Riemann’s Period matrix of $y^4 = x^4 - 1$

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1 Introduction

In Tashiro’s paper [1], he calculated Riemann’s period matrix $(\pi : \pi)$ of the closed hyperelliptic Riemann surface defined by the equation $y^2 = x^{2n+1} - 1$, and he showed that the determinants $|\pi|$ and $|\pi^*|$ are not zero and the matrix $\pi^{-1} \pi$ is symmetric.

We now pick up the closed Riemann surface defined by $y^4 = x^4 - 1$, which is not hyperelliptic, and we determine the period matrix $(\pi : \pi^*)$, and calculate the determinants $|\pi|$ and $|\pi^*|$.

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2 Preliminaries

2.1 Genus of $y^4 = x^4 - 1$

To see that the genus of $y^4 = x^4 - 1$ is 3, we note that the Riemann surface over the $x$-sphere covers each point four times, so $n = 4$. Furthermore, the total branching order $v$ is 12, since there are four branch points of order 3 at $x = 1$, $x = i$, $x = -1$, and $x = -i$. Then from $v = 2(n + g - 1)$, we get $g = 3$.

2.2 Standard basis of $R$

Let $R$ be Riemann surface defined by $y^4 = x^4 - 1$. Then there exist 6 cycles $\gamma_i, \delta_i (i = 1, 2, 3)$:

$I(\gamma_1, \gamma_2) = I(\delta_i, \delta_i) = 0$

$I(\gamma_i, \delta_j) = \varepsilon_{ij}(i = 1, 2, 3)$

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where \( I(\gamma, \delta) \) denotes the intersection number of 2 cycles \( \delta \) and \( \gamma \).

We show these standard basis in the figure fig. 1. Cutting out the surface along these paths, we get a simply connected domain, which is the fig. 2.
2.3 Integral paths of $R_3$

The surface $R_3$ is not hyperelliptic, and considered as a covering space of degree 4 over a Riemann sphere. The fig. 3 is how the 6 cycles in fig. 2 are seen. From this figure, 6 integral paths $\delta_i, \gamma_i (i = 1, 2, 3)$ are drawn in fig. 4. Branching points are $P_1 = 1, P_2 = i, P_3 = -1, P_4 = -i$. 

![Diagram of integral paths](image)

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**Fig. 5**

**Fig. 6**
2.4 Abelian differentials on $y^p = x^q - 1$

The set of all Abelian differentials on a Riemann surface of genus $g$ forms a vector space, and the dimension of the vector space is equal to the genus $g$. The fact comes from Riemann-Roch theorem.

**Theorem 1** In particular, the dimension of the space on the algebraic curve $y^p = x^q - 1$ is equal to $(p-1)(q-1)/2$.

**proof.** The points where $\frac{x^b}{y^a} \, dx$ may not be holomorphic are $y = 0$ or $\infty$. At the points, we seek an equality of $a$, $b$.

(i) $y = 0$: From $y^p = x^q - 1$, $py^{p-1} \, dy = qx^{q-1} \, dx$. Then $dx$ has a zero of order $p - 1$ at the points with $y = 0$, that is, $(x, y) = (a_i, 0)$, $a_i^q - 1 = 0$ ($i = 1, \ldots, q$). Hence $\frac{x^b}{y^a} \, dx$ is holomorphic at $x = a_i$ if

\[ p - 1 \geq a \]

from an inequality

(the order of zero points of $x^b \, dx$) $\geq$ (the order of zero points of $y^a$)

(ii) $\infty$: We may assume the equation is $y^p = x^q$. Then, $x^b \, dx = pt^{-bq-p-1} \, dt$ and $y^a = t^{-aq}$. $\frac{x^b}{y^a} \, dx$ is holomorphic if

\[ -bp \geq aq + p + 1 \]

From the inequalities above

\[ a = 1, 2, \ldots, (p-1) \]
\[ b = 0, 1, 2, \ldots, \left[ \frac{aq - (p+1)}{p} \right] \]

