mid R = g\mid R$

3) $h$ is non-degenerate

4) $S_{h} \cap f^{-1}(q)$ is finite

5) $\dim S_{h} < \dim Q$

$a_{1})$ $h\mid f^{-1}(q)$ is a homeomorphism

$a_{2})$ $h f^{-1}(q_{1}) \cap h f^{-1}(q_{2}) = \text{one point or } \phi$

for any $q_{1} \neq q_{2} \in Q$

$a)$ there is no cyclic chain in $\Gamma = \{h f^{-1}(q) \mid q \in Q\}$

$b)$ there is no simple chain of length $\geq n + 2$ in $\Gamma$.

2. In this section we shall prove the following:

**Lemma 1** Let $K$ be a complex consisting of two simplexes $\xi_{1}, \xi_{2}$ and their faces. Let $f : K \to H$ be a simplicial mapping of $K$ onto a complex $H$ and $g : K \to E^{n}$ a semi-simplicial mapping such that

$n > \dim H + 2 \max_{q\in H} \dim f^{-1}(q)$.

Let $\lambda : D \to f(\xi_{2})$ be a linear homeomorphism of a convex polyhedron $D$ of $f(\xi_{1})$ into $f(\xi_{2})$. Then for any $\epsilon > 0$ there is a semi-simplicial mapping $h : K \to E^{n}$ such that

i) $d(h, g) < \epsilon$

ii) $h(f^{-1}(q) \cap \xi_{1}) \cap h(f^{-1}(q) \cap \xi_{2})$

$\subset h(\xi_{1} \cap \xi_{2})$, for any $q \in D$.

If $P$ is a polyhedron, we denote by $E(P)$ the minimal euclidean space containing $P$. We shall consider any euclidean space $E^{1}$ as a vector space and use vector notation. Points $p, p_{1}, \ldots, p_{i}$ are said to be linearly independent if the vectors $p_{1} - p_{0}, \ldots, p_{i} - p_{0}$ are linearly independent. If $v_{1}, v_{2}, \ldots, v_{i}$ are vectors, we denote by $\rho(v_{1}, v_{2}, \ldots, v_{i})$ the maximal number of linearly independent vectors in $v_{1}, v_{2}, \ldots, v_{i}$. Vectors $v_{1}, \ldots, v_{i}, v_{i+1}, \ldots, v_{j}$ are said to be linearly independent with respect to $v_{i+1}, \ldots, v_{j}$ if
\[ \rho (v_1, v_2, \ldots, v_i, \ldots, v_j) - \rho (v_{i+1}, \ldots, v_j) = i. \]

**Proof of Lemma 1.** Since \( \lambda \) is a linear mapping of \( D \) into \( f(\xi) \). We may consider \( \lambda \) as a linear mapping of \( E(D) \) into \( E(f(\xi)) \). Furthermore we denote by \( f_1 \) and \( f_2 \) the linear mappings of \( E(f^{-1}(D) \cap \xi_1) \) onto \( E(D) \) and \( E(f^{-1}(\lambda(D) \cap \xi_2)) \) onto \( E(\lambda(D)) \) satisfying
\[
\begin{align*}
f_1 & : f^{-1}(D) \cap \xi_1 = f^{-1}(\lambda(D) \cap \xi_2) \\
f_2 & : f^{-1}(D) \cap \xi_2 = f^{-1}(\lambda(D) \cap \xi_2)
\end{align*}
\]
respectively.

We choose linearly independent points of \( E(f^{-1}(D) \cap \xi_1) \)
\[ \{a_0, a_1, \ldots, a_i, \ldots, a_k, \ldots, a_l, \ldots, a_0\} \]
and linearly independent points of \( E(f^{-1}(\lambda(D) \cap \xi_2)) \)
\[ \{a'_0, a'_1, \ldots, a'_i, \ldots, a'_{i+1}, \ldots, a'_l, \ldots, a'_0\} \]
such that
\[
\begin{align*}
\alpha) & \quad \lambda f_1(a_0) = f_2(a'_0) \\
\beta) & \quad f_1(a_0), \ldots, f_1(a_j) \text{ are linearly independent points of } E(D) \text{ and } j = \dim E(D) \\
\gamma) & \quad a_0, a_1, \ldots, a_i, a_{i+1}, \ldots, a_k \in E(\xi_1 \cap \xi_2) \\
& \quad a'_0, a'_1, \ldots, a'_i, a'_{i+1}, \ldots, a'_l \in E(\xi_1 \cap \xi_2) \\
& \quad a_{i+1} \cup a'_{i+1}, \ldots, a_j \cup a'_j \in E(\xi_1 \cap \xi_2) \\
& \quad a_{k+1}, \ldots, a_k, a'_{k+1}, \ldots, a'_l \notin E(\xi_1 \cap \xi_2).
\end{align*}
\]

