Totally Geodesic Orbits of the Isotropy Subgroups
of Compact Riemannian Symmetric Spaces

By

Shigeo AKIBA

Riemannian symmetric spaces and their submanifolds produce abundant examples for Riemannian geometry. In this paper we investigate totally geodesic orbits of the isotropy subgroups of irreducible compact Riemannian symmetric spaces and calculate their dimensions in some simple cases.

Let $U/K$ be an irreducible Riemannian symmetric space of compact type, $u$ be the Lie algebra of $U$ and $t$ be the Lie algebra of $K$. Then we have a Cartan decomposition of $u$: $u = t + p$,

where $p$ is a linear subspace of $u$ invariant by the involutive automorphism of $u$. $p$ is identified with the tangent space of $U/K$ at the origin $o$.

Let $a_\ast$ be a maximal abelian subspace of $p$, $h$ be a maximal abelian subalgebra of $u$ containing $a_\ast$ and $m$ be the centralizer of $a_\ast$ in $t$. The centralizer of $a_\ast$ in $K$ is denoted by $M$, whose Lie algebra is $m$.

For an arbitrary point $p$ of $U/K$, the orbit $K(p)$ of the isotropy subgroup $K$ is a closed submanifold of the symmetric space $U/K$. Since the mapping

$$
\Phi: (KM, H) \rightarrow \text{Exp} \text{Ad}(k)H, \quad k \in K, H \in a_\ast.
$$

is a differentiable mapping of $K/M \times a_\ast$ onto $U/K$ ([1], p. 294), the orbit $K(p)$ intersects the set $\text{Exp} a_\ast$. Thus we may suppose $p = \text{Exp} H$ for some $H \in a_\ast$.

By means of the action $\tau$ of $U$ on $U/K$, $Z \in u$ generates a curve $\tau(\exp tZ)q$ starting at $q \in U/K$ and a tangent vector $d/dt|_{t=0} \tau(\exp tZ)q$ of $U/K$ at $q$. If $q = \tau(k)\text{Exp} H$, $k \in K, H \in a_\ast$, then a tangent vector at $q$ is written:

$$
Z'(q) = d\tau(kh_h)\circ (d\pi)_h \text{Ad} (kh)^{-1} Z,
$$

where $h = \text{Exp} H$.

Let $\Sigma^+$ be the set of positive restricted roots of $(u, a_\ast)$ and $f_1[p_1]$ be the root space of $T_{\mu H} = (\text{ad} iH)^2$ in $f[p]$ belonging to a restricted root $\lambda$. Then the direct decomposition

$$
u = \Sigma f_1 + m + a + \Sigma p_1
$$

is orthogonal. Let $f_1[p,]$ denote the direct sum.

* Department of Mathematics, Faculty of Education, Yokohama National University.
\[ \sum_{\lambda} \varphi_{1+\lambda}, \]

where \( \lambda \) runs through positive restricted roots \( \lambda \) such that \( \lambda((H) \in \pi Z \).

The action of \( K \) on \( K(p) \) being transitive, mapping \( \xi \rightarrow \xi' (q) \) is a linear mapping of \( t \) onto the tangent space \( K(p)_{q} \), the kernel of the mapping is \( \Lambda k(k(m+t)) \). If \( X \in p \), mapping \( X \rightarrow X'(q) \) is an isomorphism of \( Ad(k(a+p_M)) \) onto the normal space \( K(p)^{i} \) to \( K(p) \) at \( q \).

Let \( \nabla \) be the Riemannian connection on \( U/K \) associated with the symmetric structure, i.e., preserving the Riemannian metric \( <,> \) defined by the minus of the Killing-Cartan from on \( p \times p \).