Hence, let $N$ be the number of Abelian differentials of the form $\frac{x^b}{y^a} \, dx$, then

\[ N = \sum_{a=1}^{p-1} \left( \left[ \frac{aq - (p+1)}{p} \right] + 1 \right) = \frac{(p-1)(q-1)}{2} \]

Substituting $p = q = 4$, we have $N = 3$. 
Theorem 2  *In fact, holomorphic Abelian differentials are following three forms:*

\[
\begin{align*}
\frac{dx}{y^3}, \quad \frac{x\, dx}{y^3}, \quad \frac{dx}{y^3}
\end{align*}
\]

*proof.* It is sufficient to check that the forms are holomorphic at the branching points \(x = 1, -1, i, -i\) and \(x = \infty\). At \(x = 1\), let \(t = \sqrt[4]{1-x}\). Then \(t^4 = 1 - x\) and \(4t^3\, dt = dx\).

Now, \(y^4 = t^4(t^4 + 2)(t^4 + 1 + i)(t^4 + 1 - i)\), so

\[
\begin{align*}
\frac{dx}{y^3} &= \frac{4\, dt}{[((t^4 + 2)(t^4 + 1 + i)(t^4 + 1 - i)]}, \\
\frac{x\, dx}{y^3} &= \frac{4(t^4 + 1)\, dt}{[((t^4 + 2)(t^4 + 1 + i)(t^4 + 1 - i)]}, \\
\frac{dx}{y^3} &= \frac{4\, dt}{[((t^4 + 2)(t^4 + 1 + i)(t^4 + 1 - i)]}.
\end{align*}
\]

These are all holomorphic at \(t = 0\). So it is at \(x = -1, i, -i\). At \(x = \infty\), let \(x = 1/t\), then

\[
\begin{align*}
\frac{dx}{y^3} &= \frac{-tdt}{(1-t^4)^{\frac{3}{2}}}, \quad \frac{x\, dx}{y^3} = \frac{-dt}{(1-t^4)^{\frac{3}{2}}}, \quad \frac{dx}{y^3} = \frac{-dt}{(1-t^4)^{\frac{3}{2}}}
\end{align*}
\]

and those are holomorphic at \(t = 0\), i.e., at \(x = \infty\). These three Abelian differentials are independent over \(C\), and then form a basis of the vector space of all Abelian differentials on the Riemann surface defined by \(y^4 = x^4 - 1\).

3 Definitions

Riemann's period matrix \(\Omega\) for \(R_3\) is defined as the following matrix:

\[
\Omega = ([\pi_{ij}], \pi'_{ij}) = \begin{pmatrix}
\pi_{11} & \pi_{12} & \pi_{13} \\
\pi_{21} & \pi_{22} & \pi_{23} \\
\pi_{31} & \pi_{32} & \pi_{33}
\end{pmatrix},
\]

where

\[
\begin{align*}
\pi_{ij} &= \int\omega_i \, \omega'_j = \int\omega_i \, \omega_j \, (i, j = 1, 2, 3), \\
\omega_1 &= \frac{dx}{y^3}, \quad \omega_2 = \frac{x\, dx}{y^3}, \quad \omega_3 = \frac{dx}{y^3}
\end{align*}
\]

When we integrate an Abelian differential on the paths, such as \(\delta, \gamma\) in Fig. 4, we must pay attention to the sign of the differentials.
Let \( \omega_i, \omega_2, \omega_3 \) and \( \omega_4 \) be represented as \( \omega_{ij} (i = 1, 2, 3, j = 1, 2, 3, 4) \) on the 4 Riemann surface in Fig. 4, then

\[
\begin{align*}
\omega_{11} &= \omega_1 & \omega_{21} &= \omega_2 & \omega_{31} &= \omega_3 \\
\omega_{12} &= i\omega_1 & \omega_{22} &= i\omega_2 & \omega_{32} &= -\omega_3 \\
\omega_{13} &= -\omega_1 & \omega_{23} &= -\omega_2 & \omega_{33} &= \omega_3 \\
\omega_{14} &= -i\omega_1 & \omega_{24} &= -i\omega_2 & \omega_{34} &= -\omega_3
\end{align*}
\]

