Totally real submanifolds of a complex space form with nonzero parallel mean curvature vector

By

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Abstract. Any pseudo-umbilical submanifold of a non-flat complex space form with nonzero parallel mean curvature vector is a totally real submanifold.

1. Introduction

The aim of this paper is to show that the class of totally real submanifolds of a non-flat complex space form contains all pseudo-umbilical submanifolds with nonzero parallel mean curvature vector.

Our result is as follows:

Theorem 1 Let $M^n$ be an $n$-dimensional pseudo-umbilical submanifold of a complex $m$-dimensional complex space form $\tilde{M}^m(\tilde{c})(\tilde{c} \neq 0)$ with nonzero parallel mean curvature vector. Then $2m > n$ and $M$ is a totally real submanifold of $\tilde{M}^m(\tilde{c})$.

As an immediate consequence of Theorem 1, we see the following fact: The mean curvature vector of a non-minimal, pseudo-umbilical submanifold of $\tilde{M}^m(\tilde{c})(\tilde{c} \neq 0)$ is not parallel unless it is totally real.

Moreover, we get the following

Corollary 1 Let $M^n$ be a pseudo-umbilical submanifold of $\tilde{M}^m(\tilde{c})(\tilde{c} \neq 0)$ with nonzero parallel mean curvature vector. Then the scalar curvature $\rho$ of $M$ satisfies the inequality:

$$\rho \leq n(n-1)(\frac{\tilde{c}}{4} + g(H, H)),$$

where $H$ is the mean curvature vector of $M$ in $\tilde{M}(\tilde{c})$.

If the equality sign of (1.1) holds, then $M$ is a real space form immersed in $\tilde{M}(\tilde{c})$.

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as a totally real and totally umbilical submanifold.

This case occurs only when $2m > n$.

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2. Preliminaries

Let $f$ be an isometric immersion of a Riemannian $n$–manifold $M^n$ into a Kaehlerian $m$–manifold $\tilde{M}^m$ with complex structure $J$. We denote by $g$ the metric tensor of $\tilde{M}$ as well as that induced on $M$. For all local formulas we regard $f$ as an imbedding and thus identify $x \in M$ with $f(x) \in \tilde{M}$. The tangent space $T_x(M)$ is identified with a subspace of $T_x(\tilde{M})$. The normal space $T_x^\perp(M)$ is a subspace of $T_x(\tilde{M})$ consisting of all $X \in T_x(M)$ which are orthogonal to $T_x(M)$ with respect to the Riemannian metric $g$. Let $\tilde{\nabla}$ (resp. $\nabla$) be the Riemannian connection on $\tilde{M}$ (resp. $M$). Moreover, we denote by $\sigma$ the second fundamental form of $M$ in $\tilde{M}$. Then the Gauss formula and the Weingarten formula are given respectively by

$$\sigma(X,Y) = \tilde{\nabla}_X Y - \nabla_X Y, \quad \text{for } X,Y \in T_x(M),$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \quad \text{for } \xi \in T_x^\perp(M),$$

where $-A_\xi X$ (resp. $D_X \xi$) denotes the tangential (resp. normal) component of $\tilde{\nabla}_X \xi$. A normal vector field $\xi$ is said to be parallel if $D_X \xi = 0$ for all $X \in T_x(M)$. Let $H = \frac{1}{n} \text{trace } \sigma$ be the mean curvature vector of $M$ in $\tilde{M}$. If the second fundamental form $\sigma(X,Y) = g(X,Y)H$, then $M$ is said to be totally umbilical. $M$ is said to be pseudo–umbilical if the second fundamental form $\sigma$ is of the form $g(\sigma(X,Y), H) = g(X,Y)g(H,H)$, for $X,Y \in T_x(M)$, or equivalently, $A_H = ||H||^2 I$.