Let \( \xi \) and \( \eta \) be arbitrary elements of \( t \) and \( q=\tau(k)p \) be a point of \( K(p) \). \( \xi \) generates a curve \( q_t=\tau(\exp t\xi)q \) and \( \eta \) generates a vector field \( \eta'(q_t) \) along the curve \( q_t \). Then we have

\[ \xi'(q)=\frac{d}{dt}\tau(\exp t\xi)((\sinh(-adH)(Ad(\xi))-i\eta)) \]
and

\[ \eta'(q_t)=\frac{d}{dt}\tau(\exp t\xi)(\sinh(-adH)(Ad(\exp t\xi-k^{-1}\eta))). \]

The isometry \( \tau(kh)^{-1} \) transforms a vector \( \xi'(q) \) to a vector at \( o : \)

\[ \sinh(-ad\eta)(Ad(k)-i\eta) \]

and a vector field \( \eta'(q_t) \) to a vector fields along a curve \( \tau(kh)^{-1}q \) starting at \( o : \)

\[ \frac{d}{dt}(h^{-1}k^{-1} \exp t\xi-k^{-1}\eta)(\sinh(-adH)(Ad(\exp t\xi-k^{-1}\eta))), \]

which is generated by \( Ad(kh)^{-1}\eta \). Thus by Corollary 1.3 of [2], p. 188,

\[ <\nabla_{\eta'}\eta'(q_t)=\frac{d}{dt}\tau(\exp t\xi)(\sinh(-adH)(Ad(\exp t\xi-\eta))>(\tau(kh)^{-1}q_t) \]

\[ =\frac{d}{dt}\tau(\exp t\xi)(([Ad(kh)^{-1}\eta], [Ad(kh)^{-1}\xi],) \), \]

where \([\_]\), and \([\_]_p\) denotes the \( t \)- and \( p \)-component with respect to the direct sum decomposition \( u=t+p \).

An arbitrary normal vector of \( K(p) \) at \( q \) being generated by an element \( N \in Ad(k(a_+p_M)) \) the second fundamental form \( \alpha \) of \( K(p) \) in \( U/K \) is determined by the values \( <\alpha(\xi', \eta'), N'(q)> = <\nabla_{\xi'}(\eta'), N'(q)> \) for arbitrary elements \( \xi \) and \( \eta \) of \( t \) and an arbitrary element \( N \) of \( Ad(k(a_+p_M)) \). By the formula (1) and since \( \tau(kh) \) is an isometry we obtain

\[ <\alpha(\xi', \eta'), N'(q)> = <([Ad(kh)^{-1}\eta], [Ad(kh)^{-1}\xi],) \), \[ [Ad(kh)^{-1}N]], \]

The submanifold \( K(p) \) is totally geodesic if and only if \( <\alpha(\xi', \eta'), N'(q)> = 0 \) for arbitrary \( \xi \) and \( \eta \in t \), an arbitrary \( N \in Ad(k(a_+p_M)) \) and at an arbitrary point \( q \in K(p) \). Since \( Ad(k) \) is an automorphism of \( u \), we may investigate the case \( k=e \) without loss of generality. Since \( Ad(h)^{-1}=e^{-adh} \) we have \( [Ad(h)^{-1}q]=\cosh(-adH)q, [Ad(h)^{-1}\xi]=\sinh(-adH)\xi \) and \( [Ad(h)^{-1}N]=\cosh(-adH)N \). Thus a necessary and sufficient condition for \( K(p) \) to be totally geodesic is
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\[ \langle \cosh(-\text{ad } H)\xi, \sinh(-\text{ad } H)\eta, N \rangle = 0 \]

or

\[ \langle \sinh(\text{ad } H)\xi, [\cosh(\text{ad } H)\eta, N] \rangle = 0 \]

for arbitrary \( \xi \) and \( \eta \in \mathfrak{f} \) and an arbitrary \( N \in \mathfrak{a}_s + p_N \).

**Theorem.** The orbit \( K(p) \) of \( p = \text{Exp } H \) is totally geodesic if and only if \( \lambda(iH) = n\pi/2 \) for any \( \lambda \in \Sigma^+ \). In this case the dimension of \( K(p) \) equals the sum of multiplicities of restricted roots \( \lambda \) with \( n \) odd.