Since we have
\[
\dim K \leq \dim H + \max_{q \in H} \dim f^{-1}(q) < n \quad \text{and} \quad 1 + j + (l-j) + (l-j) \leq 1 + \dim H + 2 \max_{q \in H} \dim f^{-1}(q) \leq n.
\]

We can construct a semi-simplicial mapping \( h \) of \( K \) into \( E^n \) so that
\[
\alpha') \quad d(h, g) < \varepsilon
\]
\[
\beta') \quad h \text{ is non-degenerate}
\]
\[
\gamma') \quad \text{vectors } h_1(a_0) - h_2(a'_0), \ldots, h_1(a_i) - h_2(a'_i), h_1(a_{i+1}), \ldots, h_1(a_k), h_2(a'_{i+1}), \ldots, h_2(a'_{k+1}), \ldots, h_1(a_{j+1}), \ldots, h_1(a_k), h_2(a'_{j+1}), \ldots, h_2(a'_{k+1}), \ldots, h_2(a'_{l+1}), \ldots, h_2(a'_k), \ldots, h_2(a'_l), \text{ where } h_1 \text{ and } h_2 \text{ are the linear mappings of } E(f^{-1}(D) \cap \xi_1) \text{ into } E^n \text{ and } E(f^{-1}(\lambda(D) \cap \xi_2)) \text{ into } E^n \text{ satisfying}
\begin{align*}
h_1 & : f^{-1}(D) \cap \xi_1 = h \circ f^{-1}(D) \cap \xi_1 \\
h_2 & : f^{-1}(\lambda(D) \cap \xi_2) = h \circ f^{-1}(\lambda(D) \cap \xi_2)
\end{align*}
\]
respectively.
We shall prove that \( h \) is the required semi-simplicial mapping. If \( p \in f^{-1}(q) \cap \xi_1, p' \in f^{-1}(q) \cap \xi_2, q \in D \) and \( h(p) = h(p') \), we can write

\[
p = \sum_{s=0}^i \mu_s a_s, \quad p' = \sum_{s=0}^{i'} \mu'_s a'_s,
\]

\[
\sum_{s=0}^i \mu_s = \sum_{s=0}^{i'} \mu'_s = 1 \quad \text{and} \quad \mu'_0 = \mu_0, \mu'_1 = \mu_1, \ldots, \mu'_f = \mu_f.
\]

Then \( h(p) - h(p') = \mu_0 (h_1(a_0) - h_2(a'_0)) + \cdots + \mu_f (h_1(a_f) - h_2(a'_f)) + \mu_{f+1} h_1(a_{f+1}) + \cdots + \mu_l h_1(a_l) - \mu_{f+1}' h_2(a_{f+1}') - \cdots - \mu_{l'} h_2(a_{l'}) = 0 \)

From the linear independence of vectors, we have

\[
\mu_{f+1} = \cdots = \mu_f = \mu_{k+1} = \cdots = \mu_{l} = \mu_{k'} + 1 = \cdots = \mu_{l'} = 0
\]

Then \( p, p' \in \xi_1 \cap \xi_2 \) and therefore \( h(p) = h(p') \in h(\xi_1 \cap \xi_2) \).

Hence we have proved

\[
h(f^{-1}(q) \cap \xi_1) \cap h(f^{-1}(q) \cap \xi_2) \subset h(\xi_1 \cap \xi_2), \text{ for any } q \in D
\]

and completed the proof of Lemma 1.

