Some Examples of Weakly Elliptic Singularities

By

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§ 1. Introduction.

In [6], Wagreich introduced the definition of weakly elliptic singularities, and found all the possible dual graphs that will be weakly elliptic. There are many observations of weakly elliptic singularities. In [4], Laufer developed the theory of minimally elliptic singularities.

DEFINITION 1.1. (Laufer) Let \((X, x)\) be a 2-dimensional normal singularity, and \(\pi: \tilde{X} \to X\) be the minimal resolution of \((X, x)\). If \((X, x)\) is weakly elliptic and if any connected proper subvariety of the exceptional set \(\pi^{-1}(x)\) contracts into a rational singularity, then we shall say that \((X, x)\) is minimally elliptic.

THEOREM 1.2. (Laufer) An \((X, x)\) is minimally elliptic if and only if it is strongly elliptic and whose local ring \(\mathcal{O}_{X, x}\) is Gorenstein, where Gorenstein means that there are a neighborhood of \(x\) in \(X\) and a non-vanishing holomorphic 2-form on the deleted neighborhood.

Here, a strongly elliptic singularity means an \((X, x)\) whose geometric genus \(p_g\) equal to 2.

For an \((X, x)\) whose \(p_g\) equal to 2, the following Theorem is proved.

THEOREM 1.3. (Watanabe [7], Yoshinaga and Ohyanagi [10]) Let \((X, x)\) be a 2-dimensional normal singularity \(p_g=2\), and \(\pi: \tilde{X} \to X\) be the minimal resolution of \((X, x)\). Then the following conditions are equivalent:

(i) \(\mathcal{O}_{X, x}\) is Gorenstein;

(ii) \((X, x)\) is weakly elliptic, and any connected proper subvariety of the exceptional set \(\pi^{-1}(x)\) contracts into either a rational singularity or a strongly elliptic singularity.

Recently, Yau [8, 9] developed two observations for certain weakly elliptic singularities.

DEFINITION 1.4. Let \((X, x)\) be a weakly elliptic singularity, and \(\pi: \tilde{X} \to X\) be the minimal resolution of \((X, x)\). Then there exists the minimally elliptic cycle \(E\) on \(A=\pi^{-1}(x)\); see [4]. If for all irreducible component \(A_i\) of the \(A\) that \(A_i \supset |E|\) and \(A_i \cap |E| \neq 0\), then \(A_i \cdot Z < 0\), where \(Z\) denotes the fundamental cycle on \(A\) and \(|E|\) denotes the support of \(E\), defined by \(\cup \{A_i\ \text{with non-zero coefficient in} \ E\}\). We call \((X, x)\) an almost minimally elliptic singularity.

DEFINITION 1.5. Let \(A\) be the exceptional set of the minimal good resolution \(\pi: \tilde{X} \to X\), where \(X\) is a normal 2-dimensional analytic space with \(x\) as its
only weakly elliptic singularity. $E$ and $Z$ are as in Definition 1.4. If $E \cdot Z < 0$, we say that the elliptic sequence is $\{Z\}$ and the length of elliptic sequence is equal to 1. Suppose $E \cdot Z = 0$. Let $B_1$ be the maximal connected subvariety of $A$ such that $B_1 \supseteq |E|$ and $A_i \cdot Z = 0$ for all $A_i \subseteq B_1$. Since $A$ is an exceptional set, $Z \cdot Z < 0$. So $B_1$ is properly contained in $A$. Let $Z_1$ be the fundamental cycle on $B_1$. Suppose $Z_1 \cdot E = 0$. Let $B_2$ be the maximal connected subvariety of $B_1$ such that $B_2 \supseteq E$ and $A_i \cdot Z_1 = 0$ for all $A_i \subseteq B_2$. By the same argument as above, $B_2$ is properly contained in $B_1$. Continuing this process, we finally obtain $B_m$ with $Z_m \cdot E < 0$. We call $\{Z_0 = Z, Z_1, \ldots, Z_m\}$ the elliptic sequence, and the length of the elliptic sequence is $m + 1$.

**Definition 1.6.** Let $\pi: \tilde{X} \rightarrow X$ be the minimal good resolution of 2-dimensional normal singularity $(X, x)$. Suppose that $(X, x)$ is a weakly elliptic singularity, and the canonical divisor $K$ on $X$ is given by a cycle on $A = \pi^{-1}(x)$, where the canonical divisor is the numerically equivalent divisor of the canonical line bundle of $\tilde{X}$. If its geometric genus $p_g = \text{length of the elliptic sequence}$, then $(X, x)$ is called a maximally elliptic singularity.

