A Weak Convergence Theorem for Functionals of Sums of Strong Mixing Sequences

By

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1. Introduction. Let \( \{\xi_j\} \) be a strictly stationary sequence of random variables which are defined on a probability space \( (\Omega, \mathcal{F}, P) \). For \( a \leq b \), let \( M_n^a \) denote the \( \sigma \)-algebra of events generated by \( \xi_a, \cdots, \xi_b \). We shall say that the sequence \( \{\xi_j\} \) is strong mixing (s.m.) if

\[
\alpha(n) = \sup_{A \in M_n^a, B \in M_n^b} |P(A \cap B) - P(A)P(B)| \to 0 \quad (n \to \infty).
\]

Let \( D[0, 1] \) be the space of functions on \( [0, 1] \) that are right-continuous and have left-hand limits. We give the Skorokhod \( J_1 \)-topology in \( D[0, 1] \).

Let \( \{\xi_j\} \) be a sequence of i.i.d. random variables. Skorokhod and Slobodeneuk [3] proved that

\[
\int [\xi(t)dw(t)] \quad (n \to \infty)
\]

when \( \{\xi_j\} \) is a sequence of i.i.d. random variables.

In [5] and [6], the author proved weak convergence theorems of the same type concerning martingale differences and series of independent random variables.

In this paper, we shall prove a similar weak convergence theorem when \( \{\xi_j\} \) is some strictly stationary s.m. sequence.

2. Conditions and the main result. In this and the following sections, we shall denote by the letter \( K \), with or without subscripts, various absolute positive constants.

Let \( F_M \) be the space of functions defined on \( [0, 1] \times \mathbb{R}^d \) satisfying the following condition: there exists an absolute constant \( M \) such that for \( f \in F_M \), \( f \) and its derivatives satisfy inequalities of the form

\[
|Df(s, x)| \leq M(1 + |x|^\alpha)
\]

where \( D \) denotes either the identity operator or a first derivative and \( \alpha \) is some positive constant.

Remark 2.1. If \( f \in F_M \) and \( f(s, x) = 0(|x| > C) \) for each \( s \in [0, 1] \) and for
some \( C > 0 \), then it is obvious that
\[
|f(s, x) - f(s', x')| \leq M(1 + C^n)\{|s - s'| + |x - x'|\}.
\]

Put
\[
\sigma^2 = \text{Var}(\xi_n) + 2 \sum_{j=1}^{\infty} \text{cov}(\xi_n, \xi_j)
\]
if the series is convergent.

We shall consider the following condition:

CONDITION A. \( \{\xi_n\} \) is a strictly stationary s.m. sequence of random variables such that for some \( \delta > 0 \)

\[(A1)\] \( E\xi_n = 0 \), \( E|\xi_n|^{4+\delta} < \infty \) and
\[(A2)\] \( \sum n\{a(n)\}^{\delta/(4+\delta)} < \infty. \)

REMARK 2.2. It is known that if Condition A is satisfied then the series in (2.3) converges absolutely and
\[
(2.4) \quad (n^{1/2})^{-1/S_{(n)}} \xrightarrow{\text{D}} W \quad (\text{in } D[0, 1])
\]
when \( \sigma > 0 \). (cf. Oodaira and Yoshihara [2]).

In what follows, we shall assume that \( \sigma^2 = 1 \).

THEOREM. Let \( \{\xi_n\} \) be a sequence satisfying Condition A. Let \( f_n \in F_M \) \((n=1, 2, \ldots)\) and \( f \in F_M \). Assume that the for each \( s \in [0, 1] \)

\[(2.5)\] \( Df_n(s, x) \rightarrow Df(s, x) \quad (n \to \infty) \)

uniformly in \( x \) on every finite interval. Then
\[
(2.6) \quad \sum_{i=1}^{n-1} f_n\left(\frac{i}{n}, \frac{S_i}{\sqrt{n}} \right) \xrightarrow{\text{D}} \int_{0}^{1} f(t, w(t))dw(t).
\]

Here, the stochastic interval in (2.6) is taken in the \( L^2 \) sense, and \( q = q(n) \) is a function of \( n \) such that \( \beta_n = n\{a(q)\}^{(2+\delta)/(4+\delta)} \to 0 \) as \( n \to \infty \).

3. Proof of Theorem. Firstly, we shall prove some lemmas. Let \( [s] \) be the largest integer \( m \) such that \( m \leq s \).