3. In this section we shall assume that \( f: P \rightarrow Q \) and \( g: P \rightarrow E^n \) are p.w.l. mappings and \( R \) is a subpolyhedron of \( P \) such that

\[
g: R \text{ is a homeomorphism}
\]

\[
n > \dim Q + 2 \max \dim f^{-1}(q)
\]

We shall prove the following:

**Theorem 2.** For any \( \varepsilon > 0 \) there is a p.w.l. mapping \( h: P \rightarrow E^n \) satisfying the conditions 1), 2), 3), 4), 5), \( \alpha_1 \), \( \alpha_2 \), \( \alpha \), \( \beta \) of Theorem 1.

At first we shall prove the following:

**Lemma (2, 1)** For any \( \varepsilon > 0 \) there is a p.w.l. mapping \( h_1: P \rightarrow E^n \) satisfying the conditions 1), 2), 3), 4), 5), and \( \alpha_1 \).

**Proof of Lemma** (2, 1) We choose simplicial divisions \( K \supset J \) and \( H \) of \( P \supset R \) and \( G \) such that \( f: K \rightarrow H \) is simplicial and \( g: K \rightarrow E^n \) is semi-simplicial. If we construct a semi-simplicial mapping \( h_1: K \rightarrow E^n \) sufficiently close to \( g \) and satisfying all conditions of Lemma (2, 1) except 2), \( h_1|J \) is an isomorphism. Since \( g|R \) is a
homeomorphism and then \(g|J\) is an isomorphism, Furthermore we may assume that \(h'|J\) is sufficiently close to \(g|J\) so that there is a p. w. l. homeomorphism \(\pi: E^n \to E^n\) sufficiently close to identity mapping \(1\) and satisfying \(\pi h'|R = g|R\). Put \(\pi h' = h_1\). Then it is clear that \(h_1\) is the required p. w. l. mapping. Therefore we shall construct a semi-simplicial mapping which is sufficiently close to \(g\) and satisfies 3), 4), 5), \(\alpha\). The conditions 3), 4), 5) follow respectively from following formulas:

\[
\begin{align*}
n &> \dim Q + \max_{q \in Q} \dim f^{-1}(q) \geq \dim P \\
n &> \dim Q + 2 \max_{q \in Q} \dim f^{-1}(q) \geq \dim P + \max_{q \in Q} \dim f^{-1}(q)
\end{align*}
\]

We denote \(\Delta = \{(\xi_1, \xi_2) | f(\xi_1) = f(\xi_2), \xi_1, \xi_2 \in K\}\). Let \((\xi_1, \xi_2) \in \Delta\). Then put \(f(\xi_1) = D, \lambda = 1\) and let \(K'\) be the subcomplex of \(K\) consisting of \(\xi_1, \xi_2\) and their faces. We apply [Lemma 1] to \(f|K', g|K', D = f(\xi)\) and \(\lambda = 1\). Then we get a semi-simplicial mapping \(h': K' \to E^n\) satisfying ii) of [Lemma 1] and sufficiently close to \(g\). It is clear that any semi-simplicial mapping \(h': K' \to E^n\) sufficiently close to \(h'\) satisfies ii). By induction with respect to elements of \(\Delta\) we get a semi-simplicial mapping \(h_1: K \to E^n\) satisfying the condition ii) for all elements of \(\Delta\). We shall prove that \(h_1\) satisfies \(\alpha_1\). Let \(p_1 \neq p_2 \in f^{-1}(q)\). Then there is a \((\xi_1, \xi_2) \in \Delta\) such that \(\xi_1 \ni p_1\) and \(\xi_2 \ni p_2\). If \(p_1, p_2 \in \xi_1 \cap \xi_2\), from 3) \(h_1(p_1) \neq h_1(p_2)\). If \(p_1 \in \xi_1 \cap \xi_2\), from 3) \(h_1(p_1) \neq h_1(\xi_1 \cap \xi_2)\). By ii) we have \(h_1(p_1) \neq h_1(p_2)\). Therefore \(h|f^{-1}(q)\) is a homeomorphism and then [Lemma (2, 1)] has been proved.

**Lemma (2, 2)** There is a p. w. l. mapping \(h_2: P \to E^n\) satisfying 1), 2), 3), 4), 5), \(\alpha_1\), \(\alpha_2\).