Let $\tilde{R}$ (resp. $R$) be the Riemannian curvature for $\tilde{\nabla}$ (resp. $\nabla$). Then the Gauss equation is given by

$$g(\tilde{R}(X,Y)Z,W) = g(R(X,Y)Z,W) + g(\sigma(X,Z),\sigma(Y,W)) - g(\sigma(Y,Z),\sigma(X,W)),$$

for $X,Y,Z,W \in T_x(M)$. 
A Kaehlerian manifold of constant holomorphic sectional curvature $\tilde{c}$ is called a complex space form and will be denoted by $\tilde{M}(\tilde{c})$, and a Riemannian manifold of constant sectional curvature is called a real space form. The Riemannian curvature $\tilde{R}$ of $\tilde{M}(\tilde{c})$ is given by

$$\tilde{R}(X, Y)Z = \frac{\tilde{c}}{4} \{g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY + 2g(X, JY)JZ\},$$

for all vector fields $X, Y, Z$ on $\tilde{M}(\tilde{c})$.

The submanifold $M$ is called a totally real submanifold of $\tilde{M}$ if $J(T_x(M)) \subset T^\perp_x(M)$ (see, [1][2]).

### 3. Proof of Theorem

**Proof of Theorem 1.** Let $M^n$ be a pseudo-umbilical submanifold of $\tilde{M}(\tilde{c})(\tilde{c} \neq 0)$ with nonzero parallel mean curvature vector $H$. Then we get

$$\tilde{\nabla}_X H = -A_H X = -\|H\|^2,$$

for all $X, Y \in T_x(M)$, where $\|H\|$ is a constant. Therefore we have

$$\tilde{R}(X, Y)H = \tilde{\nabla}_X \tilde{\nabla}_Y H - \tilde{\nabla}_Y \tilde{\nabla}_X H - \tilde{\nabla}_{[X,Y]} H = \|H\|^2(\tilde{\nabla}_Y X - \tilde{\nabla}_X Y + [X, Y]) = 0$$

for all $X, Y \in T_x(M)$.

On the other hand, from (2.1) we get

$$g(\tilde{R}(X, Y)H, JH) = \frac{\tilde{c}}{2} g(X, JY)g(H, H).$$

Since $\tilde{c} \neq 0$ and $g(H, H) \neq 0$, it follows from (3.1) and (3.2) that $M$ is totally real in $\tilde{M}$.

Moreover, from (2.1) we get

$$g(\tilde{R}(X, Y)H, JX) = \frac{\tilde{c}}{4} g(g(X, X)JY - g(X, Y)JX, H).$$

Choose $X$ in such a way that $g(JX, H) = 0$. Since $\tilde{c} \neq 0$, from (3.1) and (3.3) we have

$$g(JY, H) = 0 \quad \text{for all} \quad Y \in T_x(M).$$
Since $M$ is a totally real submanifold of $\tilde{M}$, the normal space $T_{x}^{\perp}(M)$ is decomposed in the following way: $T_{x}^{\perp}(M) = JT_{x}(M) \oplus \nu_{x}$ at each point $x$ of $M$, where $\nu_{x}$ denotes the orthogonal complement of $JT_{x}(M)$ in $T_{x}^{\perp}(M)$. Thus it follows from (3.4) that $H \in \nu_{x}$. This implies $2m > n$. Q.E.D.

Proof of Corollary 1. Let $M^{n}$ be a pseudo-umbilical submanifold of $\tilde{M}(\tilde{c})(\tilde{c} \neq 0)$ with nonzero parallel mean curvature vector $H$. Let $\{E_{i}\}_{i=1,\cdots,n}$ be orthonormal tangent vector fields and $\{\xi_{\alpha}\}_{\alpha=1,\cdots,2m-n}$ orthonormal normal vector fields such that the nonzero mean curvature vector $H$ is equal to $\Vert H \Vert \xi_{1}$. Theorem 1 implies that $M$ is totally real in $\tilde{M}$. By the Gauss equation the scalar curvature $\rho$ is given by

$$\rho = n(n-1)(\frac{\tilde{c}}{4} + g(H, H)) - \sum_{i,j=1}^{n} \sum_{\alpha=2}^{2m-n} g(\sigma(E_{i}, E_{j}), \xi_{\alpha})^{2}.$$  \hspace{1cm} (3.5)

Thus we get (1.1). If the equality sign of (1.1) holds, it follows from (3.5) that $M$ is totally umbilical in $\tilde{M}$. By the Gauss equation, we easily see that any totally real and totally umbilical submanifold in $\tilde{M}(\tilde{c})$ also has constant curvature $\frac{\tilde{c}}{4} + g(H, H)$. Thus we obtain $M$ is a real space form immersed in $\tilde{M}$. Q.E.D.

References


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