**Remark.** In general, for a weakly elliptic singularity, its $p_g$ is less than, or equal to length of the elliptic sequence.

In this short work, we will give some examples of weakly elliptic singularities. These are useful to make a required singularity restricted whose geometric genus, for instance, it is an almost minimally elliptic singularity, but not maximally elliptic singularity.

§ 2. Preliminaries.

Let $A$ be a connected curve. If $A$ is the exceptional set of the resolution of some 2-dimensional normal singularity, then we say that $A$ is contractible. Then $A$ consists of a finite number of irreducible components $A_i, 1 \leq i \leq n$. In what follows, by a curve, we shall mean a contractible curve as above.

By [2], the intersection matrix $(A_i \cdot A_j)$ is negative-definite and symmetric.

By a divisor, we shall mean an element of the vector space generated by curves $A_i, 1 \leq i \leq n$, over the rational numbers $\mathbb{Q}$. A divisor is called a cycle if all coefficients are integers. There is a natural partial ordering between cycles defined by comparing the coefficients.

The intersection number $D_i \cdot D_j$ of divisors $D_i$ and $D_j$ can be naturally defined.

**Definition 2.1.** For a cycle $D$, the virtual genus of $D$ is defined by $p(D) = 1 + (D \cdot D + D \cdot K)/2$, where $K$ is the canonical divisor of $A$. $D \cdot K$ may be defined as follows. Let $\omega$ be a meromorphic 2-form on a strongly pseudoconvex neighborhood of $A$, i.e., a meromorphic section of the canonical line bundle of $\tilde{X}$. Let $(\omega)$ be the divisor of $\omega$. Then $D \cdot K = D \cdot (\omega)$, and this is independent of the choice of $\omega$.

It follows immediately from the definition that if $B$ and $C$ are cycles, then
Some Examples of Weakly Elliptic Singularities

Definition 2.2. For a curve $A$, the arithmetic genus $p_a$ is defined by $\sup \{p(D) | D$ is a positive cycle on $A \}$, where a positive cycle means that all coefficients of it are non-negative.

There exists the minimal cycle among those cycles $Z$, which are positive and satisfy $A_i \cdot Z \leq 0$ for all $A_i$. We shall call this minimal cycle the fundamental cycle on $A$. We shall denote the virtual genus of the fundamental cycle by $p_f$.

The choice of the resolution does not have an effect to the values $p_a$ and $p_f$. So, for the future, we assume that the resolution is minimal. Namely, among $A_i$\'s, there is no nonsingular rational curve with self-intersection number $-1$.

Definition 2.3. The geometric genus $p_g$ of $(X, x)$ is defined by $\dim_{\mathbb{C}} (R^1 \pi_* O_X)_x$, where $\pi: (\bar{X}, \bar{A}) \to (X, x)$ is a (the minimal) resolution of $(X, x)$. Also, this is independent of the choice of the resolution.

Among $p_f$, $p_a$ and $p_g$, the following relations are known.

Theorem 2.4. (Artin [1]) (i) $0 \leq p_f \leq p_a \leq p_g$, (ii) $p_f = 0 \Rightarrow p_a = 0 \Rightarrow p_g = 0$.

Definition 2.5. If $(X, x)$ has $p_g = 0$, then it is said to be rational. If $(X, x)$ has $p_a = 0$, then it is said to be weakly elliptic. If $(X, x)$ has $p_f = 1$, then it is said to be strongly elliptic.

Remark. A weakly elliptic singularity is not always strongly elliptic.

Theorem 2.6. (Wagreich [6]) If $p_f = 1$, then $p_a = 1$.

For a curve $A = \sqcup A_i$, we denote the self-intersection number of $A_i$ by $-a_i$. In [5], the following Theorem is proved.

Theorem 2.7. (Ohyanagi) If we take $(a'_1, a'_2, \ldots, a'_n)$ with $a'_i \geq a_i$ for all $i$, then the $p_g$ (resp. $p_a$) of the curve $A$ with $(a'_i)$ is less than, or equal to the $p_g$ (resp. $p_a$) of the original curve $A$ with $(a_i)$.

§ 3. Constructions.

Let $A = \sqcup A_i$, $1 \leq i \leq n$, be a connected curve such that $A_1$ is a nonsingular elliptic curve and other $A_i$\'s are nonsingular rational curves. Let $A_1 \cdot A_2 = A_2 \cdot A_3 = \cdots = A_{n-1} \cdot A_n = 1$ and otherwise $A_i \cdot A_j = 0$, $i \neq j$. Namely $A$ has a chain-shaped dual graph described below.