**Proof.** Represent \( \xi \) and \( \eta \) by the direct sum

\[ \xi = \Sigma \xi_\lambda, \quad \eta = \Sigma \eta_\lambda. \]

By means of [1], Lemma 2.3., p. 285, there exists vectors \( X_\lambda \) and \( Y_\lambda \) for each \( \lambda \in \Sigma^+ \) such that

\[ \text{ad } H \cdot \xi_\lambda = -\lambda(iH)X_\lambda, \quad \text{ad } H \cdot X_\lambda = \lambda(iH)\xi_\lambda \]

and

\[ \text{ad } H \cdot \eta_\lambda = -\lambda(iH)Y_\lambda, \quad \text{ad } H \cdot Y_\lambda = \lambda(iH)\eta_\lambda. \]

Hence we have

\[ \sinh(\text{ad } H)\xi_\lambda = -\sin(\lambda(iH))X_\lambda, \]

\[ \cosh(\text{ad } H)\eta_\lambda = \cos(\lambda(iH))Y_\lambda \]

and thus

\[ \langle \sinh(\text{ad } H)\xi, [\cosh(\text{ad } H)\eta, N] \rangle \]

\[ = -\Sigma \lambda \sin(\lambda(iH)) \cos(\lambda(iH))X_\lambda, [Y_\lambda, N] \]. \hspace{1cm} \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (2) \]

Suppose that \( K(p) \) is totally geodesic and choose \( \xi = \eta = \xi_\lambda \neq 0 \) and \( N \in \mathfrak{a}_s \) such that \( \lambda(iN) \neq 0 \). Then we have \( \sin(\lambda(iH)) \cos(\lambda(iH)) = 0 \) and it follows that \( \lambda(iH) = n\pi/2 \) for each \( \lambda \in \Sigma^+ \).

Conversely, suppose \( \lambda(iH) = n\pi/2 \) for each \( \lambda \in \Sigma^+ \). Value of (2) vanishes for any \( \xi, \eta \in \mathfrak{f} \) and any \( N \in \mathfrak{a}_s + p_N \).

The latter assertion of the theorem follows from the fact that, for nonzero \( \xi \in \mathfrak{f}_1, \xi'(p) \) vanishes if and only if \( \lambda(iH) \in i\pi \mathbb{Z} \). This completes the proof.

**Examples.** Fix an element \( H \) of \( \mathfrak{a}_s \) with \( \lambda(iH) = n\pi/2, n \in \mathbb{Z} \), for any \( \lambda \in \Sigma^+ \). We call a restricted root \( \lambda \) even [odd, respectively] when \( n \) is even [odd, respectively]. If we designate the parity of all the simple restricted roots that of all the positive restricted roots are determined and the dimension of the totally geodesic orbit is represented as the sum of the multiplicities of odd positive restricted roots.

In the case of type \( A_1 \) we shall calculate the dimension of \( K(p) \).

In the Dynkin diagram of simple restricted roots we denote every odd root by a white circle \( \bigcirc \) and every even root by a black circle \( \bullet \). Attach each side of the diagram a dotted line of unit length followed by a white circle.

\[ \bigcirc \cdots \bullet \bigcirc \bigcirc \bullet \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \cdots \bullet \bigcirc \bigcirc \bigcirc \bigcirc \bullet \cdots \bigcirc \]
$r_i$ denote the distance between the $i$-th and the $(i+1)$-th white circles. Then the dimension of the totally geodesic orbit is $\sum r_i r_j$, where the sum is on the $(i, j)$'s such that $j-i$ are positive odd integers. A totally geodesic orbit is of dimension at least the rank $l$ of the symmetric space and is flat in this case of the least dimension. The maximum dimension is $(l+1)^2/4$ when $l$ is odd and $l(l+2)/4$ when $l$ is even. The minimum is attained (for example) in the case:

\[ \bullet - \bullet - \bullet - \cdots \]

and the maximum is attained (for example) in the case:

\[ \bullet - \bullet - \bullet - \cdots \]

In the case of type AII, since all the multiplicities are 4 the dimension of the totally geodesic orbit is 4 times of each case of type A I.

Bibliography