LEMMA 3.1. Let \( f_n \in F_M(n=1, 2, \ldots) \). Let \( u_n(n=1, 2, \ldots) \) be functions such that
\[
(3.1) \quad u_n(t, x) = f_n^{(i)}(t, x) \quad (n=1, 2, \ldots)
\]
where \( C > 0 \) is some constant. Let \( \{t_0, t_1, \ldots, t_b\} \) be any collection of nonnegative numbers such that \( 0 = t_0 < t_1 < \cdots < t_b = 1 \). Assume that \( \{\xi_n\} \) satisfies Condition A. Put
\[
(3.2) \quad P_{\xi}(\varepsilon, \gamma, n) = P\left( \sum_{i=1}^{q} u_n\left(\frac{i}{n}, T_i\right) \frac{\xi_{i+q}}{\sqrt{n}} \right)
\]
A Weak Convergence Theorem

\[- \sum_{j=1}^{b} u_n(t_j, T_{[nt_j]+q})(T_{[nt_j]+1}-T_{[nt_j]-}) > \varepsilon \]

where \( T_i = n^{-\alpha/2} S_i \). Then

\[ \lim_{\gamma+i \to 0} \lim_{n \to \infty} P_i(\varepsilon, \gamma, n) = 0 \]

where \( \gamma = \max (t_{i+1}-t_i) \).

**Proof.** Define

\[ \xi_i = \begin{cases} \xi_i & \text{if } |\xi_i| \leq N, \\ 0 & \text{if } |\xi_i| > N, \end{cases} \]

and put \( \eta_i = n^{-\alpha/2} (\xi_i - E\xi_i) \) and \( \zeta_i = (n^{-1/2} \xi_i)^{-} \eta_i \quad (i=1, \ldots, n). \)

For brevity, we write \( \sum_{i,j} \) instead of \( \sum_{i=\lfloor nt_j \rfloor+1}^{\lfloor nt_{j+1} \rfloor} \). Then

\[ P_i(\varepsilon, \gamma, n) \leq P \left( \sum_{i,j} u_n \left( \frac{i}{n}, T_i \right) \eta_{i+j} - \sum_{i=\lfloor nt_j \rfloor+1}^{\lfloor nt_{j+1} \rfloor} u_n(t_j, T_{[nt_j]+}) \right) \leq \frac{\varepsilon}{2} \]

(3.3)

\[ + P \left( \sum_{i,j} u_n \left( \frac{i}{n}, T_i \right) \zeta_{i+j} - \sum_{i=\lfloor nt_j \rfloor+1}^{\lfloor nt_{j+1} \rfloor} u_n(t_j, T_{[nt_j]+}) \right) \leq \frac{\varepsilon}{2} \]

(3.4)

\[ = P_{i1}(\varepsilon, \gamma, n) + P_{i2}(\varepsilon, \gamma, n), \quad \text{(say)}. \]

For any \( j(1 \leq j \leq b) \) and \( i (\lfloor nt_j \rfloor \leq i < \lfloor nt_{j+1} \rfloor) \), put

\[ V_{ij} = u_n \left( \frac{i}{n}, T_i \right) - u_n(t_j, T_{[nt_j]+}). \]

Then, \( V_{ij} \) is uniformly bounded since \( u_n \) is uniformly bounded and by (2.2)

(3.5)

\[ EV_{ij} \leq KE \left( \left| \frac{i}{n} - t_j \right| + |T_i - T_{[nt_j]+}| \right)^2 \]

(3.6)

\[ \leq K \left( \left| \frac{\lfloor nt_j+1 \rfloor}{n} - t_j \right|^2 + E|T_{[nt_j]+}|^2 \right) \]

\[ \leq K \left( t_{j+1} - t_j + \frac{1}{n} \right). \]

Now, from Lemma 2.1 in Davydov [1]

(3.7)

\[ |E(V_{ij} \xi_{i+j})(V_{ij} \xi_{i+j})| \leq Kn^{-i}(E|\xi_i|^{i+j+\delta_1/(i+j)} \alpha \min (i' - i, q))^{(i+j+\delta_2)/(i+j)}(i'>i), \]

and

(3.8)

\[ E(V_{ij} \xi_{i+j})(V_{ij} \xi_{i+j}) \leq Kn^{-i}(E|\xi_i|^{i+j+\delta_1/(i+j)} \alpha \min (i' - i, q))^{(i+j+\delta_2)/(i+j)}(i'>i). \]

So, for all \( n \) sufficiently large
(3.9) \[ E \{ \sum_{\ell} V_{\ell j} \eta_{j+q} \}^2 \]
\[ \leq \sum_{\ell} E V_{\ell j}^2 \eta_{j+q}^2 + Kn^{-1} \{ E \{ \xi_1 \}^{4+\delta} \}^{\frac{1}{2+\delta}} \{ \sum_{\ell_j \geq 2} \} \{ (\alpha(q))^{2+\delta} \} \]
\[ \leq N^2 n^{-1} \sum_{\ell_j} E V_{\ell j} + K \beta_n \]
and