Now, we assume the resolution is minimal, so $a_i \geq 2$ for $i \geq 2$. In this case, the resolution is also minimal good. For our curve, it is contractible if and only if $a_i - [a_2, \ldots, a_n]^{-1} > 0$.

Since $[a_2, \ldots, a_n] \geq [2, \ldots, 2] = n/(n-1)$, we shall consider the case of $(a_1, a_2, \ldots, a_n) = (1, 2, \ldots, 2)$. For other cases $(a'_1, a'_2, \ldots, a'_n)$, we have always the properties $a'_i \geq 1$ and $a'_i \geq 2$ for $i \geq 2$, so their genera are restricted by the genus of the case of $(1, 2, \ldots, 2)$; see Theorem 2.7. Then we can see that the fundamental cycle on $A$ is given by $A_1 + A_2 + \cdots + A_n$ and $p_f$ of $A$ equals to 1.
By Theorem 2.6, we have the $p_a$ equals to 1, i.e., $A$ is weakly elliptic. Since $A$ has an elliptic curve in it, the $p_a$ of $A$ is greater than, or equal to 1 for any combination of $a_i$'s. Hence for any $(a_1, a_2, \cdots, a_n)$,

$$1 \leq \text{the } p_a\text{ of } (a_i) \leq \text{the } p_a\text{ of } (1, 2, \cdots, 2) = p_f = 1.$$  

The second inequality follows from Theorem 2.7. Thus our curve $A$ is always weakly elliptic for any combination of the self-intersection numbers $a_i$.

The following equations are well-known.

\[(*) \quad A_iK = -A_iA_i + 2g_i - 2 + 2\delta_i, \text{ for all } i,\]

where $g_i$ denotes the genus of $A_i$ as a Riemann surface and $\delta_i$ denotes the conductor number of the singularities on $A_i$, in particular, $\delta_i = 0$ for a non-singular curve $A_i$.

Applying this equation to our curve with self-intersection numbers $(1, 2, \cdots, 2)$, we have that $A_1K = 1 + 2 - 2 + 0 = 1$, $A_2K \cdots = A_nK = 2 + 0 - 2 + 0 = 0$. Let the canonical divisor $K = \sum r_iA_i, 1 \leq i \leq n$, where $r_i \in \mathbb{Q}$. Then by the intersection matrix of $A$, we have the following system of equations.

\[
(*)^* \quad \begin{cases} 
A_1K = -r_1 + r_2 = 1, \\
A_2K = r_1 - 2r_2 + r_3 = 0, \\
\vdots \\
A_{i-1}K = r_{i-1} - 2r_i + r_{i+1} = 0, \\
A_nK = r_{n-1} - 2r_n = 0.
\end{cases}
\]

We can easily solve $(**)$ as follows: $r_1 = -n$, $r_2 = -(n-1)$, $\cdots$, $r_i = -(n-i+1)$, $\cdots$, $r_n = -1$. Hence $K = -nA_1 - (n-1)A_2 - \cdots - A_n$, this says that $A$ with $(1, 2, \cdots, 2)$ is numerically Gorenstein; see Definition 1.6.

REMARK. The numerically Gorenstein'ness does not follow from this for any other $A$ with $(a_1, a_2, \cdots, a_n)$.

Let $P$ denote the point of intersection between $A_1$ and $A_2$, $A_1A_2 = 1$, and $R$ denote the point of $A_1$ which gives the conormal bundle of $A_1$, $A_1A_1 = -1$. 

\[
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\]
DEFINITION 3.1. Let $P$ and $R$ be as above. If $P=R$ on $A_i$, we say that $P$ and $R$ are (linearly) equivalent of order 1. Suppose that $P \neq R$. Then if there is a meromorphic function on $A_i$ whose divisor (zeros and poles with multiplicities) is $mp-mr$, then we shall say that $P$ and $R$ are linearly equivalent of order $m$, and denote this by $mP \sim mR$.

We shall consider that there will be the difference between the curves with the different conditions for $P$ and $R$ on $A_i$.

REMARK. If $P=R$, then $mP \sim mR$ for all integers $m$.

Our curve $A$ is star-shaped, i.e., $A_i$ is the center and $A_1 \cup \cdots \cup A_n$ is the only branch. The following Theorem which gives the algorithm to calculate $p_\delta$ for a star-shaped exceptional set, is proved by Watanabe in [7].