4 Period matrix of \( y' = x^4 - 1 \)

We calculate \( \omega_i \) along the paths \( \delta_{js} \) in Fig. 1:

\[
\begin{align*}
\pi_{11} &= \int_{s_1} \omega_1 = \int_{p_1}^{p_3} \omega_1 + \int_{p_3}^{p_1} \omega_1 = \int_{p_1}^{p_3} \omega_1 + \int_{p_3}^{p_1} (i\omega_1) \\
&= \int_{p_1}^{p_3} \omega_1 - \int_{p_3}^{p_1} \omega_1 - i\int_{p_1}^{p_3} \omega_1 + i\int_{p_3}^{p_1} \omega_1 = (1 - i)(p_1 - p_3) \int_1^\infty \omega_1 \\
\pi_{12} &= \int_{s_2} \omega_1 = \int_{p_1}^{p_3} \omega_1 + \int_{p_3}^{p_1} \omega_1 = \int_{p_1}^{p_3} \omega_1 + \int_{p_3}^{p_1} (i\omega_1) \\
&= \int_{p_1}^{p_3} \omega_1 - \int_{p_3}^{p_1} \omega_1 + i\int_{p_1}^{p_3} \omega_1 - i\int_{p_3}^{p_1} \omega_1 = (1 + i)(p_1 - p_3) \int_1^\infty \omega_1 \\
\pi_{13} &= \int_{s_3} \omega_1 = \int_{p_1}^{p_4} \omega_1 + \int_{p_4}^{p_1} \omega_1 = \int_{p_1}^{p_4} (-\omega_1) + \int_{p_4}^{p_1} (-i\omega_1) \\
&= -\int_{p_1}^{p_4} \omega_1 + \int_{p_4}^{p_1} \omega_1 - i\int_{p_1}^{p_4} \omega_1 + i\int_{p_4}^{p_1} \omega_1 = (-1 + i)(p_1 - p_4) \int_1^\infty \omega_1 \\
\pi_{14} &= \int_{s_4} \omega_1 = \int_{p_1}^{p_4} \omega_1 + \int_{p_4}^{p_1} \omega_1 = \int_{p_1}^{p_4} \omega_1 + \int_{p_4}^{p_1} (i\omega_1) \\
&= \int_{p_1}^{p_4} \omega_1 - \int_{p_4}^{p_1} \omega_1 + i\int_{p_1}^{p_4} \omega_1 - i\int_{p_4}^{p_1} \omega_1 = (-1 - i)(p_1 - p_4) \int_1^\infty \omega_1 \\
\pi_{21} &= \int_{s_1} \omega_2 = \int_{p_1}^{p_3} \omega_2 + \int_{p_3}^{p_1} \omega_2 = \int_{p_1}^{p_3} \omega_2 + \int_{p_3}^{p_1} (i\omega_2) \\
&= \int_{p_1}^{p_3} \omega_2 - \int_{p_3}^{p_1} \omega_2 + i\int_{p_1}^{p_3} \omega_2 + i\int_{p_3}^{p_1} \omega_2 = (1 - i)(p_1^2 - p_3^2) \int_1^\infty \omega_2 \\
\pi_{22} &= \int_{s_2} \omega_2 = \int_{p_1}^{p_3} \omega_2 + \int_{p_3}^{p_1} \omega_2 = \int_{p_1}^{p_3} \omega_2 + \int_{p_3}^{p_1} (i\omega_2) \\
&= \int_{p_1}^{p_3} \omega_2 - \int_{p_3}^{p_1} \omega_2 + i\int_{p_1}^{p_3} \omega_2 + i\int_{p_3}^{p_1} \omega_2 = (1 + i)(p_1^2 - p_3^2) \int_1^\infty \omega_2 \\
\pi_{23} &= \int_{s_3} \omega_2 = \int_{p_1}^{p_4} \omega_2 + \int_{p_4}^{p_1} \omega_2 = \int_{p_1}^{p_4} (-\omega_2) + \int_{p_4}^{p_1} (-i\omega_2) \\
&= \int_{p_1}^{p_4} \omega_2 + \int_{p_4}^{p_1} \omega_2 - i\int_{p_1}^{p_4} \omega_2 - i\int_{p_4}^{p_1} \omega_2 = (-1 - i)(p_1^2 - p_3^2) \int_1^\infty \omega_2 \\
\pi_{24} &= \int_{s_4} \omega_2 = \int_{p_1}^{p_4} \omega_2 + \int_{p_4}^{p_1} \omega_2 = \int_{p_1}^{p_4} \omega_2 + \int_{p_4}^{p_1} (i\omega_2) \\
&= \int_{p_1}^{p_4} \omega_2 + \int_{p_4}^{p_1} \omega_2 - i\int_{p_1}^{p_4} \omega_2 - i\int_{p_4}^{p_1} \omega_2 = (-1 + i)(p_1^2 - p_3^2) \int_1^\infty \omega_2 \\
\pi_{31} &= \int_{s_1} \omega_3 = \int_{p_1}^{p_3} \omega_3 + \int_{p_3}^{p_1} \omega_3 = \int_{p_1}^{p_3} \omega_3 + \int_{p_3}^{p_1} (-\omega_3) \\
&= \int_{p_1}^{p_3} \omega_3 - \int_{p_3}^{p_1} \omega_3 + \int_{p_1}^{p_3} \omega_3 = 2(p_1 - p_3) \int_1^\infty \omega_3 \\
\pi_{32} &= \int_{s_2} \omega_3 = \int_{p_1}^{p_3} \omega_3 + \int_{p_3}^{p_1} \omega_3 = \int_{p_1}^{p_3} \omega_3 + \int_{p_3}^{p_1} (\omega_3) \\
&= \int_{p_1}^{p_3} \omega_3 + \int_{p_3}^{p_1} \omega_3 - \int_{p_1}^{p_3} \omega_3 = -2(p_1 - p_3) \int_1^\infty \omega_3 \\
\pi_{33} &= \int_{s_3} \omega_3 = \int_{p_1}^{p_4} \omega_3 + \int_{p_4}^{p_1} \omega_3 = \int_{p_1}^{p_4} \omega_3 + \int_{p_4}^{p_1} (-\omega_3) \\
&= \int_{p_1}^{p_4} \omega_3 - \int_{p_4}^{p_1} \omega_3 + \int_{p_1}^{p_4} \omega_3 = 2(p_1 - p_3) \int_1^\infty \omega_3 \\
\pi_{34} &= \int_{s_4} \omega_3 = \int_{p_1}^{p_4} \omega_3 + \int_{p_4}^{p_1} \omega_3 = \int_{p_1}^{p_4} \omega_3 + \int_{p_4}^{p_1} (\omega_3) \\
&= \int_{p_1}^{p_4} \omega_3 + \int_{p_4}^{p_1} \omega_3 - \int_{p_1}^{p_4} \omega_3 = -2(p_1 - p_3) \int_1^\infty \omega_3 \\
\end{align*}
\]
Riemann's Period matrix of $y^4 = x^4 - 1$