(3.10) \[ E \{ \sum_{\ell} V_{\ell j} \xi_{j+q} \}^2 \leq \sum_{\ell} E V_{\ell j}^2 \xi_{j+q}^2 + K \beta_n \]
\[ \leq K \{ \sum_{n \geq 2} E \xi_{n+q}^2 + \beta_n \} \leq K \{ nE \xi_{n+1}^2 \} (t_{j+1} - t_j) + \beta_n \}

Hence, from (3.9) and (3.10) it follows that

(3.11) \[ E \{ \sum_{l=1}^{n-q} u_n \left( \frac{i}{n}, T_1 \right) \eta_{j+q} - \sum_{j=1}^{n-q} u_n(t_j, T_{\lfloor nt_j \rfloor}, \lfloor nt_j \rfloor - q) \sum_{\ell \geq 2} \eta_{\ell j+q} \}^2 \]
\[ \leq \sum_{j=1}^{n-q} E \{ \sum_{\ell \geq 2} V_{\ell j} \eta_{\ell j+q} \}^2 + 2 \sum_{j \geq j_2 \geq 2} \sum_{\ell \geq 2} E \{ \sum_{\ell \geq 2} V_{\ell j} \eta_{\ell j+q} \sum_{\ell \geq 2} V_{\ell j} \eta_{\ell j+q} \}
\]
\[ \leq K \{ \gamma + b^2 \beta_n \}
\]
and

(3.12) \[ E \{ \sum_{l=1}^{n-q} u_n \left( \frac{i}{n}, T_1 \right) \xi_{j+q} - \sum_{j=1}^{n-q} u_n(t_j, T_{\lfloor nt_j \rfloor}, \lfloor nt_j \rfloor - q) \sum_{\ell \geq 2} \xi_{\ell j+q} \}^2 \]
\[ \leq K \{ nE \xi_{n+1}^2 + b^2 \beta_n \}
\]

and so we have

(3.13) \[ \lim_{r \to 0} \lim_{n \to \infty} P_{\nu}^0(\xi, \gamma, n) = 0 \]
and

(3.14) \[ \lim_{r \to 0} \lim_{n \to \infty} P_{\nu}^0(\xi, \gamma, n) \leq K \varepsilon^{-2} E |\xi - \xi_0|^2 \]

Since \( E \xi_0^2 < \infty \), letting \( N \to \infty \) (3.5) follows from (3.13) and (3.14) and the proof is completed.

**LEMMA 3.2.** Let \( f \in F_M \). Let \( u \) be the function defined by

(3.15) \[ u(s, x) = f^{(c)}(s, x) = \begin{cases} f(s, x), & \text{if } (s, x) \in [0, 1] \times [-C, C], \\ 0, & \text{otherwise}, \end{cases} \]

where \( C \) is a positive constant. Assume that the conditions of Lemma 3.1 are satisfied. For any \( \varepsilon > 0 \), let

(3.16) \[ P_{\varepsilon}(\xi, \gamma, n) = P \left( \sum_{j=1}^{n-q} u_n(t_j, T_{\lfloor nt_j \rfloor}, T_{\lfloor nt_j + q \rfloor} - T_{\lfloor nt_j \rfloor}) \right. \\
\left. - \sum_{j=1}^{n-q} u(t_j, T_{\lfloor nt_j \rfloor}, T_{\lfloor nt_j + q \rfloor} - T_{\lfloor nt_j \rfloor}) \right) > \varepsilon \]

Then

(3.17) \[ \lim_{r \to 0} \lim_{n \to \infty} P_{\varepsilon}(\xi, \gamma, n) = 0. \]
PROOF. Let $j$ be any integer such that $1 \leq j \leq b$. Let

$$v_j(x) = u_n(t_j, x) - u(t_j, x)$$

and $V_j = v_j(T_{nt+j})$. Further, let

$$X_j = \begin{cases} 
T_{nt+j} - T_{nt+j+q}, & \text{if } |T_{nt+j} - T_{nt+j+q}| < N, \\
0, & \text{otherwise,}
\end{cases}$$

$$Y_j = X_j - EX_j$$

and

$$Z_j = T_{nt+j} - T_{nt+j+q} - Y_j.$$ 

Then

$$P_\delta(\epsilon, \gamma, n) \leq P\left( \left| \sum_{j=1}^{b} V_j Y_j \right| > \frac{\epsilon}{2} \right) + P\left( \left| \sum_{j=1}^{b} V_j Z_j \right| > \frac{\epsilon}{2} \right)$$

$$\leq P\left( N \sum_{j=1}^{b} |V_j| > \frac{\epsilon}{2} \right) + P\left( \max_{1 \leq j \leq b} |Z_j| \sum_{j=1}^{b} |V_j| > \frac{\epsilon}{2} \right)$$

$$\leq 2P\left( N \sum_{j=1}^{b} |V_j| > \frac{\epsilon}{2} \right) + P\left( \max_{1 \leq j \leq b} |Z_j| > N \right).$$