THEOREM 3.2. Let $B$ be a star-shaped contractible curve with the center $B_0$ and branches $B^{(i)}$, $1 \leq i \leq n$, where we shall denote the $i$-th branch $B^{(i)}$ by a continued fraction $d_i = [b'_1, b'_2, \ldots, b'_l]$, $b'_j = -B_{b'_j} \cdot B^{(i)}_j$ for $B^{(i)} = B^{(i)}_1 \cup B^{(i)}_2 \cup \cdots \cup B^{(i)}_l$, with $B^{(i)}_0 \cap B_0 \neq \emptyset$ and $B^{(i)}_0 \cap B^{(i)}_{j+1} \neq \emptyset$.

Let $D^{(k)} = kd - \sum_{i=1}^{l} (k\delta_i/d_i) P_i$, a divisor on $B_0$, $\{P_i\} = B^{(i)} \cap B_0$. For any $a \in \mathbb{R}$, we denote the least integer greater than, or equal to a by $\{a\}$.

Then the $p_\delta$ of $B$ is $\sum_{k \in \mathbb{Z}} \dim_c H^0(B_0, O_{B_0}(D^{(k)}))$.

By Serre duality, this is rewritten as follows:

$$\sum_{k \in \mathbb{Z}} \dim_c H^0(B_0, O_{B_0}(K - D^{(k)})),$$

where $K$ is the canonical line bundle of $B$.

For our curve $A$ with $(1, 2, \ldots, 2)$, $B_0 = A_1$ and $B^{(i)} = A_1 \cup \cdots \cup A_n$, only branch with $[2, \ldots, 2] = n/(n-1)$. Then $D^{(k)} = kR - (k(n-1)/n)P$. Hence by the Theorem, $p_\delta = \sum_{k \geq 0} \dim_c H^0(A_1, O_{A_1}(K - D^{(k)}))$.

LEMMA 3.3. For any $k \geq 0$, there is a non-negative integer $m$ such that $mn \leq k < (m+1)n$. Then $(k(n-1)/n) = k - m$.

PROOF. Since $mn \leq k < (m+1)n$, $k-m-1 < k(n-1)/n \leq k-m$.

Thus $D^{(0)} = 0$, $D^{(1)} = R - P$, $D^{(2)} = 2R - 2P$, $D^{(3)} = 3R - 3P$, $D^{(n+1)} = (n-1)R - (n-1)P, D^{(n+2)} = (n+1)R - nP, \ldots, D^{(2n-1)} = 2nR - (2n-2)P$ and so on.

Hence the degree of the line bundle $K_{A_1} - D^{(k)}$ is as follows. For $k \leq (n-1)$, $\deg(K_{A_1} - D^{(k)}) = 0$, and for $k \geq n$, $\deg(K_{A_1} - D^{(k)}) < 0$.

REMARK. The $A_i$ is an elliptic curve, so $K_{A_1}$ is trivial.

Therefore $p_\delta = \sum_{0 \leq k \leq n-1} \dim_c H^0(A_1, O_{A_1}(-D^{(k)})) = \sum_{0 \leq k \leq n-1} \dim_c H^0(A_1, O_{A_1}(kR - kP))$.

If $kP \sim kR$, then $\dim_c H^0(A_1, O_{A_1}(kR - kP)) = 1$, and if $kP \not\sim kR$, then $\dim_c H^0(A_1, O_{A_1}(kR - kP)) = 0$.

Hence we have that
Thus $1 \leq p_g \leq n$. For $p_g = n$, we must have $P = R$. For $p_g = 1$, we must have $P \neq R$, $2P \neq 2R$, $\cdots$, and $(n-1)P \neq (n-1)R$.

In this assertion, we have not said whether $mP \sim mR$ or $mP \neq mR$ for all $m \geq n$. So we cannot distinguish between $A$ with $\{k_1, \cdots, k_l\}$, $k_lP \sim k_lR$, and $mP \sim mR$ for $m \geq n$, and $A$ with the same $\{k_1, \cdots, k_l\}$ and $mP \neq mR$, under the information of $p_g$.

There is introduced the geometric plurigenera $d_m$, $m \geq 1$, by Watanabe [7]. One can distinguish the above difference with $d_m$, $m \geq 1$; see Example 4.5.

§ 4. Applications.

4.1. Gorenstein singularities.

DEFINITION 4.1.1. Let $(X, x)$ be an $n$-dimensional normal isolated singularity. If the following conditions are satisfied, we say that $(X, x)$ is a Gorenstein singularity.

(i) The local ring $\mathcal{O}_{X, x}$ is Cohen-Macaulay, i.e., depth $\mathcal{O}_{X, x} = \dim \mathcal{O}_{X, x}$.