$$
\pi_{33} = \int_{\gamma_3} \omega_3 = \int_{p_3}^{p_4} \omega_3 + \int_{p_3}^{p_1} \omega_3 = \int_{p_4}^{p_3} (\omega_3)
= \int_{p_1}^{p_4} \omega_3 - \int_{p_4}^{p_1} \omega_3 + \int_{p_1}^{p_3} \omega_3 - \int_{p_3}^{p_1} \omega_3 = 2(p_1 - p_4) \int_{1}^{\infty} \omega_3
$$

Let $A$, $B$, and $C$ be defined as follows:

$$
A = \int_{1}^{\infty} \omega_1 = \int_{1}^{\infty} \frac{dx}{\sqrt{x^4 - 1}},
B = \int_{1}^{\infty} \omega_2 = \int_{1}^{\infty} \frac{x dx}{\sqrt{x^4 - 1}},
C = \int_{1}^{\infty} \omega_3 = \int_{1}^{\infty} \frac{dx}{\sqrt{x^4 - 1}}
$$

$$
\Pi = \begin{pmatrix}
A & 0 & 0 \\
0 & B & 0 \\
0 & 0 & C
\end{pmatrix}
\begin{pmatrix}
(1 - i)(p_1 - p_3) & (1 - i)(p_1 - p_3) & -1 \\
(1 - i)(p_1^2 - p_3^2) & (1 - i)(p_1^2 - p_3^2) & -2(p_1 - p_3) \\
2(p_1 - p_3) & 2(p_1 - p_3) & 2(p_1 - p_3)
\end{pmatrix}
$$