**Proof of Lemma (2, 2)** Let \(h_1\) be the p. w. l. mapping of \(P\) into \(E^n\) sufficiently close to \(g\) and satisfying 1), 2), 3), 4), 5), \(\alpha_1\). Let \(K \supset J\) and \(H\) be the simplicial division of \(P \supset R\) and \(Q\) respectively such that \(f: K \to H\) is simplicial and \(h_1: K \to E^n\) is semi-simplicial. If \(h_2: K \to E^n\) is sufficiently close to \(h_1\) and satisfies all conditions of [Lemma (2, 2) except 2], similarly as the proof of [Lemma (2, 1)] we can modify the p. w. l. mapping \(h_2: P \to E^n\) so that it satisfies the condition 2) too. Furthermore if the semi-simplicial mapping \(h_2: K \to E^n\) is sufficiently close to \(h_1\), it is clear that \(h_2\) satisfies the conditions 3), 4), 5), \(\alpha_1\). Therefore we shall construct a semi-simplicial mapping \(h_2: K \to E^n\) which is sufficiently close to \(h_1\) and satisfies \(\alpha_2\). Assume that \((\xi_{11}, \xi_{12}), (\xi_{21}, \xi_{22}) \in \Delta\).
and $h_1(\xi_{12}) \cap h_1(\xi_{21}) = C \not\approx \phi$. Then $C$ is a convex polyhedron and $h_1^{-1}(C) \cap \xi_1, h_1^{-1}(C) \cap \xi_2$ are also convex polyhedra. The condition 3) implies that $\tilde{h}_1 = h_1 | h_1^{-1}(C) \cap \xi_1 \rightarrow C$ and $\tilde{h}_2 = h_1 | h_1^{-1}(C) \cap \xi_2 \rightarrow C$ are linear homeomorphisms. Furthermore the condition 4) implies that $f(h_1^{-1}(C) \cap \xi_1) = D_1$ and $f(h_1^{-1}(C) \cap \xi_2) = D_2$ are convex polyhedra and $\tilde{f}_1 = f | h_1^{-1}(D) \cap \xi_1 \rightarrow D_1, \tilde{f}_2 = f | h_1^{-1}(C) \cap \xi_2 \rightarrow D_2$ are linear homeomorphisms. Therefore

$$\tilde{h}_1 \tilde{f}_1^{-1} : D_1 \rightarrow D_2 \text{ is a linear homeomorphism.}$$

We apply [Lemma 1] to $f | \xi_{11} \cup \xi_{22}$, $h_1 | \xi_{11} \subseteq \xi_{22} , \lambda, D = D_1$. Then by induction we get a semi-simplicial mapping $h_2 | K \rightarrow E^n$ satisfies the condition ii of [Lemma 1] for all such pairs $\{(\xi_{11}, \xi_{22})\}$. We shall prove that $h_2$ satisfies $\alpha_2$. If $q_1 \neq q_2 \in Q$ and $h_2 f^{-1}(q_1) \cap h_2 f^{-1}(q_2) \ni p$, from $\alpha_1$ $h_2^{-1}(p) \cap f^{-1}(q_1) = p$, $i=1,2$, is a point. If $p_1 \neq p_2 \in f^{-1}(q_1)$ and $p_1 \neq p_2 \in f^{-1}(q_2)$, there are four simplexes $\xi_{21}$ and $p_1', p_2', p_1, p_2 \in \xi_{21} \cap \xi_{22} \ni p_1, p_2 \not\in h_2(f^{-1}(q_2))$, such that $(\xi_{11}, \xi_{12}), (\xi_{21}, \xi_{22}) \in \Delta$. If $p_1', p_2' \in \xi_{11} \cap \xi_{22}$, from 3 we have $h_2(p_1') \not\approx h_2(p_2')$. Therefore $h_2 f^{-1}(q_1) \cap h_2 f^{-1}(q_2) \ni p$ and we have proved [Lemma (2, 2)]

By induction with respect to $i$ we shall prove the following:

**Lemma (2, i), $i \geq 2$.** There is a p. w. l. mapping $h_1 : P \rightarrow E^n$ satisfying 1), 2), 3), 4), 5), $\alpha_1, \alpha_2$ and

$\alpha_1$) There is no cyclic chain of length $\leq i$ in $\Gamma_i = \{h_1 f^{-1}(q) | q \in Q\}$,

$\beta_i$) $\dim X_i \leq \dim Q+1-i$,

where $X_i =$ closure of $\{q | q \in Q, h_1 f^{-1}(q) \text{ is an element of a simple chain of length} \geq i \text{ in } \Gamma_i\}$.