(ii) There exist a neighborhood $U$ of $x$ in $X$ and a non-vanishing holomorphic $n$-form $\omega$ on $U - \{x\}$.

In particular, for a 2-dimensional normal singularity $(X, x)$, the $\mathcal{O}_{X, x}$ is always Cohen-Macaulay.

THEOREM 4.1.2. (Watanabe [7]) Let $(X, x)$ be a 2-dimensional Gorenstein singularity and $\pi: (\bar{X}, \bar{A}) \rightarrow (X, x)$ be the minimal resolution. Then for any connected proper subvariety $A'$ of $A$, we have $p_{\bar{A}}(A') < p_{\bar{A}}(A)$, where $p_{\bar{A}}(A)$ (resp. $p_{\bar{A}}(A')$) denotes the geometric genus of $(X, x)$ (resp. the geometric genus of the 2-dimensional normal singularity obtained from $A'$ via blow-down).

THEOREM 4.1.3. (Watanabe [7]) Let $(X, x)$ be a weakly elliptic singularity and $\pi: (\bar{X}, \bar{A}) \rightarrow (X, x)$ be the minimal resolution. If for any connected proper subvariety $A'$ of $A$, $p_{\bar{A}}(A') < p_{\bar{A}}(A)$, then $(X, x)$ is Gorenstein.

Our curve $A$ constructed in section 3, is weakly elliptic. Hence our curve $A$ with $(1, 2, \cdots, 2)$ and $(n-1)P \sim (n-1)R$ is a Gorenstein singularity. If $A' \cong A_1$, $A'$ is the exceptional set of the minimal resolution of a cyclic quotient singularity, i.e., $p_{\bar{A}}(A') = 0$. If $A' = A_1 \cup A_2 \cdots \cup A_l$, $l < n$, $p_{\bar{A}}(A') = \#\{k | 0 \leq k \leq l-1, kP \sim kR\}$. Since $(n-1)P \sim (n-1)R$, $\#\{k | 0 \leq k \leq l-1, kP \sim kR\} < \#\{k | 0 \leq k \leq n-1, kP \sim kR\} = p_{\bar{A}}(A)$.

4.2. Almost minimally elliptic singularities.

Let $n = 2$. Then $Z = A_1 + A_2$ and $E = A_1$. We have that $A_2 \cong |E| = A_1$ and $A_2 \cap |E| = A_2 \cap A_1 \neq 0$. Then $A_2 \cdot Z = A_2 \cdot (A_1 + A_2) = A_1 \cdot A_2 + A_2 \cdot A_2 = 1 - 2 = -1 < 0$. So this is an almost minimally elliptic singularity. In this case, the $p_g$ is 1 or 2.

4.3. Maximally elliptic singularities.

Let $A$ be as in (4.2). The elliptic sequence is $\{A_1 + A_2, A_1\}$. So the length of it is 2. Let $P = R$. Then $p_g = \#\{k | 0 \leq k \leq 1, kP \sim kR\} = \#\{0, 1\} = 2$, this is
maximally elliptic. Now this curve $A$ with $P=R$ gives an example which is not only almost minimally elliptic but also maximally elliptic.

Nextly, let $P\neq R$, then we have $p_2=1$. Since the length of the elliptic sequence is 2, this is not maximally elliptic, but this is almost minimally elliptic.

4.4. The following curve is neither almost minimally elliptic nor maximally elliptic. Let $n\geq 3$. Then this is not almost minimally elliptic, since $A_2$ satisfies $A_2\subseteq |E|$ and $A_2 \cap |E| \neq 0$, but $A_3 \cdot Z = 0$.

Let $(n-1)P \sim (n-1)R$ and $kP \not\sim kR$ for some $1 \leq k \leq n-2$, then $p_3 < n$. But the elliptic sequence is $\{A_1 + A_2 + \cdots + A_{n-2}, A_1 + A_2 + \cdots + A_{n-1}, \cdots, A_1\}$, so the length of it is $n$. Hence this is not maximally elliptic.

**Remark.** $A$ is maximally elliptic if and only if $P=R$.

4.5. Geometric plurigenera. Let $n=2$. Then we can distinguish the following different cases with $\delta_m = \dim_c H^q(\check{X} - A, O(mK))/L^{2/m}(\check{X} - A)$, $K$ is the canonical line bundle on $\check{X}$ and $L^{2/m}(\check{X} - A)$ is the subspace of locally square integrable holomorphic $m$-ple 2-forms on $\check{X} - A$, in particular $\delta_1 = p_2$; [3].

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