Substituting $p_1 = 1$, $p_2 = i$, $p_3 = -1$ and $p_4 = 1$, we have

$$
||\Pi|| = -8ABC(1 - i)
$$

In the following, we calculate the integrals in another representation of $\omega_i$'s.

$$
\int_{s_1} \omega_1 = \int_{p_2}^{p_3} \omega_1 + \int_{p_3}^{p_1} \omega_1 = \int_{p_2}^{p_3} \omega_1 + \int_{p_3}^{p_1} (i \omega_1)
= \int_{p_2}^{p_3} \omega_1 + i \int_{p_3}^{p_2} \omega_1 - \int_{p_3}^{p_1} \omega_1 = (1 - i)(p_2 - p_3) \int_{1}^{\infty} \omega_1
$$

$$
\int_{s_2} \omega_1 = \int_{p_2}^{p_3} \omega_1 + \int_{p_3}^{p_1} \omega_1 = \int_{p_2}^{p_3} \omega_1 + \int_{p_3}^{p_1} (-\omega_1)
= i \int_{p_2}^{p_3} \omega_1 + i \int_{p_3}^{p_1} \omega_1 - i \int_{p_3}^{p_1} \omega_1 = (1 - i)(p_2 - p_3) \int_{1}^{\infty} \omega_1
$$

$$
\int_{s_3} \omega_1 = \int_{p_1}^{p_3} \omega_1 + \int_{p_3}^{p_3} \omega_1 = \int_{p_1}^{p_3} \omega_1 + \int_{p_3}^{p_1} (-\omega_1)
= - i \int_{p_1}^{p_3} \omega_1 + i \int_{p_3}^{p_1} \omega_1 - i \int_{p_3}^{p_1} \omega_1 = (1 + i)(p_2 - p_3) \int_{1}^{\infty} \omega_1
$$

$$
\int_{s_1} \omega_2 = \int_{p_2}^{p_3} \omega_2 + \int_{p_3}^{p_3} \omega_2 = \int_{p_2}^{p_3} \omega_2 + \int_{p_3}^{p_3} (i \omega_2)
= \int_{p_2}^{p_3} \omega_2 + i \int_{p_3}^{p_3} \omega_2 + i \int_{p_3}^{p_1} \omega_2 = (1 - i)(p_2^2 - p_3^2) \int_{1}^{\infty} \omega_2
$$

$$
\int_{s_2} \omega_2 = \int_{p_2}^{p_3} \omega_2 + \int_{p_3}^{p_3} \omega_2 = \int_{p_2}^{p_3} \omega_2 + \int_{p_3}^{p_3} (-\omega_2)
= i \int_{p_2}^{p_3} \omega_2 + i \int_{p_3}^{p_3} \omega_2 - i \int_{p_3}^{p_1} \omega_2 = (1 + i)(p_2^2 - p_3^2) \int_{1}^{\infty} \omega_2
$$

$$
\int_{s_3} \omega_2 = \int_{p_1}^{p_3} \omega_2 + \int_{p_3}^{p_3} \omega_2 = \int_{p_1}^{p_3} \omega_2 + \int_{p_3}^{p_3} (-\omega_2)
= - i \int_{p_1}^{p_3} \omega_2 + i \int_{p_3}^{p_3} \omega_2 + i \int_{p_3}^{p_1} \omega_2 = (1 + i)(p_2^2 - p_3^2) \int_{1}^{\infty} \omega_2
$$

$$
\int_{s_1} \omega_3 = \int_{p_2}^{p_3} \omega_3 + \int_{p_3}^{p_1} \omega_3 = \int_{p_2}^{p_3} \omega_3 + \int_{p_3}^{p_1} (-\omega_3)
$$