Since by (2.5) $v_j(x) \to 0$ ($n \to \infty$) uniformly in $x \in [-C, C]$ for each $j(1 \leq j \leq b)$, so

$$EV_j = \int_{-C}^{C} \{v_j(x)\} dF_j(x) \to 0 \quad (n \to \infty)$$

where $F_j$ is the df of $T_{nt+j}$. Hence

$$P\left( N \sum_{j=1}^{b} |V_j| > \frac{\epsilon}{2} \right) \leq 4N^2 \epsilon^{-b} \max_{1 \leq j \leq b} EV_j \to 0 \quad (n \to \infty).$$

On the other hand, noting that $E|Z_j| = n(1+o(1))$, we have

$$P\left( \max_{1 \leq j \leq b} |Z_j| > N \right) \leq \sum_{1 \leq j \leq b} P(|Z_j| > N)$$

$$\leq \sum_{1 \leq j \leq b} N^{-b}EZ_j \leq \sum_{1 \leq j \leq b} N^{-b}E|T_{nt+j} - T_{nt+j+q}|$$

$$\leq N^{-b}K \sum_{j=1}^{b} (t_j - t_{j-1}) \leq KN^{-\epsilon}.$$

From (3.19)-(3.21) it follows that for all $N$

$$\lim_{n \to \infty} P_\delta(\epsilon, \gamma, n) \leq KN^{-\epsilon}.$$ 

Thus, we have the lemma.

**Lemma 3.3.** Under the conditions of Lemma 3.2

$$\sum_{j=1}^{b} u(t_j, T_{nt+j})(T_{nt+j} - T_{nt+j+q}) \xrightarrow{D} \sum_{j=1}^{b} u(t_j, w(t_j))(w(t_{j+1}) - u(t_j)) \quad (n \to \infty).$$


PROOF. (3.23) follows from Remark 2.2, since \( u \) is continuous.

The following lemma was proved in Yoshihara [5].

**Lemma 3.4.** Let \( u \) be the same function as the one in Lemma 3.2. For any \( \varepsilon > 0 \), let

\[
P_\varepsilon(r) = P\left( \left| \sum_{j=1}^{n} u(t_j, w(t_j)) - \int_0^t u(t, w(t)) \, dw(t) \right| > \varepsilon \right).
\]

Then

\[
\lim_{r \to 0} P_\varepsilon(r) = 0.
\]

**Proof of Theorem.** For any \( C > 0 \), let \( f^{(C)} \) and \( f_n^{(C)}(n=1, 2, \ldots) \) be functions defined in the preceding lemmas. Then for any \( \varepsilon > 0 \)

\[
P\left( \left| \sum_{i=1}^{n} f_n\left( \frac{i}{n}, T_i \right) \frac{\xi_{i+1}}{\sqrt{n}} - \int_0^1 f(t, w(t)) \, dw(t) \right| > \varepsilon \right)
\]

\[
\leq P\left( \left| \sum_{i=1}^{n} f^{(C)}\left( \frac{i}{n}, T_i \right) \frac{\xi_{i+1}}{\sqrt{n}} - \int_0^1 f^{(C)}(t, w(t)) \, dw(t) \right| > \varepsilon \right)
\]

\[+ P\left( \max_{1 \leq i \leq n} |T_i| > C \right) + P\left( \sup_{0 \leq t \leq 1} |w(t)| > C \right).
\]

Let \( \{C_n\} \) be an increasing sequence of positive numbers such that \( C_n \to \infty \). Since \( S_n, \ldots, S_n \) are partial sums of mixing sequences, so by Theorem 3 in Yoshihara [4] we have

\[
\lim_{C_n \to \infty} P\left( \max_{1 \leq i \leq n} |T_i| > C_n \right) \leq \lim_{C_n \to \infty} C_n E\left( \max_{1 \leq i \leq n} |T_i| \right) = 0.
\]

On the other hand, it is clear that

\[
\lim_{C_n \to \infty} P\left( \sup_{0 \leq t \leq 1} |w(t)| > C_n \right) = 0
\]

holds. Hence, (2.6) is obtained from Lemmas 3.1-3.4 and (3.26)-(3.28).

**4. Concluding remark.** It is desirable to prove that

\[
\sum_{i=1}^{n-1} f_n\left( \frac{i}{n}, S_i \right) \frac{\xi_{i+1}}{\sqrt{n}} \xrightarrow{\mathcal{D}} \int_0^1 f(t, w(t)) \, dw(t)
\]

instead of (2.6). Obviously, under conditions of Theorem (4.1) holds for \( f_n(t, x) \) of special type, such as \( f_n(t, x) = g_n(t) + ax + b \). But, it seems to be difficult to prove (4.1) for \( f_n(t, x) \) of general type.

**References**