**Proof of Lemma (2, i).** We have already proved [Lemma (2, 2)]. In fact it is clear that $X_2 = f(S_{22})$ and then the condition $\beta_2$ is equivalent to the condition $\beta_2$. Therefore we assume that [Lemma (2, i-1)] is true. Let $h_{i-1}$ be the p. w. l. mapping satisfying all conditions of [Lemma (2, i-1)] and sufficiently close to $g$. Let $K \supset J$ and $H$ be simplicial subdivision of $P \supset R$ and $Q$ respectively such that $f : K \rightarrow H$ is simplicial and $h_{i-1} | K \rightarrow E^n$ is semi-simplicial. If $(\xi_{11}, \xi_{12}), \cdots, (\xi_{i1}, \xi_{i2}) \in \Delta$ and $h_{i-1}(\xi_{j2}) \cap h_{i-1}(\xi_{j+11}) = C_i \not\approx \phi$. We put $\tilde{h}_{j2} = h_{i-1}(C_i) \cap \xi_{i2} \rightarrow C_j$ and $\tilde{h}_{j+11} = h_{i-1}(C_i) \cap \xi_{j+11} \rightarrow C_j$, from 3) $\tilde{h}_{j2}$ and $\tilde{h}_{j+11}$ are linear homeomorphisms. Put $\tilde{f}_2 = f | h_{i-1}(C_i) \cap \xi_{j2}$ and $\tilde{f}_{j+11} = f | h_{i-1}(C_i) \cap \xi_{j+11}$. Then from 4) $\tilde{f}_2$ and $\tilde{f}_{j+11}$ are linear homeomorphisms. $D_{j+1} = \tilde{f}_{j+11}, \tilde{h}_{j+11}$ is a convex polyhedron in $\sigma_{j+1} = f(\xi_{j+11}) = f(\xi_{j+12})$ and $\lambda_j = \tilde{f}_{j+11}, \tilde{h}_{j+11}, \tilde{f}_2$ is a...
linear homeomorphism of the convex polyhedron \( f_{J_{2}}h_{J_{2}}(C_{i}) \) onto \( D_{J+1} \). If \( D = \lambda_{i}^{-1}(D_{2} \cap \lambda_{i}^{-1}(D_{3} \cap \cdots \cap \lambda_{i}^{-1}(D_{i-1} \cap \lambda_{i}^{-1}(D_{i}) \cdots)) \neq \phi \). We apply Lemma 1 to \( f|_{\xi_{11} \cup \xi_{12}, h_{i^{-1}}|_{\xi_{11} \cup \xi_{12}, D, \lambda=\lambda_{i-1} \cdots \lambda_{2} \lambda_{1}} \). Then by induction we get a semi-simplicial mapping \( h_{i}|K \rightarrow E^{n} \), which is sufficiently close to \( h_{i-1} \) and satisfying ii) of Lemma 1 for all such pairs \( \{(\xi_{11}, \xi_{12})\} \). If \( q_{1}, q_{2}, \cdots, q_{i} \epsilon Q, q_{j} \neq q_{k}, \) and \( h_{i}|f^{-1}(q_{j}) \cap h_{i-1}|f^{-1}(q_{j}) = r_{j}, j = 1, \cdots, i-1, \) there are \( p_{j} \epsilon f^{-1}(q_{j}) \) and \( p'_{j+1} \epsilon f^{-1}(q_{j+1}) \) such that \( h_{i}(p_{j}) = h_{i}(p'_{j+1}) = r_{j} \). If \( p_{j} \epsilon P_{i}, p'_{j} \epsilon P_{i} \) and \( P_{i} \neq P'_{i} \). We can choose \( (\xi_{11}, \xi_{12}), \cdots, (\xi_{i-1}, \xi_{i}) \epsilon \Delta \) and \( \lambda \) which are similar as the above ones and satisfies that \( p_{j} \epsilon \xi_{11}, p'_{j} \epsilon \xi_{12} \) and \( \lambda(q_{1}) = q_{1} \). From ii) and 3) we have \( h_{i}(p_{j}) \neq h_{i}(p'_{j}) \). Therefore we have \( h_{i}|f^{-1}(q_{j}) \cap h_{i}|f^{-1}(q_{j}) = \phi \) and then we have proved that \( \alpha_{i} \) there is no cyclic chain of length \( \geq i \) in \( \Gamma_{i} = \{h_{i}|f^{-1}(q)|q \epsilon Q\} \). Since
\[
\text{dim } f^{-1}(X_{i-1}) + \text{dim } P - n
\leq \text{dim } X_{i-1} + \text{Max } \text{dim } f^{-1}(q) + \text{dim } Q + \text{Max } \text{dim } f^{-1}(q) - n
\geq \text{dim } Q + 1 - (i-1) + \text{dim } Q + 2 \text{Max } \text{dim } f^{-1}(q) - n
\leq \text{dim } Q + 1 - i.
\]
We can modify \( h_{i} \) so that \( \beta_{i} \) \( \text{dim } X_{i} \leq \text{dim } Q + 1 - i \). Then we have proved Lemma (2, i).