$$
= - i \int_{p_2}^{p_3} \omega_3 + i \int_{p_3}^{p_3} \omega_3 + i \int_{p_3}^{p_1} \omega_3 = (1 - i)(p_2^2 - p_3^2) \int_{1}^{\infty} \omega_3
$$

$$
\int_{s_2} \omega_3 = \int_{p_2}^{p_3} \omega_3 + \int_{p_3}^{p_3} \omega_3 = \int_{p_2}^{p_3} \omega_3 + \int_{p_3}^{p_3} (-\omega_3)
= i \int_{p_2}^{p_3} \omega_3 + i \int_{p_3}^{p_3} \omega_3 - i \int_{p_3}^{p_1} \omega_3 = (1 - i)(p_2^2 - p_3^2) \int_{1}^{\infty} \omega_3
$$

$$
\int_{s_3} \omega_3 = \int_{p_1}^{p_3} \omega_3 + \int_{p_3}^{p_3} \omega_3 = \int_{p_1}^{p_3} \omega_3 + \int_{p_3}^{p_3} (-\omega_3)
= - i \int_{p_1}^{p_3} \omega_3 + i \int_{p_3}^{p_3} \omega_3 - i \int_{p_3}^{p_1} \omega_3 = (1 - i)(p_2^2 - p_3^2) \int_{1}^{\infty} \omega_3
$$
\[= \int_{p_2}^{p_3} w_3 + \int_{p_3}^{p_1} w_3 - \int_{p_1}^{w_2} w_3 = 2(p_2 - p_3) \int_{p_1}^{w_3} \]

\[\int_{p_2}^{w_3} w_3 = \int_{p_2}^{p_3} w_2 + \int_{p_3}^{p_2} w_3 = \int_{p_2}^{p_3} (-w_3) + \int_{p_3}^{p_2} (p_2 - p_3) \int_{p_3}^{w_3} \]

\[\int_{p_1}^{w_3} w_3 = \int_{p_1}^{p_2} w_3 + \int_{p_2}^{p_1} w_3 = \int_{p_1}^{p_2} w_3 + \int_{p_2}^{p_1} (-w_3) \]

\[= \int_{p_1}^{w_3} w_3 + \int_{p_1}^{p_2} w_3 - \int_{p_2}^{p_1} w_3 = 2(p_1 - p_2) \int_{p_2}^{w_3} \]

Let \(A, B\) and \(C\) be defined as follows:

\[A = \int_{p_2}^{p_3} \frac{dx}{\sqrt{x^2 - 1}}, \quad B = \int_{p_3}^{w_3} \frac{dx}{\sqrt{x^2 - 1}}, \quad C = \int_{w_3}^{p_1} \frac{dx}{\sqrt{x^2 - 1}} \]

\[= \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} = \begin{pmatrix} (1 - i)(p_2 - p_3) & (1 + i)(p_2 - p_3) & (-1 + i)(p_1 - p_3) \\ (1 - i)(p_2^2 - p_3^2) & (1 + i)(p_2^2 - p_3^2) & (-1 + i)(p_1^2 - p_3^2) \\ 2(p_3 - p_2) & 2(p_2 - p_3) & 2(p_1 - p_3) \end{pmatrix} \]

\[
\|\Pi\| = 32ABC(1 + i)
\]

\[
\Pi^{-1} = \frac{1}{16ABC(1 + i)} \begin{pmatrix} (1 + i) & -1 & (1 + i) \\ (1 - i) & 1 + i & (1 - i) \\ 1 + i & -i & 0 \end{pmatrix}
\]

Next, we calculate \(\omega_1\) along the paths \(\gamma_i's\).