**Proof of Theorem 2.** Since \( \text{dim } X_{n+2} \leq \text{dim } Q + 1 - (n+2) < 0 \), \( X_{n+2} = \phi \) and then there is no simple chain of length \( \geq n+2 \) in \( \Gamma_{n+2} \). It is clear that \( \alpha_{n+2} \) and \( \beta_{n+2} \) implies that there is no cyclic chain in \( \Gamma_{n+2} \). Therefore \( h = h_{n+2} \) is the required p. w. l. mapping.

4. **Proof of Theorem 1.** Let \( \{C_{i''}, \cdots, C_{i'}\}, \{C_{i'}, \cdots, C_{i'}\}, \{C_{i}, \cdots, C_{i}\} \) be families of (combinatorial) \( n \)-cell of \( M \) such that
\[
C_{i''} \cap C_{i''} \cap \cdots \cap C_{i''} = M
\]
\[
C_{i''} \subset C_{i'} \subset C_{i'} \subset C_{i} \quad i = 1, \cdots, l.
\]
Let \( \gamma \) be a positive number such that
\[
\gamma < d(C_{i'}, \hat{C}_{i'}) / n+2, \quad d(C_{i'}, \hat{C}_{i}).
\]
Then \( \gamma \) is the requied number. In fact if \( f, g, P, Q, R \) satisfy the assumption of Theorem 1, put \( P_{i} = \cup \{f^{-1}(q)|q \epsilon Q, g f^{-1}(q) \cap C_{i} = \phi\}, R_{i} = P_{i} \cap R, f(P_{i}) = Q_{i}. \)

Then \( \hat{C}_{i} \supset g(P_{i}) \) and \( \hat{C}_{i} \) is p. w. l. homeomorphic to \( E^{n} \). From Theorem 1 and by induction with respect to \( i \) we have a p. w. l. mapping \( h : P \rightarrow \hat{M} \) such that \( h_{i} = h|P_{i} \)
satisfies the conditions 1), 2), 3), 4), 5), \(\alpha_1\), \(\alpha_2\), \(\alpha\), \(\beta\) for \(P_i, R_i, Q_i\) and \(g_i=g|P_i, f_i=f|P_i\). If \(\{h f^{-1}(q_i), \cdots, h f^{-1}(q_{n+2})\}\) is a chain, from \(\cup C''=M\) there is an \(i\) such that \(C'' \cap g f^{-1}(a_1) \neq \emptyset\). Since \(\text{dia} \cup_{j=1}^{n+2} g f^{-1}(q_j) < (n+2) \gamma < d(C', C')\). We have \(\cup_{j=1}^{n+2} g f^{-1}(q_j) \subseteq C'\) and then \(\cup_{j=1}^{n+2} f^{-1}(q_j) \subseteq P_i\). From the condition \(\beta\) for \(h_i\) \(\{h_i f^{-1}(q_i), \cdots, h_i f^{-1}(q_{n+2})\}\) is not a simple chain. Hence we have proved that \(\beta\) \(\Gamma\) has no simple chain of length \(\geq n+2\). From \(\beta\) it is clear that \(\Gamma\) has no cyclic chain of length \(\geq n+3\). Furthermore the condition \(\alpha\) for \(h_i\) implies that \(\Gamma\) has no chain of length \(\leq n+2\). Therefore we have proved [Theorem 1].

REFERENCE


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