\[\pi'_{11} = \int_{p_1}^{p_2} \omega_1 = \int_{p_3}^{p_2} \omega_1 + \int_{p_2}^{p_3} \omega_1 = \int_{p_3}^{p_2} (i\omega_1) + \int_{p_2}^{p_3} \omega_1
\]

\[= -\int_{p_3}^{p_2} w_1 + \int_{p_2}^{p_3} w_1 + i\int_{p_2}^{w_1} w_1 - i\int_{p_2}^{w_1} w_1 = (-1 + i)(p_3 - p_1) \int_{p_3}^{w_1} \omega_1
\]

\[\pi'_{12} = \int_{p_2}^{p_3} \omega_1 = \int_{p_3}^{p_2} \omega_1 + \int_{p_2}^{p_3} \omega_1 = \int_{p_3}^{p_2} (-\omega_1) + \int_{p_2}^{p_3} (-i\omega_1)
\]

\[= -\int_{p_3}^{w_1} w_1 + \int_{p_2}^{w_1} w_1 + i\int_{p_2}^{w_1} w_1 - i\int_{p_2}^{w_1} w_1 = (-1 + i)(p_3 - p_1) \int_{p_3}^{w_1} \omega_1
\]

\[\pi'_{13} = \int_{p_3}^{w_1} \omega_1 = \int_{p_2}^{p_3} \omega_1 + \int_{p_3}^{p_2} \omega_1 = \int_{p_2}^{p_1} w_1 + \int_{p_2}^{p_1} (-i\omega_1)
\]

\[= \int_{p_3}^{w_2} w_1 - \int_{p_3}^{p_2} w_1 + i\int_{p_2}^{w_2} w_1 - i\int_{p_2}^{w_2} w_1 = (1 + i)(p_1 - p_2) \int_{p_3}^{w_2} \omega_1
\]

\[\pi'_{21} = \int_{p_1}^{p_2} \omega_2 = \int_{p_3}^{p_2} \omega_2 + \int_{p_2}^{p_3} \omega_2 = \int_{p_3}^{p_2} (i\omega_2) + \int_{p_2}^{p_3} (\omega_2)
\]

\[= -\int_{p_3}^{p_2} w_2 + \int_{p_2}^{p_3} w_2 + i\int_{p_2}^{p_3} w_2 - i\int_{p_2}^{p_3} w_2 = (-1 + i)(p_2^2 - p_3^2) \int_{p_3}^{w_2} \omega_2
\]

\[\pi'_{22} = \int_{p_2}^{p_3} \omega_2 = \int_{p_3}^{p_2} \omega_2 + \int_{p_2}^{p_3} \omega_2 = \int_{p_3}^{p_2} (-\omega_2) + \int_{p_2}^{p_3} (-i\omega_2)
\]

\[
= \int_{p_3}^{p_2} (i\omega_2) + \int_{p_2}^{p_3} (\omega_2)
\]
Let $A$, $B$ and $C$ be defined as follows:

\[
A = \int_1^\infty \omega_1 = \int_1^\infty \frac{dx}{\sqrt{x^4 - 1}}, \quad B = \int_1^\infty \omega_2 = \int_1^\infty \frac{x dx}{\sqrt{x^4 - 1}}, \quad C = \int_1^\infty \omega_3 = \int_1^\infty \frac{dx}{\sqrt{x^4 - 1}}
\]

\[
II = \begin{pmatrix}
  A & 0 & 0 \\
  0 & B & 0 \\
  0 & 0 & C
\end{pmatrix}
\begin{pmatrix}
  (-1 - i) (p_3 - p_4) & (-1 + i) (p_3 - p_4) & (1 + i) (p_1 - p_2) \\
  (-1 + i) (p_3^2 - p_2^2) & (-1 + i) (p_3^2 - p_2^2) & (1 + i) (p_1^2 - p_2^2) \\
  -2 (p_3 - p_4) & 2 (p_3 - p_4) & 2 (p_1 - p_2)
\end{pmatrix}
\]

Substituting $p_1 = 1$, $p_2 = i$, $p_3 = -1$ and $p_4 = 1$, we have

\[
II' = 2 \begin{pmatrix}
  A & 0 & 0 \\
  0 & B & 0 \\
  0 & 0 & C
\end{pmatrix}
\begin{pmatrix}
  -i & -i & 1 \\
  -1 + i & -1 + i & 1 + i \\
  1 - i & -1 + i & 1 - i
\end{pmatrix}
\]

\[
|II'| = 16ABC (1 - i)
\]
References